# On the geometric-arithmetic mean inequality for matrices

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**Abstract**. *In this paper refinements and converses of matrix forms of the geometric-arithmetic mean inequality are given.*

**Key words:** *arithmetic, geometric means, Hermitian matrices*

**Sažetak**.*U radu se daje poboljšanje i konverzija nejednakosti između geometrijske i aritmetičke sredine za matrice.*

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## 1. Introduction

Let $\alpha \in ]0,1[$ and let $A, B$ be two $n \times n$ positive definite Hermitian matrices. The weighted arithmetic, geometric and harmonic means of $A$ and $B$ are respectively:

$$A \bigtriangledown B = \alpha A + (1-\alpha)B; \quad A\#B = B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^{\alpha}B^{1/2}; \quad A!B = \left(\alpha A^{-1} + (1-\alpha)B^{-1}\right)^{-1}. \tag{1}$$

The matrix form of the geometric-arithmetic mean inequality is well known:

$$A!B \leq A\#B \leq A \bigtriangledown B; \tag{2}$$

where the matrix inequality $A \geq B$ means that $A - B$ is positive semi-definite.

A converse inequality to a part of (2),

$$A!B \leq A \bigtriangledown B,$$

was obtained recently in [1].

In this paper we obtain further converses and refinements of (2).

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## 2. Preliminary lemmas

An essential tool in the proof of our main converse result is the following converse of the geometric- arithmetic mean inequality due to Dočev.

**Lemma 1.** *If* $0 < m \le a_i \le M,\ 1 \le i \le n,\ w_i > 0, 1 \le i \le n$ *and* $W_n = \sum_{i=1}^n w_i$ *then*

$$\frac{1}{W_n}\sum_{i=1}^n w_i a_i \le K\left(\prod_{i=1}^n a_i^{w_i}\right)^{1/W_n}; \tag{3}$$

*where if* $\beta = M/m$ *then*

$$K = \frac{(\beta-1)\beta^{1/(\beta-1)}}{e\log\beta}. \tag{4}$$

*Furthermore, K regarded as a function of M is increasing, and regarded as a function of m is decreasing.*

**Proof.** The inequality (3) is stated in [2, p.124]. The method of proof is that of [2, p.28, Remark(14)]; not as stated in the first reference. The last remark is an easy consequence of the proof and the concavity of the logarithmic function. □

The following generalization of Bernoulli's inequality is due to Gerber [3].

**Lemma 2.** *If* $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ *,and* $x > -1$ *then*

$$\operatorname{sgn}\left((1+x)^\alpha - \sum_{i=0}^n \binom{\alpha}{i} x^i\right) = \operatorname{sgn}\left(\binom{\alpha}{n+1} x^{n+1}\right). \tag{5}$$

Substituting $t-1$ for $x$ in (5) leads to

$$\binom{\alpha}{n+1}\left(t^\alpha - \sum_{i=0}^n \binom{\alpha}{i}(t-1)^i\right) \ge 0 \quad if \quad t \ge 1 \tag{6}$$

$$(-1)^{n+1}\binom{\alpha}{n+1}\left(t^\alpha - \sum_{i=0}^n \binom{\alpha}{i}(t-1)^i\right) \ge 0 \quad if \quad 1 \ge t > 0. \tag{7}$$

## 3. Converse results

**Theorem 1.** *Let A and B be two positive definite Hermitian matrices and let* $0 < \alpha < 1$, *then*

$$A \triangledown B \le K(A\#B),$$

*where K is given by (4) with* $\beta = \max\{\lambda_1,\ \lambda_n^{-1}\}$, *where* $\lambda_1 \ge \cdots \ge \lambda_n$ *are the eigenvalues of* $B^{-1/2}AB^{-1/2}$.

**Proof.** Apply *Lemma 1* with $n = 2,\ w_1 = \alpha, w_2 = 1-\alpha, a_1 = \lambda_i, a_2 = 1$ to get

$$\alpha\lambda_i + 1 - \alpha \le K_i\lambda_i^\alpha,$$

where $K_i$ is given by (4) with

$$\beta_i = \frac{\max\{\lambda_i,\ 1\}}{\min\{\lambda_i,\ 1\}} = \max\{\lambda_i,\ \lambda_i^{-1}\}.$$

Since $\lambda_1 \geq \cdots \geq \lambda_n$ we have, from *Lemma 1* that $K_i \leq K$, $1 \leq i \leq n$ where $K$ is given by (4) with $\beta$ as in the statement of the theorem. Hence

$$\alpha\lambda_i + 1 - \alpha \leq K\lambda_i^{\alpha},\ 1 \leq i \leq n. \tag{8}$$

If a positive definite matrix $C$ has the representation $C = \Gamma D_\lambda \Gamma^*$, where $\Gamma$ is unitary and $D_\lambda$ is the diagonal matrix $\mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$, then (8) implies that

$$\alpha D_\lambda + (1-\alpha)I \leq K D_\lambda^{\alpha}. \tag{9}$$

Pre- and post-multiplication of (9) by $\Gamma$ and $\Gamma^*$ respectively gives

$$\alpha C + (1-\alpha)I \leq K C^{\alpha}. \tag{10}$$

Then setting $C = B^{-1/2}AB^{-1/2}$ in (10) we get

$$\alpha B^{-1/2}AB^{-1/2} + (1-\alpha)I \leq \left(B^{-1/2}AB^{-1/2}\right)^{\alpha}. \tag{11}$$

Pre- and post-multiplication of (11) by $B^{1/2}$ completes the proof of the theorem. □

**Theorem 2.** *With the hypotheses and notations of* Theorem 1*,*

$$A\#B \leq K(A!B).$$

**Proof.** Apply *Theorem 1* with $A, B$ replaced by $A^{-1}, B^{-1}$, respectively. □

## 4. Refinements

In this section we use the notations in section 1 but allow $\alpha$ to be an arbitrary real number.

**Theorem 3.** *Let $A$ and $B$ be two positive definite Hermitian matrices and let $\alpha \in \mathbb{R}$ then*

$$\binom{\alpha}{n+1}\left(A\#B - A\triangledown B\right) \leq \binom{\alpha}{n+1}\sum_{i=2}^{n}\binom{\alpha}{i}B^{1/2}\left(B^{-1/2}(A-B)B^{-1/2}\right)^{i}B^{1/2}\ \textit{if}\ A > B; \tag{12}$$

$$(-1)^{n+1}\tbinom{\alpha}{n+1}\left(A\#B - A\triangledown B\right) \geq (-1)^{n+1}\tbinom{\alpha}{n+1}\textstyle\sum_{i=2}^{n}\tbinom{\alpha}{i}B^{1/2}\left(B^{-1/2}(A-B)B^{-1/2}\right)^{i}B^{1/2}\ \textit{if}\ A < B. \tag{13}$$

**Proof.** Let $C$ be a positive definite Hermitian matrix then as in the proof of *Theorem 1*, but using (6) and (7), we get

$$\binom{\alpha}{n+1}\left(C^{\alpha} - \sum_{i=0}^{n}\binom{\alpha}{i}(C-I)^{i}\right) \geq 0 \quad \textit{if} \quad C > I \tag{14}$$

$$(-1)^{n+1}\binom{\alpha}{n+1}\left(C^{\alpha}-\sum_{i=0}^{n}\binom{\alpha}{i}(C-I)^{i}\right)\geq 0 \quad if \quad I>C>0. \tag{15}$$

Simple rewriting of (14) and (15) give:

$$\binom{\alpha}{n+1}\left(C^{\alpha}-\alpha C-(1-\alpha)I\right)\geq\binom{\alpha}{n+1}\sum_{i=2}^{n}\binom{\alpha}{i}(C-I)^{i} \quad if \quad C>I \tag{16}$$

$$(-1)^{n+1}\tbinom{\alpha}{n+1}\left(C^{\alpha}-\alpha sC-(1-\alpha)I\right)\geq(-1)^{n+1}\tbinom{\alpha}{n+1}\textstyle\sum_{i=2}^{n}\tbinom{\alpha}{i}(C-I)^{i} \quad if \quad I>C>0. \tag{17}$$

Substitution in (16) and 17) of $C = B^{-1/2}AB^{-1/2}$ and then pre- and post-multiplication by $B^{1/2}$ leads to (12) and (13). □

The special case $n = 1$ of *Theorem 3* was proved in [4].

The case $n = 2$ of *Theorem 3* gives

**Corollary 1.** *Let $A$ and $B$ be two positive definite Hermitian matrices and let $\alpha \in \mathbb{R}$. If $A \geq B$ and if either $0 < \alpha < 1$ or $\alpha > 2$, then*

$$A \bigtriangledown B - A\#B \leq \frac{\alpha(1-\alpha)}{2}B^{1/2}\big(B^{-1/2}(A-B)B^{-1/2}\big)^{2}B^{1/2}.$$

*If $\alpha < 0$ or $1 < \alpha < 2$, then the reverse inequality holds.*

Using a generalization of Bernoulli's inequality proved in [5] we can generalize the corollary as follows.

**Theorem 4.** *Let $A$ and $B$ be two positive definite Hermitian matrices and let $\alpha, K \in \mathbb{R}$. If $A \geq KB$, with $0 < K \leq 1$ and if either $0 < \alpha < 1$ or $\alpha > 2$, then*

$$A \bigtriangledown B - A\#B \leq \frac{\alpha(1-\alpha)}{2}K^{\alpha-2}B^{1/2}\big(B^{-1/2}(A-B)B^{-1/2}\big)^{2}B^{1/2}.$$

*If $\alpha < 0$ or $1 < \alpha < 2$, then the reverse inequality holds. The inequality also holds if $A \leq KB$, with $K \geq 1$ and if either $\alpha < 0$ or $1 < \alpha < 2$; the reverse inequality then holds if $0 < \alpha < 1$ or $\alpha > 2$.*

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