

A summability method in some strong laws of large numbers *

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Abstract. *This survey paper contains several results concerning applications of some summability methods in ergodic theory and in generalizations of some strong laws of large numbers.*

Key words: *strongly regular matrix, locally convex vector space, Banach space, ergodic transformation, random variable, random element*

Sažetak. Jedna metoda sumabilnosti u nekim jakim zakonima velikih brojeva. *Ovaj ekspozitorni članak daje pregled nekoliko novijih rezultata o primjenama nekih metoda sumabilnosti u ergodskoj teoriji i u generalizacijama nekih jakih zakona velikih brojeva.*

Ključne riječi: *jako regularna matrica, lokalno konveksan vektorski prostor, Banachov prostor, ergodska transformacija, slučajna varijabla, slučajni element*

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1. Introduction

Bernoulli law of large numbers states the following: If $Z_n \sim B(n, p)$ ($n \in \mathbb{N}$) is a sequence of binomial distributions, then the sequence $(\frac{Z_n}{n}, n \in \mathbb{N})$ of relative frequencies of the success (in n Bernoulli trials) converges in probability to probability p of obtaining the success in one trial, i.e. we have: $(P) \lim_{n \rightarrow \infty} \frac{Z_n}{n} = p$.

Borel law of large numbers generalizes this result in the sense that we have: (a.s.) $\lim_{n \rightarrow \infty} \frac{Z_n}{n} = p$.

For each $n \in \mathbb{N}$, we have $Z_n = \sum_{j=1}^n X_j$, where $(X_n, n \in \mathbb{N})$ is a sequence of independent Bernoulli distributions with parameter p , i.e. $X_n = \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$ ($n \in \mathbb{N}$). Therefore, we can look at the mentioned laws of large numbers as results

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about Cesàro $(c, 1)$ summability (in the sense of convergence in probability or almost surely) of the sequence $(X_n, n \in \mathbb{N})$. Indeed, the sequence $(\frac{Z_n}{n}, n \in \mathbb{N})$ is obtained from the sequence $(X_n, n \in \mathbb{N})$ by the matrix transformation with **Cesàro** $(c, 1)$ **summability matrix** $C = [c_{nj}]$ $(n, j \in \mathbb{N})$ defined by

$$c_{nj} = \begin{cases} \frac{1}{n}, & 1 \leq j \leq n \\ 0, & j > n \end{cases}, \quad n, j \in \mathbb{N}. \quad (1)$$

The strong law of large numbers due to Kolmogorov is well known, which generalizes the above results: Let $(X_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables. Then the sequence $(\frac{1}{n} \sum_{j=1}^n X_j, n \in \mathbb{N})$ converges almost surely if and only if EX_1 is finite, and in that case we have

$$(\text{a.s.}) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n X_j = EX_1. \quad (2)$$

Instead of the usual Cesàro $(c, 1)$ summability of a sequence $(X_n, n \in \mathbb{N})$ (where X_n could be random elements in some Banach space) we can consider the problem of the summability of this sequence by strongly regular matrices. Also, we can consider the problem of the summability of a sequence of iterates $(T^n, n \in \mathbb{N})$ by strongly regular matrices, where T is a linear operator with some additional properties.

In this paper we study the problems of such type.

2. Some results from summability theory

By $C^{\mathbb{N}}$ we denote the complex vector space of all sequences $x = (x_n, n \in \mathbb{N})$ of complex numbers. We denote by l_{∞} and \mathcal{C} subspaces of \mathcal{S} consisting of all bounded and convergent sequences respectively. Then we have $\mathcal{C} \subset l_{\infty} \subset \mathcal{S}$. The mapping $\lim : \mathcal{C} \rightarrow \mathbb{C}$ defined by

$$\lim(x) = \lim_{n \rightarrow \infty} x_n, \quad x = (x_n, n \in \mathbb{N}) \in \mathcal{C}$$

is a linear form on \mathcal{C} .

Let $A = [a_{nj}]$ $(n, j \in \mathbb{N})$ be a complex infinite matrix. Put

$$\mathcal{S}(A) = \{x \in \mathcal{S} : \text{the series } (Ax)_n = \sum_{j=1}^{\infty} a_{nj}x_j, \text{ converges for every } n \in \mathbb{N}\}.$$

$\mathcal{S}(A)$ is a subspace of \mathcal{S} consisting of all sequences $x \in \mathcal{S}$ such that the transformed sequence $Ax = ((Ax)_n, n \in \mathbb{N}) = (\sum_{j=1}^{\infty} a_{nj}x_j, n \in \mathbb{N})$ is well defined. Put

$$\mathcal{C}(A) = \{x \in \mathcal{S}(A) : Ax \in \mathcal{C}\}.$$

The set $\mathcal{C}(A)$ is a subspace of $\mathcal{S}(A)$ and it is called **the convergence domain** of the matrix A . We define a linear form $A - \lim = \lim \circ A$ on $\mathcal{C}(A)$ by

$$A - \lim(x) = A - \lim x_n = \lim(Ax) = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj}x_j, \quad x = (x_n) \in \mathcal{C}(A).$$

The elements of $\mathcal{C}(A)$ are called A – **convergent sequences**.

Definition 1. We say that a matrix A is **conservative** if it transforms convergent sequences into convergent sequences, i.e. if $\mathcal{C} \subset \mathcal{C}(A)$.

Definition 2. A matrix A is **regular** or **Toeplitz matrix** if it is conservative and if $A - \lim|_{\mathcal{C}} = \lim$, i.e.

$$A - \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_n, \quad \text{for every } x = (x_n) \in \mathcal{C}.$$

The following theorem characterizes regular matrices (for the proof see [7]).

Theorem 1. (Silverman-Toeplitz) A matrix A is regular if and only if the following three conditions are satisfied:

$$M = \sup_{n \in \mathbb{N}} \sum_{j=1}^{\infty} |a_{nj}| < \infty. \tag{A}$$

$$\lim_{n \rightarrow \infty} a_{nj} = 0, \quad \text{for every } j \in \mathbb{N}. \tag{B}$$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} = 1. \tag{C}$$

It is easy to check that the Cesàro $(c, 1)$ summability matrix C defined by (1) is regular.

Remark 1. Notice that Theorem 1 is valid in locally convex vector spaces. For Banach spaces the proof is analogous to the proof of Theorem 1 and the extension to locally convex vector spaces is immediate.

For $r \in \mathbb{N}$ we define the shifted Cesàro matrix $C^{(r)} = [c_{nj}^{(r)}]$ by

$$c_{nj}^{(r)} = \begin{cases} \frac{1}{n}, & r < j \leq r + n \\ 0, & j \leq r \text{ or } j > r + n \end{cases}, \quad n, j \in \mathbb{N}.$$

It is well known (see [5]) that for every $r \in \mathbb{N}$ we have $\mathcal{C}(C) = \mathcal{C}(C^{(r)})$ and that

$$C^{(r)} - \lim (x) = C - \lim (x), \quad \text{for every } x \in \mathcal{C}(C).$$

Therefore, for every $x \in \mathcal{C}(C)$ we have

$$C - \lim (x) = \lim_{n \rightarrow \infty} \frac{1}{n} (x_{r+1} + \dots + x_{r+n}), \quad r \in \mathbb{N}_0.$$

If the above convergence is uniform with respect to r , then we say that the sequence $x = (x_n, n \in \mathbb{N})$ is **almost convergent**. By AC we denote the space of all almost convergent sequences. It can be proved (see [5], *Lemma 2.2* and *Lemma 2.3*) that we have

$$\mathcal{C} \subset AC \subset l_{\infty}.$$

Definition 3. A matrix A is called **strongly regular** if it is regular and if $AC \subset \mathcal{C}(A)$.

The following theorem gives equivalent conditions for strong regularity (for the proof see [5], *Theorem 2.5*).

Theorem 2. Let $A = [a_{nj}](n, j \in \mathbb{N})$ be a regular matrix. The following four conditions are equivalent:

$$A \text{ is strongly regular.} \quad (\alpha)$$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} |a_{nj} - a_{n,j+1}| = 0. \quad (\beta)$$

$$\lim_{m \rightarrow \infty} \sum_{j=m}^{\infty} |a_{nj} - a_{n,j+1}| = 0, \text{ uniformly in } n \in \mathbb{N}. \quad (\gamma)$$

For every $x \in l_{\infty}$ we have

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} (a_{nj} - a_{n,j+1})x_j = 0. \quad (\delta)$$

In this case we have $A - \lim|_{AC} = C - \lim$.

It is easy to check that the Cesàro $(c, 1)$ summability matrix C defined by (1) satisfies the condition (β) and therefore C is a stochastic strongly regular matrix.

Example 1. (i) Let $A = [a_{nj}](n, j \in \mathbb{N})$ be defined by

$$\begin{aligned} a_{nj} &= \frac{\binom{r+j-1}{j}}{\binom{n+r}{n}}, \quad r > 0, \quad 0 \leq j \leq n, \\ a_{nj} &= 0, \quad j > n. \end{aligned}$$

One can prove (see [6]) that A is a strongly regular matrix.

(ii) Let $(\varepsilon_n, n \in \mathbb{N})$ be a sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Let $A = [a_{nj}](n, j \in \mathbb{N})$ be defined by

$$a_{nj} = (e^{\varepsilon_n} - 1)e^{-j\varepsilon_n}, \quad n, j \in \mathbb{N}.$$

It can be proved (see [18]) that A is a stochastic strongly regular matrix.

(iii) **The Abel matrix** $A = [a_{nj}](n \in \mathbb{N}, j \in \mathbb{N}_0)$ is defined by

$$a_{nj} = \frac{n^j}{(n+1)^{j+1}}, \quad n \in \mathbb{N}, \quad j \in \mathbb{N}_0.$$

It is easy to prove that A is a stochastic strongly regular matrix (A satisfies conditions (A), (B), (C) and (β)).

(iv) **The Borel matrix** $A = [a_{nj}] (n \in \mathbb{N}, j \in \mathbb{N}_0)$ is defined by

$$a_{nj} = \frac{n^j e^{-n}}{j!}, \quad n \in \mathbb{N}, \quad j \in \mathbb{N}_0.$$

It is obvious that A is a stochastic Toeplitz matrix. Further we have

$$\begin{aligned} \sum_{j=0}^{\infty} |a_{nj} - a_{n,j+1}| &= e^{-n} \sum_{j=0}^{\infty} \frac{n^j}{(j+1)!} |j+1-n| \\ &= e^{-n} \left(\sum_{j=0}^{n-1} \frac{n^{j+1}}{(j+1)!} - \sum_{j=0}^{n-1} \frac{n^j}{j!} + \sum_{j=n}^{\infty} \frac{n^j}{j!} - \sum_{j=n}^{\infty} \frac{n^{j+1}}{(j+1)!} \right) \\ &= e^{-n} (2 \frac{n^n}{n!} - 1) \end{aligned}$$

which, by the Stirling's formula, tends to 0, and therefore (β) holds true. It follows that A is strongly regular.

3. Applications of strongly regular matrices in ergodic theory

1.° We first recall some definitions and known results from classical ergodic theory.

Let X be a vector space and let $T : X \rightarrow X$ be a linear operator. By $R(T)$ we denote the range of T and by $N(T)$ we denote the null-subspace of T . We denote by I the identity operator on X . If X is a topological space and if K is a subset of X , we denote by \bar{K} the closure of K .

Let (X, \mathcal{A}, μ) be a measure space and let T be a linear operator defined on $L^p = L^p(X, \mathcal{A}, \mu)$, where $1 \leq p < \infty$. The operator T is said to be:

- (a) **a contraction**, if $\|Tf\|_p \leq \|f\|_p$ for all $f \in L^p$.
- (b) **positive**, if $Tf \geq 0$ whenever $f \geq 0$.

Suppose now that (X, \mathcal{A}, μ) is a measure space and let $T : X \rightarrow X$ be a measurable map. In this case T is said to be **a transformation**. T is called **measure preserving** if $\mu \circ T^{-1} = \mu$ or equivalently if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$.

A set $A \in \mathcal{A}$ is called **T -invariant** if $T^{-1}(A) = A$. The collection \mathcal{A}_T of all T -invariant sets in \mathcal{A} is a σ -algebra on X . The transformation T is called **ergodic**, if for every $A \in \mathcal{A}_T$ we have either $\mu(A) = 0$ or $\mu(A^c) = 0$.

For $1 \leq p \leq \infty$ the symbol $\|T\|_p$ will be used for the norm of T when it is considered as a bounded linear operator in L^p .

Theorem 3. (Birkhoff's pointwise ergodic theorem, see 8.4 in [8]). Let (X, \mathcal{A}, μ) be a σ -finite measure space, and let T be a measure-preserving transformation of X . Then for every $f \in L^1$ there exists $f^* \in L^1(X, \mathcal{A}_T, \mu)$ satisfying $\|f^*\|_1 \leq \|f\|_1$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j = f^*, \quad \mu - \text{a.e.}$$

If T is ergodic, then f^* equals (a.e.) some constant c .

2.° Now we generalize the ergodic theorem for locally convex vector spaces due to Yosida (see [20]).

Theorem 4. *Let X be a sequentially complete, weakly sequentially compact locally convex vector space with topology generated by family \mathcal{E} of seminorms and let $T : X \rightarrow X$ be a continuous linear operator such that the family of operators $\{T^n, n \in \mathbb{N}\}$ is equicontinuous in the sense that for every seminorm $q \in \mathcal{E}$, there exists a continuous seminorm q' on X such that*

$$\sup_{n \in \mathbb{N}} q(T^n x) \leq q'(x), \quad \text{for every } x \in X. \quad (3)$$

Further, let $A = [a_{nj}](n, j \in \mathbb{N})$ be a strongly regular matrix and

$$T_n = \sum_{j=1}^{\infty} a_{nj} T^j \quad (n \in \mathbb{N}).$$

Then, $T_n (n \in \mathbb{N})$ are well defined on X and for every $x \in X$ there exists $\lim_{n \rightarrow \infty} T_n x$. If we put $T_0 x = \lim_{n \rightarrow \infty} T_n x$ ($x \in X$) then T_0 is a continuous linear operator not depending on A , such that

$$T_0 = T_0^2 = T T_0 = T_0 T. \quad (4)$$

$$R(T_0) = N(I - T). \quad (5)$$

$$N(T_0) = \overline{R(I - T)} = R(I - T_0). \quad (6)$$

Moreover, we have a decomposition into a direct sum:

$$X = \overline{R(I - T)} \oplus N(I - T). \quad (7)$$

Proof. By using (3), for $q \in \mathcal{E}$ and every $n \in \mathbb{N}$ we get

$$q\left(\sum_{j=s}^t a_{nj} T^j x\right) \leq \sum_{j=s}^t |a_{nj}| q(T^j x) \leq q'(x) \sum_{j=s}^t |a_{nj}| \rightarrow 0, \quad \text{as } t, s \rightarrow \infty.$$

Since X is a sequentially complete space, it follows that T_n are defined on X . It is obvious that the operators $T_n (n \in \mathbb{N})$ are linear and continuous. For $n \in \mathbb{N}$ we have

$$T_n(I - T) = a_{n1} T_1 + \sum_{j=1}^{\infty} (a_{n,j+1} - a_{nj}) T^{j+1}.$$

If $w \in R(I - T)$ then there exists $x \in X$ such that $w = (I - T)x$, so by using (3) for every $q \in \mathcal{E}$ we get

$$q(T_n w) = q(T_n(I - T)x) \leq q'(x) \left(|a_{n1}| + \sum_{j=1}^{\infty} |a_{n,j+1} - a_{nj}|\right).$$

Since A is a strongly regular matrix we conclude

$$\lim_{n \rightarrow \infty} T_n w = 0, \quad \text{for every } w \in R(I - T). \quad (8)$$

Let us prove that we have

$$\overline{R(I - T)} \subset \{x \in X : \lim_{n \rightarrow \infty} T_n x = 0\} . \quad (9)$$

Suppose that $z \in \overline{R(I - T)}$. Since q' is a continuous seminorm on X , then for every $\varepsilon > 0$ there exists $w \in R(I - T)$ such that $q'(z - w) < \frac{\varepsilon}{M}$. Using (1) we get

$$q(T_n(z - w)) \leq q'(z - w) \sum_{j=1}^{\infty} |a_{nj}| < \varepsilon .$$

It follows from here

$$q(T_n z) \leq q(T_n w) + q(T_n(z - w)) \leq q(T_n w) + \varepsilon, \quad \text{for every } q \in \mathcal{E} ,$$

so by (8) we conclude that (9) holds true.

In fact (9) (see [12], *Theorem 3*) is an equality.

Suppose now, that $x \in X$ is arbitrary. Since X is weakly sequentially compact, it follows that there exists a subsequence $(T_{n_k} x, k \in \mathbb{N}) \subset (T_n x, n \in \mathbb{N})$ such that the weak limit $w - \lim_{k \rightarrow \infty} T_{n_k} x = x_0$ exists. It follows from (9) that

$$\lim_{n \rightarrow \infty} T_n(x - Tx) = 0 . \quad (10)$$

Since T and T_n commute, the operators T_{n_k} and T satisfy conditions of Cohen's lemma (see [6]) and therefore $Tx_0 = x_0$. From $T^j x_0 = x_0$ ($j \in \mathbb{N}$) it follows that $T^j x = x_0 + T^j(x - x_0)$ so we have

$$T_n x = \left(\sum_{j=1}^{\infty} a_{nj} \right) x_0 + T_n(x - x_0), \quad n \in \mathbb{N} . \quad (11)$$

By the continuity of T we have

$$\begin{aligned} x - T_{n_k} x &= \left(1 - \sum_{j=1}^{\infty} a_{n_k j} \right) x + \sum_{j=1}^{\infty} a_{n_k j} (I - T)(I + T + T^2 + \dots + T^{j-1})x \quad (12) \\ &= \left(1 - \sum_{j=1}^{\infty} a_{n_k j} \right) x + (I - T)z_{n_k}, \quad k \in \mathbb{N}, \end{aligned}$$

where $z_{n_k} = \sum_{j=1}^{\infty} a_{n_k j} (I + T + T^2 + \dots + T^{j-1})x$. Taking the weak limit in (12) (as $k \rightarrow \infty$) we get $x - x_0 = w - \lim_{k \rightarrow \infty} (I - T)z_{n_k}$. Since $(I - T)z_{n_k} \in R(I - T)$, then $x - x_0$ is an element of the weak closure of $R(I - T)$. But in locally convex vector spaces the weak closure of a convex subset is equal to the original closure of this subset (see [17], *Theorem 3.12* p.64) so we conclude $x - x_0 \in \overline{R(I - T)}$. Now, by (9) we have $\lim_{n \rightarrow \infty} T_n(x - x_0) = 0$, so by (11) we obtain $\lim_{n \rightarrow \infty} T_n x = x_0$. Put

$x_0 = T_0x = \lim_{n \rightarrow \infty} T_nx = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj}T^jx, x \in X$. Linearity of T_0 is obvious. By use of (3), for $q \in \mathcal{E}$ we get

$$q(T_nx) \leq \sum_{j=1}^{\infty} |a_{nj}|q(T^jx) \leq Mq'(x),$$

for every $x \in X$ and every $n \in \mathbb{N}$.

Therefore, the family of operators $\{T_n, n \in \mathbb{N}\}$ is equicontinuous in the sense of (3). It follows easily from here that T_0 is a continuous operator. Now, from $Tx_0 = x_0$ and $x_0 = T_0x$ we get

$$Tx_0 = TT_0x = x_0 = T_0x, \text{ for every } x \in X$$

and therefore $TT_0 = T_0 \implies T^jT_0 = T_0$ for every $j \in \mathbb{N}$. Now we have $T_nT_0 = (\sum_{j=1}^{\infty} a_{nj})T_0$, for every $n \in \mathbb{N}$, which implies $T_0^2 = T_0$. On the other hand, we have

$$T_n - T_nT = a_{n1}T + \sum_{j=1}^{\infty} (a_{n,j+1} - a_{nj})T^{j+1}.$$

By use of (3) and the fact that A is a strongly regular matrix we obtain $T_0 = T_0T$, so (4) holds true.

Let us prove that the limiting operator T_0 does not depend on a strongly regular matrix A . For this purpose let $B = [b_{nj}](n, j \in \mathbb{N})$ be another strongly regular matrix and let $W_n = \sum_{j=1}^{\infty} b_{nj}T^j (n \in \mathbb{N})$. From the previous discussion it follows that there exists the continuous linear operator T_1 on X such that

$$T_1x = \lim_{n \rightarrow \infty} W_nx, \quad x \in X, \quad (13)$$

and

$$T_1 = T_1^2 = TT_1 = T_1T. \quad (14)$$

It follows from (14) that for every $j \in \mathbb{N}$ we have

$$T_1 = T_1^2 = T^jT_1 = T_1T^j. \quad (15)$$

Multiplying (15) by a_{nj} and summing over j we get

$$\left(\sum_{j=1}^{\infty} a_{nj}\right)T_1 = T_nT_1 = T_1T_n, \quad n \in \mathbb{N}. \quad (16)$$

By letting $n \rightarrow \infty$ we obtain $T_1 = T_0T_1 = T_1T_0$. In the same way we get $T_0 = T_1T_0 = T_0T_1$ and therefore $T_0 = T_1$.

Relations (5), (6) and (7) follow now from *Corollary* in [20], p. 214. \square

Corollary 1. *Let X be a Banach space. If the strong limit $T_0 = \lim_{n \rightarrow \infty} T_n$ exists, it is a projection on X on the subspace $\{x \in X : Tx = x\}$ of all fixed points of T and its complementary projection has $R(I - T)$ as its range. Moreover, projection T_0 does not depend on a strongly regular matrix A .*

3.° Here we prove two mean ergodic theorems for contractions in the spaces $L^p(1 \leq p < \infty)$.

Theorem 5. *Let T be a positive contraction in L^p , where $1 < p < \infty$, and let $A = [a_{nj}](n, j \in \mathbb{N})$ be a strongly regular matrix. Then there exists a positive contraction P in L^p satisfying*

$$P = P^2 = TP = PT, \quad (17)$$

and such that

$$\sum_{j=1}^{\infty} a_{nj} T^j f \longrightarrow Pf \quad \text{in } L^p, \quad (18)$$

for every $f \in L^p$, as $n \rightarrow \infty$. Moreover, contraction P does not depend on a strongly regular matrix A .

Proof. We can use the results of *Theorem 4* for the case of the Banach space L^p . Since T is a contraction, the condition (3) is satisfied. From the proof of *Theorem 4*, it follows that for the proof of (18) it is sufficient to prove that the set

$$K = \left\{ \sum_{j=1}^{\infty} a_{nj} T^j f : n \in \mathbb{N} \right\} \quad (19)$$

is weakly sequentially compact for every $f \in L^p$. By [9], IV. 8.4, p. 289, this is equivalent to the boundedness of K . We have

$$\left\| \sum_{j=1}^{\infty} a_{nj} T^j f \right\|_p \leq M \|f\|_p, \quad n \in \mathbb{N},$$

and therefore (18) holds true. The relation (17) follows from (4). Moreover, it follows from *Theorem 4* that P is a bounded linear operator not depending on a strongly regular matrix A . Therefore, we get the same limit P if we take the usual Cesàro $(c, 1)$ summability of the sequence of iterates $(T^n, n \in \mathbb{N})$. Thus we have

$$\frac{1}{n} \sum_{j=1}^n T^j f \longrightarrow Pf \quad \text{in } L^p, \quad (20)$$

for every $f \in L^p$, as $n \rightarrow \infty$. Now, by *Theorem 2.1.1* in [10], p. 19, we conclude that if T is a positive contraction, then the same is true for P . Notice also that since P is a contraction and also an idempotent (relation (17)) we conclude that $\|P\| = 1$ or $P = 0$. \square

Remark 2. Suppose that the operator T in Theorem 5 is not a contraction, but the sequence of iterates $(T^n, n \in \mathbb{N})$ is uniformly bounded, that is $M_1 = \sup_{n \in \mathbb{N}} \|T^n\| < \infty$ (in that case we say that operator T is power bounded). Then Theorem 5 remains true (this follows from Theorem 4). The limiting operator P is bounded (not necessarily a contraction) and we have $\|P\| \leq MM_1$.

Now we prove the mean ergodic theorem for the space L^1 .

Theorem 6. Let T be a contraction in L^1 and suppose that there exists a strictly positive $g \in L^1$ such that

$$|f| \leq g \implies |Tf| \leq g, \quad f \in L^1. \quad (21)$$

Further, let $A = [a_{nj}](n, j \in \mathbb{N})$ be a strongly regular matrix. Then there exists a contraction P in L^1 satisfying (17) and such that

$$\sum_{j=1}^{\infty} a_{nj} T^j f \longrightarrow Pf \quad \text{in } L^1, \quad (22)$$

for every $f \in L^1$, as $n \rightarrow \infty$. Moreover, contraction P does not depend on a strongly regular matrix A .

Proof. As in the proof of Theorem 5 for proving (22) it is sufficient to prove that the set K in (19) is weakly sequentially compact for every $f \in L^1$. By [9], IV. 8.9, p. 292, this is equivalent to the boundedness of K and the condition that for each decreasing sequence $(E_k, k \in \mathbb{N}) \subset \mathcal{A}$ with void intersection we have

$$\lim_{k \rightarrow \infty} \int_{E_k} h d\mu = 0, \quad \text{uniformly for } h \in K.$$

The proof that K is bounded is the same as in Theorem 5. Therefore, for the proof of (22) it is sufficient to prove that for every $f \in L^1$ we have

$$\lim_{k \rightarrow \infty} \int_{E_k} \left(\sum_{j=1}^{\infty} a_{nj} T^j f \right) d\mu = 0, \quad \text{uniformly in } n \in \mathbb{N}. \quad (23)$$

We have

$$\left| \int_{E_k} \left(\sum_{j=1}^{\infty} a_{nj} T^j f \right) d\mu \right| \leq \sum_{j=1}^{\infty} |a_{nj}| \int_{E_k} |T^j f| d\mu. \quad (24)$$

Since the function $g \in L^1$ is strictly positive, then for every $f \in L^1$ and any $\varepsilon > 0$ there is a constant $c > 0$ and a splitting

$$f = f_c + \tilde{f}_c$$

with $|f_c| \leq cg$ and $\|\tilde{f}_c\|_1 < \varepsilon$ (see [1], Theorem 2.4.13, p. 88). By use of (21) and the fact that T is a contraction we obtain

$$\int_{E_k} |T^j f| d\mu \leq c \int_{E_k} g d\mu + \int_X |T^j \tilde{f}_c| d\mu \leq c \int_{E_k} g d\mu + \varepsilon. \quad (25)$$

(23) follows from (24), (25) and (A) since $\lim_{k \rightarrow \infty} \int_{E_k} g d\mu = 0$. Therefore, (22)

holds true. It follows from *Theorem 4* that P is a bounded linear operator in L^1 not depending on a strongly regular matrix A . Therefore, we get the same limit P if we take for A the usual Cesàro $(c, 1)$ summability matrix. By a consequence of *Theorem 1.1* in [16] (p. 72 and 73) we then conclude that P is a contraction. In the same way as in *Theorem 5* we conclude that $\|P\| = 1$ or $P = 0$. \square

Remark 3. *Similarly, as in Remark 2, we conclude that Theorem 6 remains true if we suppose that the operator T is not a contraction, but T is power bounded.*

4.° Now we generalize the well known classical pointwise ergodic theorem due to G.D. Birkhoff (see [4]).

We first need the following lemma.

Lemma 1. *Let (X, \mathcal{A}, μ) be a σ -finite measure space and let T be a linear operator in L^1 with $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$. If $A = [a_{nj}](n, j \in \mathbb{N})$ is a stochastic strongly regular matrix, then for $1 \leq p < \infty$ and $f \in L^p$ we have*

$$\sup_{n \in \mathbb{N}} \left| \sum_{j=1}^{\infty} a_{nj} (T^j f)(x) \right| < \infty, \quad \mu - \text{a.e.}$$

Proof. Put $T_n = \sum_{j=1}^{\infty} a_{nj} T^j$ ($n \in \mathbb{N}$). Since the space L^1 is complete and for $f \in L^1$

$$\left\| \sum_{j=k}^r a_{nj} T^j f \right\|_1 \leq \|f\|_1 \sum_{j=k}^r a_{nj},$$

it follows from the fact that A is a stochastic matrix that T_n ($n \in \mathbb{N}$) are defined on L^1 . Further $\|T_n\|_1 \leq 1$ ($n \in \mathbb{N}$). Let g be a bounded function in L^p . Since $\|T\|_\infty \leq 1$ we have

$$\|T_n g\|_\infty \leq \sum_{j=1}^{\infty} a_{nj} \|g\|_\infty = \|g\|_\infty.$$

Thus in proving the lemma we may assume as in *Lemma VIII.6.5* in [9] (p. 675) that f is in L^1 , $f \geq 0$ and that T is a positive operator. Now, the proof of the lemma is similar to the proof of *Lemma VIII.6.5* in [9]. \square

Theorem 7. (Pointwise ergodic theorem). *Let (X, \mathcal{A}, μ) be a σ -finite measure space and let T be a linear operator in L^1 with $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$. If $A = [a_{nj}](n, j \in \mathbb{N})$ is a stochastic strongly regular matrix, then for every p with $1 \leq p < \infty$ and every function $f \in L^p$, the limit*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} (T^j f)(x), \tag{26}$$

exists for almost all $x \in X$ and the limiting function does not depend (μ - a.e.) on the matrix A .

Proof. Since $\|T\|_\infty \leq 1$ and $\|T\|_1 \leq 1$, it follows from the corollary of the Riesz convexity theorem that $\|T\|_p \leq 1$ for all p , $1 \leq p \leq \infty$ (see [9], VI.10.12, p. 526). For p with $1 < p < \infty$ the space L^p is reflexive ([9], IV.8.2, p. 288). We have

$$\|T_n f\|_p = \left\| \sum_{j=1}^{\infty} a_{nj} T^j f \right\|_p \leq \|f\|_p ,$$

since A is a stochastic matrix. Therefore, the set $\{T_n f : n \in \mathbb{N}\}$ is bounded in L^p for every $f \in L^p$ and by II.3.28 in [9] (p. 68) it follows that this set is weakly sequentially compact. Since A is a strongly regular matrix we conclude from *Theorem 4* and *Corollary 1* that the sequence of operators $(T_n, n \in \mathbb{N})$ converges strongly to the bounded linear operator $T_0 : L^p \rightarrow L^p$, not depending on A , which is a projection on the subspace $\{f \in L^p : Tf = f\}$ of all fixed points of T . From *Corollary 1* it follows that vectors

$$h = f^* + (I - T)g ,$$

with $f^*, g \in L^p, Tf^* = f^*$ and g bounded are dense in L^p . Moreover, the vector f^* is uniquely determined by h . For such a vector h we have

$$T_n h = f^* + a_{n1} Tg + \sum_{j=1}^{\infty} (a_{n,j+1} - a_{nj}) T^{j+1} g ,$$

and therefore

$$\|T_n h - f^*\|_\infty \leq (a_{n1} + \sum_{j=1}^{\infty} |a_{n,j+1} - a_{nj}|) \|g\|_\infty .$$

Since A is a strongly regular matrix, it follows

$$\|T_n h - f^*\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty ,$$

and therefore $T_n h \rightarrow f^*$, uniformly μ -a.e. as $n \rightarrow \infty$, for every h in a dense set in L^p . By *Lemma 1* $\sup_{n \in \mathbb{N}} |(T_n f)(x)| < \infty$ almost everywhere for every $f \in L^p$. Thus,

by *Theorem IV.11.2* in [9], p. 332, the sequence $\sum_{j=1}^{\infty} a_{nj} T^j f$ converges μ -a.e. for every $f \in L^p$.

For $f^* \in N(I - T)$ we have $Tf^* = f^*$ and therefore

$$T^j f^* = f^* (j \in \mathbb{N}) \implies T_n f^* = \left(\sum_{j=1}^{\infty} a_{nj} \right) f^* = f^* (n \in \mathbb{N})$$

and since $T_n f^* \rightarrow T_0 f^*$ in L^p (as $n \rightarrow \infty$) we conclude that

$$T_0 f^* = f^* .$$

On the other hand, using the relation $T_0 = T_0 T$ (see (4)) we get

$$T_n h = T_n f^* + T_n (g - Tg) \xrightarrow{\text{in } L^p} T_0 f^* + T_0 g - T_0 Tg = T_0 f^* = f^* .$$

We conclude from here that the limiting function in (26) does not depend ($\mu - \text{a.e.}$) on a stochastic strongly regular matrix A .

Since L^p is dense in L^1 , we may apply *Lemma 1* and *Theorem IV.11.2* in [9] to see that the sequence $\sum_{j=1}^{\infty} a_{nj}T^j f$ converges $\mu - \text{a.e.}$ for every $f \in L^1$. \square

4. A generalization of Kolmogorov’s strong law of large numbers

We first prove the following theorem:

Theorem 8. *Let T be any measure-preserving transformation of a σ -finite measure space (X, \mathcal{A}, μ) and let $A = [a_{nj}](n, j \in \mathbb{N})$ be a stochastic strongly regular matrix. Then for every $f \in L^1$ there exists $f^* \in L^1(X, \mathcal{A}_T, \mu)$ such that*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} f \circ T^{j-1} = f^* , \mu - \text{a.e.} \tag{27}$$

Moreover the function f^* does not depend ($\mu - \text{a.e.}$) on matrix A and we have $\|f^*\|_1 \leq \|f\|_1$. If T is ergodic, then f^* equals ($\mu - \text{a.e.}$) some constant c .

Proof. Put $Uf := f \circ T, f \in L^1$. Then $U : L^1 \rightarrow L^1$ is a positive linear contraction on L^1 , since by the image measure theorem we have

$$\|Uf\|_1 = \int_X |f \circ T| d\mu = \int_X |f| d(\mu \circ T^{-1}) = \int_X |f| d\mu = \|f\|_1 , f \in L^1 .$$

We have $U^j f = f \circ T^j, j \in \mathbb{N}, f \in L^1$. Since T is a measure-preserving transformation, we also obtain $\|U\|_{\infty} \leq 1$. Therefore, we can apply *Theorem 7*, so we get that the limit

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} (U^{j-1} f)(x) = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} f(T^{j-1} x) = f^*(x) ,$$

exists for almost all $x \in X$ and the limiting function f^* does not depend ($\mu - \text{a.e.}$) on matrix A . Therefore, we get the same limit f^* if we take the usual Cesàro $(c, 1)$ summability of the sequence $(f \circ T^{n-1}, n \in \mathbb{N})$. The remaining statements follow now from the Birkhoff’s pointwise ergodic theorem (see *Theorem 3*). \square

We also need the following example:

Example 2. *Let (X, \mathcal{A}, P) be a probability space and let $X^{\mathbb{N}}$ be the space of all sequences $(x_n, n \in \mathbb{N})$ of elements $x_n \in X$. Then on $X^{\mathbb{N}}$ there is a product σ -algebra and a product probability $P^{\mathbb{N}}$ of copies of P in the well known sense. The **shift** transformation T is defined by $T((x_n, n \in \mathbb{N})) = (x_{n+1}, n \in \mathbb{N})$. Then T is a measurable, measure preserving transformation and it can be proved (see 8.4. in [8] that T is ergodic.*

The next theorem is a generalization of Kolmogorov's strong law of large numbers for independent identically distributed random variables.

Theorem 9. *Let $(X_n, n \in \mathbb{N})$ be a sequence of independent identically distributed random variables with finite mean and let $A = [a_{nj}] (n, j \in \mathbb{N})$ be a stochastic strongly regular matrix. Then we have*

$$(\text{a.s.}) \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j = EX_1. \quad (28)$$

Proof. Let (Ω, \mathcal{F}, P) be the probability space on which variables X_n are defined. Consider the function $Y : \Omega \rightarrow \mathbb{R}^{\mathbb{N}}$, defined by $Y(\omega) = (X_n(\omega), n \in \mathbb{N}), \omega \in \Omega$. Then by independence the image measure $P \circ Y^{-1}$ equals $Q^{\mathbb{N}}$, where $Q = P_{X_1}$ is the probability law of X_1 . So we may assume $\Omega = \mathbb{R}^{\mathbb{N}}$, X_n are the coordinates and $P = Q^{\mathbb{N}}$. By *Example 2* the shift transformation T in $\Omega = \mathbb{R}^{\mathbb{N}}$ is ergodic. If we take for f in (27) the first projection, we conclude that

$$(\text{a.s.}) \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j \quad (29)$$

exists and that this limit does not depend on stochastic strongly regular matrix A . Therefore, we get the same limit if we take the usual Cesàro $(c, 1)$ (a.s.) summability of the sequence $(X_n, n \in \mathbb{N})$. Now (28) follows from Kolmogorov's strong law of large numbers. \square

5. A generalization of Beck's strong law of large numbers for random elements in Banach spaces

1.° We first recall some definitions and known results.

Let (Ω, \mathcal{F}, P) be a probability space and let S be a separable Banach space (for some reasons of measurability we suppose that Banach space S is separable).

A function $X : \Omega \rightarrow S$ is a random element (in S) if $X^{-1}(B) \in \mathcal{F}$ for every $B \in \mathcal{B}$, where \mathcal{B} is the σ -algebra of Borel subsets of S .

X is a random element in S iff $f(X)$ is a random variable for each $f \in S^*$, where S^* is the dual space of S .

Random elements X and Y are **identically distributed** (i.d.) if

$$P\{X \in B\} = P\{Y \in B\}, \text{ for each } B \in \mathcal{B}.$$

Random elements X_1, X_2, \dots, X_n are **independent** if

$$P\{X_1 \in B_1, \dots, X_n \in B_n\} = \prod_{k=1}^n P\{X_k \in B_k\}, \text{ for all } B_1, \dots, B_n \in \mathcal{B}.$$

We have the following well known results: (i) Random elements X and Y are i.d. iff $f(X)$ and $f(Y)$ are i.d. random variables for each $f \in S^*$.

(ii) Random elements X and Y are independent iff $f(X)$ and $g(Y)$ are independent random variables for all $f, g \in S^*$.

The proof of these results is based on the fact that the family $\mathcal{C} = \{ \{x : f(x) < t\} : f \in S^*, t \in \mathbb{R} \}$ is a determining class for \mathcal{B} .

2.° Using *Theorem 7* one can prove the following generalization of the well known Beck–Schwartz ergodic type theorem (see [3]) which we need in the proof of the main result in this section.

Theorem 10. *Let S be a reflexive Banach space and let (X, \mathcal{A}, μ) be a σ -finite measure space. Further, let $T_x : X \rightarrow \mathcal{L}(S)$ (the space of all bounded linear operators on S) be a strongly measurable function such that $\|T_x\| \leq 1$ for each $x \in X$. If $h : X \rightarrow X$ is a measure preserving transformation and if $A = [a_{nj}] (n, j \in \mathbb{N})$ is a stochastic strongly regular matrix, then for every $f \in L^1(X, S)$ there is $f^* \in L^1(X, S)$ such that*

$$(a.e.) \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} T_x T_{h(x)} \dots T_{h^{j-1}(x)} (f(h^j(x))) = f^*(x), \quad (30)$$

in the norm topology of S , and

$$f^*(x) = T_x(f^*(h(x))), \text{ a.e. (on } X). \quad (31)$$

Moreover, if $\mu(X) < \infty$, then the limit in (30) holds true also in the mean of order 1.

Theorem 11. (A generalization of Beck’s strong law of large numbers).

Let $(X_j, j \in \mathbb{N})$ be a sequence of independent random elements in a separable reflexive Banach space S , and let $T : S \rightarrow S$ be a linear operator such that $\|T\| = 1$. If $EX_j = 0$ for all $j \in \mathbb{N}$, and if X_j and $T^{j-1}(X_1)$ are identically distributed for all $j \in \mathbb{N}$, then for an arbitrary stochastic strongly regular matrix $A = [a_{nj}] (n, j \in \mathbb{N})$ we have

$$(a.s.) \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j = 0. \quad (32)$$

(The expectation is defined in the sense of the Pettis integral, that is $EX \in S$ and it satisfies

$$E(f(X)) = f(EX), \text{ for each } f \in S^*.$$

For separable spaces and Bochner integrable random elements, mathematical expectation is equal to the Bochner integral).

Proof of Theorem 11. Let (Ω, \mathcal{F}, P) be probability space on which the sequence $(X_j, j \in \mathbb{N})$ is defined. Put

$$(\Omega', \mathcal{F}', P') = \left(\prod_{j=-\infty}^{\infty} \Omega_j, \prod_{j=-\infty}^{\infty} \mathcal{F}_j, \prod_{j=-\infty}^{\infty} P_j \right), \quad (33)$$

where $\Omega_j = \Omega, \mathcal{F}_j = \mathcal{F}, P_j = P, \forall j \in \mathbb{Z}$. Every point $\omega \in \Omega'$ is a two-sided sequence $(\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$ of points of Ω . Now, we may define a sequence of random elements $(X_j^*, j \in \mathbb{N})$ on Ω' with

$$\begin{aligned} X_1^*(\omega) &= X_1(\omega_1) \\ X_j^*(\omega) &= T^{j-1}(X_1(\omega_j)) = T^{j-1}(X_1^*(h^{j-1}(\omega))), \end{aligned} \quad (34)$$

where h is a “shift” transformation on Ω' taking $(\dots, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots)$ into a sequence having in its i -th place the point ω_{i+1} . h is obviously a measure preserving transformation in Ω' and it can be proved (see *Example 2*.) that h is ergodic. Therefore, for $B_1, \dots, B_r \in \mathcal{B}$ we have

$$\begin{aligned} P\{X_1 \in B_1, \dots, X_r \in B_r\} &= \prod_{j=1}^r P\{X_j \in B_j\} = \prod_{j=1}^r P\{T^{j-1}X_1 \in B_j\} \\ &= \prod_{j=1}^r P'\{\omega \in \Omega' : T^{j-1}(X_1^*(h^{j-1}(\omega))) \in B_j\} \\ &= \prod_{j=1}^r P'\{\omega \in \Omega' : X_j^*(\omega) \in B_j\} \\ &= P'\{X_1^* \in B_1, \dots, X_r^* \in B_r\}. \end{aligned}$$

It follows from here

$$P\left\{\sum_{j=1}^{\infty} a_{nj} X_j \in B\right\} = P'\left\{\sum_{j=1}^{\infty} a_{nj} X_j^* \in B\right\}, \forall B \in \mathcal{B}, \forall n \in \mathbb{N}. \quad (35)$$

If we have a sequence $(Y_n, n \in \mathbb{N})$ of random elements we may define

$$\begin{aligned} C_A\{Y_n\} &= \operatorname{ess\,sup}_{\Omega} \limsup_n \left\| \sum_{j=1}^{\infty} a_{nj} Y_j \right\| \\ &= \inf\{r \in \mathbb{R}_+ : \limsup_n \left\| \sum_{j=1}^{\infty} a_{nj} Y_j \right\| \leq r, \text{ a.s.}\} \end{aligned} \quad (36)$$

It is clear that $C_A\{Y_n\} = 0$ iff the sequence (Y_n) satisfies the strong law of large numbers, that is

$$A - \lim_{n \rightarrow \infty} Y_n = \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} Y_j = 0 \quad (\text{a.s. in } \Omega) \quad (37)$$

(in the norm topology of S). From (35) we have

$$C_A\{X_n\} = C_A\{X_n^*\}. \quad (38)$$

Now, from *Theorem 10* we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j^* &= \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} T^{j-1}(X_1^*(h^{j-1}(\cdot))) \\ &= \bar{X}, \text{ a.s. in } \Omega' \end{aligned} \quad (39)$$

(in the norm topology of S) and the convergence in the above relation is also in $L^1(\Omega', \mathcal{F}', P', S)$

Since \bar{X} is measurable and invariant under h , and h is ergodic, it follows from *Theorem 8* that \bar{X} is a constant. Since $EX_1 = 0$ (this we get by the use of the dominated convergence theorem) we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} a_{nj} T^{j-1}(EX_1) &= \sum_{j=1}^{\infty} a_{nj} E(T^{j-1}X_1) = \sum_{j=1}^{\infty} a_{nj} EX_j^* = \sum_{j=1}^{\infty} a_{nj} \int_{\Omega'} X_j^* dP' \\ &= \int_{\Omega'} \left(\sum_{j=1}^{\infty} a_{nj} X_j^* \right) dP' = 0, \end{aligned}$$

so by (39) we have

$$\int_{\Omega'} \bar{X} dP' = \lim_{n \rightarrow \infty} \int_{\Omega'} \left(\sum_{j=1}^{\infty} a_{nj} X_j^* \right) dP' = 0. \quad (40)$$

Since \bar{X} is a constant, we conclude from (40) that $\bar{X}(\omega) = 0$ for all $\omega \in \Omega'$. Now, from (39) we get

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j^* = 0, \text{ a.s. in } \Omega'. \quad (41)$$

(in the norm topology of S) and therefore $C_A\{X_n^*\} = 0$. By (38) we conclude $C_A\{X_n\} = 0$, i.e.

$$(\text{a.s.}) A - \lim_{n \rightarrow \infty} X_n = (\text{a.s.}) \lim_{n \rightarrow \infty} \sum_{j=1}^{\infty} a_{nj} X_j = 0.$$

□

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