Abel–type inequalities, complex numbers and Gauss–Pólya type integral inequalities

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Abstract. *We obtain inequalities of Abel type but for nondecreasing sequences rather than the usual nonincreasing sequences. Striking complex analogues are presented. The inequalities on the real domain are used to derive new integral inequalities related to those of Gauss–Pólya type.*

Key words: *Abel's identity, Abel's inequality, Gauss–Pólya inequality*

Sažetak. Nejednakosti Abelovog tipa, kompleksni brojevi i integralne nejednakosti Gauss-Polyinog tipa.

Ključne riječi: *Abelov identitet, Abelova nejednakost, Gauss-Polyina nejednakost*

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1. Introduction

The following result is well–known in the literature as Abel's inequality (see [1], p. 335).

Theorem 1. *Let* $\mathbf{p}$ *be a real* n*–tuple and* $\mathbf{a}$ *a nonnegative, nonincreasing* n*–tuple. Then for* $P_k := \sum_{i=1}^{k} p_i$*, we have*

$$a_1 \min_{1 \le k \le n} P_k \le \sum_{i=1}^{n} p_i a_i \le a_1 \max_{1 \le k \le n} P_k.$$

This early result was subsequently generalized by Bromwich (see [1], p. 337), who derived the following theorem.

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Theorem 2. *For a given real n–tuple* $\mathbf{p}$ *and given integer* v $(1 \le v \le n)$, *define* $H_1 = h_1 = 0$, $H_v = \max(P_1, \dots, P_{v-1})$, $h_v = \min(P_1, \dots, P_{v-1})$, $H_v^{'} = \max(P_v, \dots, P_n)$, $h_v^{'} = \min(P_v, \dots, P_n)$. *If* $\mathbf{a}$ *is a positive, nonincreasing n–tuple, then*

$$h_v(a_1 - a_v) + h_v^{'} a_v \le \sum_{i=1}^{n} p_i a_i \le H_v(a_1 - a_v) + H_v^{'} a_v.$$

These inequalities contain in their proof the identities

$$\sum_{i=1}^{n} p_i a_i = a_1 \sum_{i=1}^{n} p_i + \sum_{i=2}^{n} \left(\sum_{k=i}^{n} p_k \right) \Delta a_{i-1} = a_n \sum_{i=1}^{n} p_i - \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \right) \Delta a_i \tag{1}$$

(where $\Delta a_i := a_{i+1} - a_i$) due to Abel.

This nest of relations is a surprisingly fertile one despite its simplicity. Recently the Abel motif has been exploited to effect by Pearce, Pečarić and Šunde [2] in connection with the Chebyshev and Popoviciu inequalities.

In this note we ring the changes and take $\mathbf{a}$ as a nondecreasing rather than a nonincreasing n–tuple. In *Section 2* we present Abel–type inequalities for this case. In *Section 3* we derive some analogous results in the complex domain. These are striking in that although there are constraints involved on the complex n–tuples $\mathbf{z}$ and $\mathbf{a}$, the relations hold for any complex n–tuples $\mathbf{w}$ whatsoever.

Finally, in *Section 4*, we use the results of *Section 2* to derive an integral inequality. Recently a number of results have been derived extending the classical Gauss–Pólya inequalities to yield various results connecting weighted means of a set of functions and their derivatives (see [3], [5–7]). The methods are here analogous to some used in that connection, but the inequalities found are new and different.

2. Inequalities for real numbers

We start with the following theorem.

Theorem 3. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{p} = (p_1, \dots, p_n)$ *be n-tuple of real numbers such that* $a_1 \le \dots \le a_n$ *and* $\sum_{k=i}^{n} p_k \ge 0$ *for* $i = 2, \dots, n$. *Then*

$$\sum_{i=1}^{n} p_i a_i \ge a_1 P_n + \left| \sum_{i=1}^{n} p_i |a_i| - |a_1| P_n \right|. \tag{2}$$

Proof. As $\mathbf{a}$ is nondecreasing we have that

$$\Delta a_{i-1} = a_i - a_{i-1} = |a_i - a_{i-1}| \ge \; ||a_i| - |a_{i-1}|| = |\Delta |a_{i-1}|| \ge 0$$

for all $i = 2, \dots, n$ and

$$0 \le \sum_{k=i}^{n} p_k = \left| \sum_{k=i}^{n} p_k \right|$$

for all $i = 2, \ldots, n$. Thus, by the first equality in (1), we have

$$\begin{aligned}\sum_{i=1}^{n} p_i a_i - a_1 P_n = &\sum_{i=2}^{n} \left(\sum_{k=i}^{n} p_k \right) \Delta a_{i-1} = \sum_{i=2}^{n} \left| \sum_{k=i}^{n} p_k \right| |\Delta a_{i-1}| \\ \geq &\sum_{i=2}^{n} \left| \sum_{k=i}^{n} p_k \right| |\Delta |a_{i-1}|| = \sum_{i=2}^{n} \left| \left(\sum_{k=i}^{n} p_k \right) \Delta |a_{i-1}| \right| \\ \geq &\left| \sum_{i=2}^{n} \left(\sum_{k=i}^{n} p_k \right) \Delta |a_{i-1}| \right|.\end{aligned}$$

By Abel's identity for $|\mathbf{a}| := (|a_1|, \ldots, |a_n|)$, we have also that

$$\sum_{i=1}^{n} p_i |a_i| - |a_1| \sum_{i=1}^{n} p_i = \sum_{i=2}^{n} \left(\sum_{k=i}^{n} p_k \right) \Delta |a_{i-1}|.$$

Thus

$$\sum_{i=1}^{n} p_i a_i - a_1 \sum_{i=1}^{n} p_i \geq \left| \sum_{i=1}^{n} p_i |a_i| - |a_1| P_n \right| \geq 0$$

and (2) is proved. □

The second result is embodied in the following theorem.

Theorem 4. *Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{p} = (p_1, \ldots, p_n)$ be n–tuples of real numbers such that $a_1 \leq \ldots \leq a_n$ and $\sum_{k=1}^{i} p_k \geq 0$ $(i = 1, \ldots, n-1)$. Then*

$$a_n P_n - \sum_{i=1}^{n} p_i a_i \geq \left| |a_n| P_n - \sum_{i=1}^{n} p_i |a_i| \right| \geq 0. \tag{3}$$

Proof. By the second identity in (1) we can write

$$a_n P_n - \sum_{i=1}^{n-1} p_i a_i = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \right) \Delta a_i.$$

Since

$$\Delta a_i = a_{i+1} - a_i = |a_{i+1} - a_i| \geq ||a_{i+1}| - |a_i|| = |\Delta |a_i||$$

and $\sum_{k=1}^{i} p_k \geq 0$ for $i = 1, \ldots, n-1$, we have that

$$\begin{aligned}\sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \right) \Delta a_i = &\sum_{i=1}^{n-1} \left| \sum_{n=1}^{i} p_k \right| |\Delta a_i| \geq \sum_{i=1}^{n-1} \left| \sum_{k=1}^{i} p_k \right| |\Delta |a_i|| \\ = &\sum_{i=1}^{n-1} \left| \sum_{n=1}^{i} p_k \Delta |a_i| \right| \geq \left| \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \right) \Delta |a_i| \right|.\end{aligned}$$

By Abel's identity for $|\mathbf{a}|$ we have also

$$\sum_{i=1}^{n} p_i|a_i| = |a_n|P_n - \sum_{i=1}^{n-1}\left(\sum_{k=1}^{i} p_k\right)\Delta|a_i|,$$

whence we derive (3). □

Remark 1. *The condition* $\sum_{k=i}^{n} p_k \geq 0$ $(i = 2,\dots,n)$ *is equivalent to* $P_n - P_{i-1} \geq 0$ $(i = 2,\dots,n)$ *or* $P_n \geq P_i$ *for* $i = 1,\dots,n-1$ *and the condition* $\sum_{k=1}^{i} p_k \geq 0$, $(i = 1,\dots,n-1)$ *is equivalent to* $P_i \geq 0$ $(i = 1,\dots,n-1)$.

The following corollary also holds.

Corollary 1. *Suppose* $\mathbf{a}$ *is nondecreasing and* $\mathbf{p} \in I\!R^n$ *with* $P_n \geq P_i \geq 0$ *for all* $i = 1,\dots,n-1$. *Then*

$$a_nP_n - \left| |a_n|P_n - \sum_{i=1}^{n} p_i|a_i| \right| \geq \sum_{i=1}^{n} p_i a_i \geq a_1P_n + \left| \sum_{i=1}^{n} p_i|a_i| - |a_1|P_n \right|.$$

Remark 2. *The above inequality is similar to Abel's result as it provides an upper and a lower bound for* $\sum_{i=1}^{n} p_i a_i$ *when the sequence* $\mathbf{a}$ *is nondecreasing and* $\mathbf{p}$ *is such that* $0 \leq P_i \leq P_n$ *for all* $i = 1,\dots,n-1$.

3. Inequalities for complex numbers

We now derive some similar results valid for complex numbers.

Theorem 5. *Suppose* $\mathbf{z} = (z_1,\dots,z_n) \in C^n$ *and* $\mathbf{a} = (a_1,\dots,a_n) \in I\!R^n$ *are such that*

$$|z_i - z_{i-1}| \leq a_i - a_{i-1} \quad \forall i = 2,\dots,n. \tag{4}$$

Then for all $\mathbf{w} = (w_1,\dots,w_n) \in C^n$, *we have*

$$\sum_{i=1}^{n} |w_i|a_i - a_1\sum_{i=1}^{n} |w_i| \geq \max\left\{ \left|\sum_{i=1}^{n} w_i z_i - z_1\sum_{i=1}^{n} w_i\right|, \left|\sum_{i=1}^{n} w_i|z_i| - |z_1|\sum_{i=1}^{n} w_i\right|, \left|\sum_{i=1}^{n} w_i|z_i| - |z_1|\sum_{i=1}^{n} |w_i|\right|, \left|\sum_{i=1}^{n} |w_i||z_i| - |z_1|\sum_{i=1}^{n} |w_i|\right| \right\}.$$

Proof. By Abel's identity and (4) we have that

$$\sum_{i=1}^{n} |w_i|a_i - a_1\sum_{i=1}^{n} |w_i| = \sum_{i=2}^{n}\left(\sum_{k=i}^{n} |w_k|\right)\Delta a_{i-1} \geq \sum_{i=2}^{n}\left(\sum_{k=i}^{n} |w_k|\right)|\Delta z_{i-1}| =: A.$$

By the properties of the modulus mapping, we have further that

$$\sum_{k=i}^{n} |w_k| \geq \left|\sum_{k=i}^{n} w_k\right|,$$

and so

$$A \geq \sum_{i=2}^{n} \left| \left(\sum_{k=i}^{n} w_k \right) \Delta z_{i-1} \right| \geq \left| \sum_{i=2}^{n} \left(\sum_{k=i}^{n} w_k \right) \Delta z_{i-1} \right| = \left| \sum_{i=1}^{n} w_i z_i - z_1 \sum_{i=1}^{n} w_i \right| .$$

Also, we can write

$$|\Delta z_{i-1}| \geq |\Delta |z_{i-1}||$$

for $i = 2, \dots, n+1$. Thus

$$A \geq \sum_{i=2}^{n} \left| \left(\sum_{k=i}^{n} w_k \right) \Delta z_{i-1} \right| \geq \left| \sum_{i=2}^{n} \left(\sum_{k=i}^{n} w_k \right) \Delta |z_{i-1}| \right| = \left| \sum_{i=1}^{n} w_i |z_i| - |z_1| \sum_{i=1}^{n} w_i \right| .$$

In the same way, we have

$$\begin{aligned} A = \; & \sum_{i=2}^{n} \left(\sum_{k=i}^{n} |w_k| \right) |\Delta z_{i-1}| = \sum_{i=2}^{n} \left| \left(\sum_{k=i}^{n} |w_k| \right) \Delta z_{i-1} \right| \\ \geq \; & \left| \sum_{i=2}^{n} \left(\sum_{k=i}^{n} |w_k| \right) \Delta z_{i-1} \right| = \left| \sum_{i=1}^{n} |w_i| z_i - z_1 \sum_{i=1}^{n} |w_i| \right| \end{aligned}$$

and

$$\begin{aligned} A = \; & \sum_{i=2}^{n} \left(\sum_{k=i}^{n} |w_k| \right) |\Delta z_{i-1}| \geq \sum_{i=2}^{n} \left(\sum_{k=i}^{n} |w_k| \right) |\Delta |z_{i-1}|| \\ \geq \; & \left| \sum_{i=2}^{n} \left(\sum_{k=i}^{n} |w_k| \right) \Delta |z_{i-1}| \right| = \left| \sum_{i=1}^{n} |w_i| |z_i| - |z_1| \sum_{i=1}^{n} |w_i| \right| , \end{aligned}$$

and we are done. □

In the same way, the second part of Abel's identity leads to the following theorem.

Theorem 6. *Under the conditions of* Theorem 5.*, we have*

$$\begin{aligned} a_n \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| a_i \geq \; & \max \left\{ \left| z_n \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_i z_i \right| , \left| |z_n| \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} |z_i| w_i \right| , \right. \\ & \left. \left| z_n \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| z_i \right| , \left| |z_n| \sum_{i=1}^{n} |w_i| - \sum_{i=1}^{n} |w_i| |z_i| \right| \right\} . \end{aligned}$$

4. Application to integral inequalities

Theorem 7. *Let* $f : [a,b] \to I\!R$ *be a nonnegative, increasing function and* $x_i : [a,b] \to I\!R$ *functions with a continuous first derivative such that*

1) $x_1(t) \leq \cdots \leq x_n(t)$, $t \in [a,b]$,

2) $x_1'(t) \leq \cdots \leq x_n'(t)$, $t \in [a,b]$.

Suppose also $p_i \geq 0$ *and* $\sum_{i=1}^{n} p_i = 1$. *Then*

$$\left| \sum_{i=1}^{n} p_i \left(\left| \int_a^b x_i'(t) f(t) dt \right| - \left| \int_a^b x_1'(t) f(t) dt \right| \right) \right| + \left| \sum_{i=1}^{n} p_i \int_a^b \left(|x_i(t)| - |x_1(t)| \right) df(t) \right|$$
$$\leq f(b) \sum_{i=1}^{n} p_i \left[x_i(b) - x_1(b) \right] - f(a) \sum_{i=1}^{n} p_i \left[x_i(a) - x_1(a) \right]. \tag{5}$$

Proof. By integration by parts,

$$\sum_{i=1}^{n} p_i \int_a^b x_i'(t) f(t) dt = f(b) \sum_{i=1}^{n} p_i x_i(b) - f(a) \sum_{i=1}^{n} p_i x_i(a) - \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t) \right) df(t). \tag{6}$$

We can apply (2) to obtain

$$\sum_{i=1}^{n} p_i \int_a^b x_i'(t) f(t) dt \geq \int_a^b x_1'(t) f(t) dt + \left| \sum_{i=1}^{n} p_i \left| \int_a^b x_i'(t) f(t) dt \right| - \left| \int_a^b x_i'(t) f(t) dt \right| \right| \tag{7}$$

and

$$\sum_{i=1}^{n} p_i x_i(t) \geq x_1(t) + \left| \sum_{i=1}^{n} p_i \left| x_i(t) \right| - \left| x_1(t) \right| \right|$$

for all $t \in [a, b]$.

Integrating this last inequality, we deduce that

$$\begin{aligned} \int_a^b \left(\sum_{i=1}^{n} p_i x_i(t) \right) df(t) \geq & \int_a^b x_1(t) df(t) + \int_a^b \left| \sum_{i=1}^{n} p_i \left| x_i(t) \right| - \left| x_1(t) \right| \right| df(t) \\ = & f(t) x_1(t) |_a^b - \int_a^b x_1'(t) f(t) dt \\ & + \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i(t) \right| df(t) - \int_a^b \left| x_1(t) \right| df(t) \right|. \end{aligned} \tag{8}$$

Using (6)–(8), we derive

$$\begin{aligned} \int_a^b x_1' f(t) dt \quad & + \left| \sum_{i=1}^{n} p_i \left| \int_a^b x_i' f(t) dt \right| - \left| \int_a^b x_1' f(t) dt \right| \right| \\ \leq \quad & f(b) \sum_{i=1}^{n} p_i x_i(b) - f(a) \sum_{i=1}^{n} p_i x_i(a) - \left(f(b) x_1(b) - f(a) x_1(a) \right) \\ & + \int_a^b x_1' f(t) dt - \left| \sum_{i=1}^{n} p_i \int_a^b \left| x_i'(t) \right| df(t) - \int_a^b \left| x_1(t) \right| df(t) \right|, \end{aligned}$$

which is equivalent to (5). □

Remark 3. *Similar results can be obtained from the second Abel–type inequality (3).*

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