Sensitivity analysis in models of data envelopment analysis[*†]

LUKA NERALIĆ[‡]

Abstract. *Sensitivity analysis in Data Envelopment Analysis (DEA) is studied in this paper for the Charnes-Cooper-Rhodes (CCR) ratio model and for the additive model. Different cases of additive or proportionate changes of inputs or/and of outputs of an efficient Decision Making Unit (DMU) according to the CCR model or according to the additive model are considered. Sufficient conditions for an efficient DMU to preserve its efficiency after the corresponding changes of its inputs or/and of outputs are presented for these cases. Similar results for arbitrary (or nonnegative) additive perturbations of data of all DMUs in the additive model are described, too.*

Key words: *data envelopment analysis, efficiency, additive (proportionate) change of inputs or/and of outputs, sensitivity analysis, linear programming*

Sažetak. **Analiza osjetljivosti u modelima analize omeđivanja podataka.** *U ovom radu razmatra se analiza osjetljivosti Charnes-Cooper-Rhodesovog (CCR) modela i aditivnog modela analize omeđivanja podataka. Promatrani su različiti slučajevi aditivne ili proporcionalne promjene inputa i/ili outputa efikasnog donosioca odluke prema CCR modelu ili prema aditivnom modelu. Navedeni su dovoljni uvjeti uz koje se pri odgovarajućoj promjeni inputa i/ili outputa efikasnog donosioca odluke čuva njegova efikasnost u tim slučajevima. Također su prikazani slični rezultati za proizvoljne (ili nenegativne) aditivne perturbacije podataka svih donosilaca odluke u aditivnom modelu.*

Ključne riječi: *analiza omeđivanja podataka, efikasnost, aditivna (proporcionalna) promjena inputa i/ili outputa, analiza osjetljivosti, linearno programiranje*

AMS subject classifications: 90C05, 90C31, 90C50

Received June 12, 1998

---

[*]The lecture presented at the MATHEMATICAL COLLOQUIUM in Osijek organized by Croatian Mathematical Society - Division Osijek, January 21, 1998.

[†]This research was partially supported by the Research Council of Croatia.

[‡]Faculty of Economics, University of Zagreb, Kennedyjev trg 6, HR-10 000 Zagreb, Croatia, e-mail: `neralic@oliver.efzg.hr`

1. Introduction

Sensitivity analysis in Data Envelopment Analysis (DEA) for the Charnes–Cooper–Rhodes (CCR) ratio model (see [6], [9], [3]) was studied in [8] for the case of the change of a single output. Sufficient conditions for an efficient Decision Making Unit (DMU) to continue to be efficient after the change of a single output were found first [8]. The generalizations of that result for the case of the simultaneous change of all outputs, the case of the simultaneous single output and single input changes, the case of the simultaneous change of all inputs and the case of the simultaneous change of all inputs and outputs for the CCR ratio model were given in [12], [13], [17]. Similar results for the additive model (see [7], [3], [9]) were found in [14]. Sufficient conditions for an efficient DMU to preserve efficiency after the proportionate change of inputs (or outputs) in the CCR model were given in [18]. Similar results for the case of the simultaneous proportionate change of inputs and outputs of the CCR model were established in [15], [16]. The cases of the proportionate change of inputs and/or outputs of the additive model were studied in [23] (see also [25], [24], [28]). Sensitivity analysis for the case of discretionary and nondiscretionary inputs and outputs of the additive model was studied in [20] (see also [19]). An alternative approach to sensitivity analysis in DEA, with change of inputs or/and of outputs of all DMUs, has been used in [30]. Stability of efficiency evaluations in DEA was studied in [21] (see also [10]). New results in sensitivity of efficiency classifications in the additive model were established in [11] (see also [4]) and [31]. Sensitivity analysis of the additive model for arbitrary perturbations of all data was studied in [27].

The aim of this paper is to review some results in sensitivity analysis in DEA for the CCR ratio model and for the additive model.

The paper is organized as follows. Sensitivity analysis of the CCR model is studied in *Section 2*. The case of the additive changes of inputs and outputs of an efficient DMU preserving its efficiency is studied first. After that, the cases of the proportionate changes of inputs or/and outputs with two coefficients of proportionality (one for inputs, the other for outputs) are studied. The case of the proportionate change of inputs and outputs with different coefficients of proportionality for each input and for each output is studied, too. Sufficient conditions for an efficient DMU to preserve its efficiency after the corresponding changes in these cases are presented. Similar results are presented in *Section 3* for the additive model. Sensitivity analysis of the additive model is studied for the cases of additive or proportionate changes of inputs or/and outputs of an efficient DMU. The cases of arbitrary additive perturbations and of nonnegative additive perturbations of data of all DMUs in the additive model preserving efficiency of an efficient DMU are studied, too. The last Section contains summary, some conclusions and suggestions for further research.

2. Sensitivity analysis of the CCR model

2.1. Let us suppose that there are n Decision Making Units (DMUs) with m inputs and s outputs. Let x_{ij} be the observed amount of the ith type of input of the jth DMU ($x_{ij} > 0$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$) and let y_{rj} be the observed amount of output of the rth type for the jth DMU ($y_{rj} > 0$, $r = 1, 2, \ldots, s, j = 1, 2, \ldots, n$). Let $Y_j,\ X_j$ be the observed vectors of outputs and inputs of the DMU_j, respectively, $j = 1, 2, \ldots, n$. Let e be the column vector of ones and let T as a superscript denote the transpose. In order to see if the $\mathrm{DMU}_{j0} = \mathrm{DMU}_0$ is efficient according to the CCR ratio model the following linear programming problem should be solved:

$$\min\ 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - \varepsilon e^T s^+ - \varepsilon e^T s^- + \theta$$

subject to

$$\begin{array}{rcccccl} Y_1\lambda_1 + \cdots + Y_0\lambda_0 + \cdots + Y_n\lambda_n & - & s^+ & & & = & Y_0 \\ -X_1\lambda_1 - \cdots - X_0\lambda_0 - \cdots - X_n\lambda_n & & & - & s^- + X_0\theta & = & 0 \\ \lambda_1, \ldots, \lambda_n, s^+, s^- & \geq & 0, & & & & \end{array} \tag{1}$$

with $Y_0 = Y_{j_0}, X_0 = X_{j_0}, \lambda_0 = \lambda_{j_0}$ and θ unconstrained. DMU_0 is DEA efficient if and only if for the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of the linear programming problem (1) both of the following are satisfied (for details see [6]):

$$\begin{array}{lcccccl} \min & \theta & = & \theta^* & = & 1 & \\ & s^{+*} & = & s^{-*} & = & 0, & \text{in all alternative optima.} \end{array} \tag{2}$$

Let us consider changes of inputs or/and outputs of an efficient DMU_0 preserving its efficiency. An increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient DMU_0. Similarly, a decrease of any input cannot worsen an already achieved efficiency rating. Downward variations are not possible in the efficiency rating for an efficient DMU_0. Hence, we can restrict attention to downward variations of outputs and upward variations of inputs. These variations can be written for outputs as

$$\widehat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad \alpha_r \geq 0, \qquad r = 1, 2, \ldots, s, \tag{3}$$

and for inputs as

$$\widehat{x}_{i0} = x_{i0} + \beta_i, \quad \beta_i \geq 0, \qquad i = 1, 2, \ldots, m. \tag{4}$$

We will also be interested in the proportionate change (decrease) of all outputs

$$\widehat{y}_{r0} = \widehat{\alpha} y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, s, \tag{5}$$

or/and of the proportionate change (increase) of all inputs

$$\widehat{x}_{i0} = \widehat{\beta} x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, m, \tag{6}$$

of an efficient DMUU0 preserving its efficiency.

We will also consider the proportionate change (decrease) of all outputs with a different coefficient of proportionality for each output

$$\widehat{y}_{r0} = \widehat{\alpha}_r y_{r0}, \quad 0 < \widehat{\alpha} \leq 1, \quad r = 1, 2, \ldots, s, \tag{7}$$

or/and of the proportionate change (increase) of all inputs with different coefficient of proportionality for each input

$$\widehat{x}_{i0} = \widehat{\beta}_i x_{i0}, \quad \widehat{\beta} \geq 1, \quad i = 1, 2, \ldots, m, \tag{8}$$

of an efficient DMU_0 preserving its efficiency.

2.2. For an efficient DMU_0 according to the CCR model vectors

$$\begin{bmatrix} Y_0 \\ -X_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ X_0 \end{bmatrix}, \tag{9}$$

must occur in some optimal basis of linear programming problem (1), what means that there is a basic optimal solution to (1) in which $\lambda_0^* = 1$ and $\theta^* = 1$. Changes (3)-(4), (5)-(6) or (7)-(8) are accompanied by alterations in the inverse

$$B^{-1} = \left[b_{ij}^{-1} \right], \quad i, \ j = 1, 2, \ldots, s+m,$$

of the optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 & 0 \\ -X_B & 0 & -I_B^- & X_0 \end{bmatrix}, \tag{10}$$

which corresponds to the optimal solution $(\lambda^*, s^{+*}, s^{-*}, \theta^*)$ of (1) with $\lambda_0^* = 1$ and $\theta^* = 1$.

Let $P_j, \ j = 1, 2, \ldots, n+s+m+1$ be the columns of the matrix and let P_0 be the right-hand side of the linear programming problem (1). We will use the following notation:

$$\begin{aligned} \Gamma_j &= B^{-1} P_j, \ j = 0, 1, \ldots, n+s+m+1, \\ \omega^T &= c_B^T B^{-1}, \\ z_j &= c_B^T B^{-1} P_j \\ &= \omega^T P_j, \ j = 0, 1, \ldots, n+s+m+1. \end{aligned}$$

Simultaneous changes (3) of all outputs and changes (4) of all inputs of an efficient DMU_0 means the following perturbation of the optimal basis B

$$\widehat{B} = B + \triangle B \tag{11}$$

with

$$\triangle B = \begin{array}{c} \begin{array}{ccccccc} & & & \overset{k}{\downarrow} & & & \overset{s+m}{\downarrow} \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_1 & 0 & \cdots & \beta_1 \\ 0 & \cdots & 0 & -\beta_2 & 0 & \cdots & \beta_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\beta_m & 0 & \cdots & \beta_m \end{pmatrix} \end{array} \tag{12}$$

and the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + [-\alpha_1 \ -\alpha_2 \ \ldots - \alpha_s \ 0 \ \ldots 0\ ]^T, \tag{13}$$

where the index k corresponds to the basic variable $\lambda_0^* = \lambda_k$ and $s+m$ corresponds to the basic variable θ^*.

Using matrices

$$U_{(s+m)\times 2} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \vdots & \vdots \\ \alpha_s & \alpha_s \\ \beta_1 & 0 \\ \beta_2 & 0 \\ \vdots & \vdots \\ \beta_m & 0 \end{bmatrix} \tag{14}$$

and

$$V^T_{2\times(s+m)} = \begin{array}{c} \begin{array}{ccccccccc} & & & k & & & & s+m \\ & & & \downarrow & & & & \downarrow \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \end{array}, \tag{15}$$

we can write the perturbation matrix (12) as

$$\triangle B = UV^T. \tag{16}$$

Because of (11) and (16) we can use the Sherman–Morrison–Woodbury formula (see, for example, [22], p. 3) in order to get the following perturbed basis inverse

$$\begin{aligned} (\widehat{B})^{-1} &= (B + UV^T)^{-1} \\ &= B^{-1} - B^{-1}U(I + V^T B^{-1} U)^{-1} V^T B^{-1}. \end{aligned} \tag{17}$$

Using the abbreviation

$$D = U(I + V^T B^{-1} U)^{-1} V^T \tag{18}$$

we can write (17) as

$$\begin{aligned} (\widehat{B})^{-1} &= B^{-1} - B^{-1} D B^{-1} \\ &= B^{-1}(I - DB^{-1}) \\ &= (I - B^{-1}D)B^{-1}. \end{aligned} \tag{19}$$

Let

$$M = I + V^T B^{-1} U \tag{20}$$

where matrix M is nonsingular with

$$\begin{aligned} \det M &= 1 - \sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t + \sum_{t=1}^{m}(-b^{-1}_{k,s+t} + b^{-1}_{s+m,s+t})\beta_t + \\ &+ (\sum_{t=1}^{s} b^{-1}_{s+m,t}\alpha_t)(\sum_{t=1}^{m} b^{-1}_{k,s+t}\beta_t) - (\sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t)(\sum_{t=1}^{m} b^{-1}_{s+m,s+t}\beta_t), \end{aligned} \tag{21}$$

and

$$D = UM^{-1}V^T. \tag{22}$$

Now we can prove the following

Theorem 1. *Let us suppose that* DMU_0 *is efficient. Conditions*

$$\omega^T D\Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables,} \tag{23}$$

are sufficient for DMU_0 *to be efficient after the simultaneous changes of outputs (3) and inputs (4).*

For the proof and details see [17], *Theorem 1.*

From *Theorem 1* it is easy to get the following

Corollary 1. *If for* $\det M$ *in (21) holds* $\det M > 0$, *conditions (23) can be written as*

$$\gamma_k \Gamma_{j,k} + \gamma_{s+m}\Gamma_{s+m,k} \geq (z_j - c_j)\det M, \tag{24}$$
$$j \text{ an index of nonbasic variables.}$$

with

$$\gamma_k = -(1 + \sum_{t=1}^{m} b^{-1}_{s+m,s+t}\beta_t)(\sum_{t=1}^{s}\omega_t\alpha_t) + (-1 + \sum_{t=1}^{s} b^{-1}_{s+m,t}\alpha_t)(\sum_{t=1}^{m}\omega_{s+t}\beta_t), \tag{25}$$

$$\gamma_{s+m} = (\sum_{t=1}^{m} b^{-1}_{k,s+t}\beta_t)(\sum_{t=1}^{s}\omega_t\alpha_t) + (1 - \sum_{t=1}^{s} b^{-1}_{k,t}\alpha_t)(\sum_{t=1}^{m}\omega_{s+t}\beta_t). \tag{26}$$

2.3. For fixed outputs let us consider the proportionate change (increase) of all inputs (6) of an efficient DMU_0 preserving its efficiency. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the change (6). We also want to find the maximal value $\widehat{\beta}^*$ of $\widehat{\beta}$ for which the efficiency of DMU_0 is preserved after the change (6). In that case in order to get the perturbed optimal basis inverse $(\widehat{B})^{-1}$ we can use *Theorem 6* in the subsection 3.2 below instead of the Sherman–Morrison–Woodbury formula (17).

Theorem 2. *Let us suppose that for*

$$p = 1 + \sum_{t=1}^{m}(-b^{-1}_{k,s+t} + b^{-1}_{s+m,s+t})\beta_t, \tag{27}$$

with $\beta_t = (\widehat{\beta} - 1)x_{t0}, t = 1, 2, \ldots, m$ *holds* $p > 0$. *Let*

$$a = \sum_{t=1}^{s}\omega_{s+t}x_{t0}, \quad b = \sum_{t=1}^{m}(-b^{-1}_{k,s+t} + b^{-1}_{s+m,s+t})x_{t0}, \tag{28}$$

$$d_j = a(-\Gamma_{kj} + \Gamma_{s+m,j}) - b\bar{c}_j, \quad j = 1, 2, \ldots, n+s+m+1, \tag{29}$$

with $\bar{c}_j = z_j - c_j$. *Then the conditions*

$$\widehat{\beta} d_j \geq d_j + \bar{c}_j, \quad j \text{ an index of nonbasic variables,} \tag{30}$$

are sufficient for DMU_0 *to preserve efficiency after the proportionate change (6) of inputs.*

For the proof and details see *Theorem 3.1* in [18].

Remark 1. *For*

$$J_1 = \{j \mid d_j < 0, j \text{ an index of nonbasic variables } \},$$

it follows from (6) and (30) that

$$1 \leq \widehat{\beta} \leq 1 + \min\{\frac{\bar{c}_j}{d_j} \mid j \in J_1\}. \tag{31}$$

This means that the maximal value $\widehat{\beta}^*$ of $\widehat{\beta}$ for which the efficiency of DMU_0 is preserved after change (6) is

$$\widehat{\beta}^* = 1 + \min\{\frac{\bar{c}_j}{d_j} \mid j \in J_1\}. \tag{32}$$

The maximal percentage of increase of all inputs preserving efficiency od DMU_0 after the change (6) is $\beta^* \cdot 100\% = (\widehat{\beta}^* - 1) \cdot 100\%$.

Remark 2. *For the case $p < 0$ instead of $p > 0$ in conditions (30) the inequality sign $\geq$ should be changed into $\leq$.*

2.4. For fixed inputs we are interested in the proportionate change (decrease) of all outputs (5) of an efficient DMU_0 preserving its efficiency. We want to find the sufficient conditions for DMU_0 to preserve efficiency and also the minimal value $\widehat{\alpha}^*$ of $\widehat{\alpha}$ for which efficiency of DMU_0 is preserved after the change (5) of outputs. We can also use *Theorem 6* below instead of the Sherman–Morrison–Woodbury formula.

Theorem 3. *Let us suppose that for*

$$p_1 = -\sum_{t=1}^{s} b_{k,t}^{-1} \alpha_t \tag{33}$$

with $\alpha_t = (1 - \widehat{\alpha}) y_{t0}, t = 1, 2, \ldots, s$ holds $1 + p_1 > 0$. Let

$$g_j = \sum_{t=1}^{s} (\bar{c}_j b_{k,t}^{-1} - \Gamma_{jk} \omega_t) y_{t0}, \quad j \text{ an index of nonbasic variables,} \tag{34}$$

with $\bar{c}_j = z_j - c_j$. Conditions

$$\widehat{\alpha} g_j \leq g_j - \bar{c}_j, \quad j \text{ an index of nonbasic variables,} \tag{35}$$

are sufficient for DMU_0 to preserve efficiency after the proportionate change (5) of outputs.

For the proof and details see *Theorem 4.1* in [18].

Remark 3. *For*

$$J_2 = \{j \mid g_j < 0, j \text{ an index of nonbasic variables } \},$$

using (5) and (35) we can get

$$1 - \min\{\frac{\bar{c}_j}{g_j} \mid j \in J_2\} \leq \widehat{\alpha} \leq 1. \tag{36}$$

According to (36), the minimal value $\widehat{\alpha}^*$ of $\widehat{\alpha}$ for which the efficiency of DMU$_0$ is preserved after change (5) is

$$\widehat{\alpha}^* = 1 - \min\{\frac{\bar{c}_j}{g_j} \mid j \in J_2\}. \tag{37}$$

The maximal percentage of decrease of all outputs preserving efficiency of DMU$_0$ after change (5) is $\alpha^* \cdot 100\% = (1 - \widehat{\alpha}^*) \cdot 100\%$.

Remark 4. *If there holds $1 + p_1 < 0$ instead of $1 + p_1 > 0$, then in conditions (35) the inequality sign should be changed from $\leq$ to $\geq$.*

2.5. Let us consider the simultaneous proportionate change (increase) of all outputs (5) and the proportionate change (decrease) of all inputs (6) of an efficient DMU$_0$ preserving efficiency. We are interested in sufficient conditions for DMU$_0$ to preserve efficiency after the simultaneous changes (5) and (6). In this case we have to use the Sherman–Morrison–Wodbury formula (17).

Theorem 4. *Let us suppose that DMU$_0$ is efficient. Let for M in (20) with $\alpha_t = y_{t0}(1 - \widehat{\alpha}_t), t = 1, 2, \ldots, s$ and $\beta_t = x_{t0}(\widehat{\beta}_t - 1), t = 1, 2, \ldots, m$ there hold*

$$\det M = 1 - a_1(1 - \widehat{\alpha}) + (-b_1 + b_2)(\widehat{\beta} - 1) + (a_2 b_1 - a_1 b_2)(1 - \widehat{\alpha})(\widehat{\beta} - 1) > 0, \tag{38}$$

with

$$a_1 = \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^{s} b_{s+m,t}^{-1} y_{t0}, \quad b_1 = \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}, \quad b_2 = \sum_{t=1}^{m} b_{s+m,s+t}^{-1} x_{t0}. \tag{39}$$

Let

$$a_3 = \sum_{t=1}^{s} \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^{m} \omega_{s+t} x_{t0}, \tag{40}$$

$$d_j = -a_3 \Gamma_{kj} + a_1 \bar{c}_j, \quad e_j = -b_3(\Gamma_{kj} - \Gamma_{s+m,j}) - (-b_1 + b_2)\bar{c}_j, \tag{41}$$

$$f_j = (a_2 b_3 - a_3 b_2)\Gamma_{kj} + (a_3 b_1 - a_1 b_3)\Gamma_{s+m,j} - (a_2 b_1 - a_1 b_2)\bar{c}_j, \tag{42}$$

$$j = 1, 2, \ldots, n + s + m + 1,$$

with $\bar{c}_j = z_j - c_j$. Then the conditions

$$d_j(1 - \widehat{\alpha}) + e_j(\widehat{\beta} - 1) + f_j(1 - \widehat{\alpha})(\widehat{\beta} - 1) \geq \bar{c}_j, \tag{43}$$

$$j \text{ an index of nonbasic variables},$$

are sufficient for DMU$_0$ to preserve efficiency after the simultaneous proportionate changes of outputs (5) and inputs (6).

For the proof and details see *Theorem 2* in [16].

Remark 5. *For the case det $M < 0$ instead of det $M > 0$ in (38), the inequality sign $\geq$ in conditions (43) should be changed into $\leq$.*

Remark 6. *The system of inequalities (43) together with conditions (5), (6) and (38) for $\widehat{\alpha}$ and $\widehat{\beta}$ gives the area $\widehat{A}_0$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. For each point $(\widehat{\alpha}, \widehat{\beta})$ in the area $\widehat{A}_0$ efficiency of DMU$_0$ will be preserved after the simultaneous proportionate changes of inputs (5) and outputs (6).*

Remark 7. *We can use the area $\widehat{A}_0$ for ranking among efficient DMUs. For example, if for efficient DMU$_1$ and DMU$_2$ there holds $\widehat{A}_1 > \widehat{A}_2$ it can be said that "DMU$_1$ is relatively more efficient than DMU$_2$" because DMU$_1$ is less sensitive to the simultaneous proportionate change of inputs and outputs preserving efficiency than DMU$_2$.*

2.6. Let us consider the simultaneous proportionate change (decrease) of all outputs (7) and the proportionate change (increase) of all inputs (8) of an efficient DMU$_0$ preserving efficiency. We are interested in sufficient conditions for DMU$_0$ to preserve efficiency after the simultaneous changes (7) of outpus and (8) of inputs.

Let us introduce the following notation:

$$A_1 = \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1 - \widehat{\alpha}_t), A_2 = \sum_{t=1}^{s} b_{s+m,t}^{-1} y_{t0}(1 - \widehat{\alpha}_t),$$
$$A_3 = \sum_{t=1}^{s} \omega_t y_{t0}(1 - \widehat{\alpha}_t), \tag{44}$$

$$B_1 = \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1), B_2 = \sum_{t=1}^{m} b_{s+m,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1),$$
$$B_3 = \sum_{t=1}^{m} \omega_{s+t} x_{t0}(\widehat{\beta}_t - 1). \tag{45}$$

Theorem 5. *Let us suppose that DMU$_0$ is efficient and let for M in (20) with $\alpha_t = y_{t0}(1 - \widehat{\alpha}_t), t = 1, 2, \ldots, s$ and $\beta_t = x_{t0}(\widehat{\beta}_t - 1), t = 1, 2, \ldots, m$ there hold*

$$\det\ M = 1 - A_1 - B_1 + B_2 + A_2 B_1 - A_1 B_2 > 0. \tag{46}$$

Then the conditions

$$\begin{aligned} & (A_3 - B_3 + A_2 B_3 - A_3 B_2)\Gamma_{kj} + \\ + \quad & (B_3 + A_3 B_1 - A_1 B_3)\Gamma_{s+m,j} + \\ + \quad & (A_1 + B_1 - B_2 - A_2 B_1 + A_1 B_2)\bar{c}_j \geq \bar{c}_j, \end{aligned} \tag{47}$$
$$j \textit{ an index of nonbasic variables,}$$

with $\bar{c}_j = z_j - c_j$, are sufficient for DMU$_0$ to preserve efficiency after the simultaneous proportionate changes of outputs (7) and inputs (8).

The proof is similar to the proof of *Theorem 2* in [16]. See also *Theorem 2* in [28].

3. Sensitivity analysis of the additive model

3.1. We will use the same notation as in *Section 2*. In order to see if for $j = j_0$, $\mathrm{DMU}_{j_0} = \mathrm{DMU}_0$ with vector of inputs $X_{j_0} = X_0$ and vector of outputs $Y_{j_0} = Y_0$ is Pareto-Koopmans efficient according to the additive model, the following linear programming problem should be solved

$$\min 0\lambda_1 + \cdots + 0\lambda_0 + \cdots + 0\lambda_n - e^T s^+ - e^T s^-$$

subject to

$$\begin{array}{rcrcrcl} Y_1\lambda_1 + \cdots + Y_0\lambda_0 + \cdots + Y_n\lambda_n & - & s^+ & & & = & Y_0 \\ -X_1\lambda_1 - \cdots - X_0\lambda_0 - \cdots - X_n\lambda_n & & & - & s^- & = & -X_0 \\ \lambda_1 + \cdots + \lambda_0 + \cdots + \lambda_n & & & & & = & 1 \\ \lambda_1, \ldots, \lambda_n, s^+, s^- \geq 0. & & & & & & \end{array} \tag{48}$$

DMU_0 is Pareto-Koopmans efficient if and only if for the optimal solution $(\lambda^*, s^{+*}, s^{-*})$ of the linear programming problem (48) holds

$$\min(-e^T s^+ - e^T s^-) = -e^T s^{+*} - e^T s^{-*} = 0 \tag{49}$$

(for details see [7], [9], [3]).

3.2. Let us consider the simultaneous change (decrease) of outputs (3) and change (increase) of inputs (4) of a Pareto-Koopmans efficient DMU_0 preserving its efficiency. We are interested in sufficient conditions for DMU_0 to preserve its efficiency under the simultaneous changes (3) and (4).

It is easy to see that for Pareto-Koopmans efficient DMU_0 there is a basic optimal solution $(\lambda^*, s^{+*}, s^{-*})$ of (48) with optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 \\ -X_B & 0 & -I_B^+ \\ e^T & 0 & 0 \end{bmatrix}.$$

Let the inverse of the matrix B be

$$B^{-1} = [b_{ij}^{-1}], \quad i, j = 1, 2, \ldots, s + m + 1.$$

The simultaneous change of outputs (3) and change of inputs (4) means the following perturbation of the optimal basis matrix B

$$\widehat{B} = B + D, \tag{50}$$

with

$$D = \begin{array}{c} \begin{array}{ccccccc} & & & \overset{k}{\downarrow} & & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\alpha_s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\beta_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & -\beta_m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} \tag{51}$$

where the index k corresponds to the optimal basic variable $\lambda_0^* = \lambda_k^*$. It also means the following change of the right-hand side vector of the linear programming problem (48)

$$\widehat{P}_0 = P_0 + [\ -\alpha_1 - \alpha_2 \ldots - \alpha_s - \beta_1 - \beta_2 \ldots - \beta_m\ 0\ ]^T . \tag{52}$$

It is easy to show that for matrices B^{-1} and D holds

$$B^{-1}DB^{-1}D = pB^{-1}D, \tag{53}$$

where

$$p = -(\sum_{t=1}^{s} b_{kt}^{-1}\alpha_t + \sum_{t=1}^{m} b_{k,s+t}^{-1}\beta_t). \tag{54}$$

Because of (53) we will use the following theorem of Charnes and Cooper, which is proved in a more general form in [5].

Theorem 6. *Let B be a $k \times k$ matrix with inverse B^{-1}. Let D be a $k \times k$ matrix such that $B^{-1}DB^{-1}D = pB^{-1}D$ for some real scalar p. If σ is any scalar such that $p\sigma \neq -1$, then*

$$\begin{aligned} (B + \sigma D)^{-1} &= B^{-1}(I + \tau DB^{-1}) \\ &= (I + \tau B^{-1}D)B^{-1}, \end{aligned}$$

where

$$\tau = -\sigma(1 + p\sigma)^{-1}.$$

Let us suppose that for p in (54) there holds $p \neq -1$. Because of (53) it follows from *Theorem 6* for $\sigma = 1$, that the inverse of the perturbed optimal basis $\widehat{B}$ in (50) is given by

$$\begin{aligned} (\widehat{B})^{-1} &= (B + D)^{-1} \\ &= B^{-1}(I + \tau DB^{-1}) \\ &= (I + \tau B^{-1}D)B^{-1}, \end{aligned} \tag{55}$$

with

$$\tau = -\frac{1}{1+p}. \tag{56}$$

Now we can prove the following

Theorem 7. *Conditions*

$$-\tau\omega^T D\Gamma_j \geq z_j - c_j, \quad j \textit{ an index of nonbasic variables}, \tag{57}$$

are sufficient for Pareto-Koopmans efficient DMU_0 to preserve efficiency after the simultaneous changes (3) and (4).

For the proof and details see *Theorem 2* in [14].

From *Theorem 7* it is easy to get the following

Corollary 2. *Let us suppose that for p in (54) there holds $1 + p > 0$. Conditions*

$$\sum_{t=1}^{s}(b_{kt}^{-1}\bar{c}_j - \omega_t\Gamma_{kj})\alpha_t + \sum_{t=1}^{m}(b_{k,s+t}^{-1}\bar{c}_j - \omega_{s+t}\Gamma_{kj})\beta_t \geq \bar{c}_j, \tag{58}$$

$$j \text{ an index of nonbasic variables,}$$

with $\bar{c}_j = z_j - c_j$ are sufficient for DMU_0 to preserve its efficiency after the simultaneous changes (3) and (4).

3.3. Let us consider the simultaneous proportionate change (decrease) (5) of outputs and proportionate change (increase) (6) of inputs. We are interested in sufficient conditions for DMU_0 to preserve efficiency after the simultaneous changes (5) and (6). We are also interested in the area $\widehat{A}_0$ which is the solution set of the corresponding system of inequalities in $\widehat{\alpha}$ and $\widehat{\beta}$ in the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. The size of that area is a measure of stability of efficiency for DMU_0.

Theorem 8. *Let us suppose that DMU_0 is Pareto - Koopmans efficient. Let us denote*

$$a_1 = \sum_{t=1}^{s} b_{kt}^{-1}y_{t0}, \quad b_1 = \sum_{t=1}^{m} b_{k,s+t}^{-1}x_{t0}, \quad a_2 = \sum_{t=1}^{s}\omega_t y_{t0}, \quad b_2 = \sum_{t=1}^{m}\omega_{s+t}x_{t0}.$$

Let for p in (54) with $\alpha_t = (1 - \widehat{\alpha})y_{t0},\ t = 1, 2, \ldots, s$ and $\beta_t = (\widehat{\beta} - 1)x_{t0},\ t = 1, 2, \ldots, m$ there hold

$$1 + p = 1 - \Big[(1 - \widehat{\alpha})a_1 + (\widehat{\beta} - 1)b_1\Big] > 0. \tag{59}$$

Then conditions

$$(1 - \widehat{\alpha})d_j + (\widehat{\beta} - 1)e_j \geq \bar{c}_j, \quad j \text{ an index of nonbasic variables} \tag{60}$$

with

$$d_j = -a_2\Gamma_{kj} + a_1\bar{c}_j, \quad e_j = b_1\bar{c}_j - b_2\Gamma_{kj}, \tag{61}$$

$$j = 1, 2, \ldots, n + s + m.$$

and $\bar{c}_j = z_j - c_j$, are sufficient for DMU_0 to continue to be efficient after the simultaneous proportionate change (5) of outputs and proportionate change (6) of inputs.

For the proof and details see *Theorem 4* in [25].

Remark 8. *The system of ineqalities (60) together with conditions (5), (6) and (59) for $\widehat{\alpha}$ and $\widehat{\beta}$ gives the area $\widehat{A}_0$ in the plane with the coordinate system $\widehat{\alpha}\widehat{O}\widehat{\beta}$. For each point $(\widehat{\alpha}, \widehat{\beta})$ in the area $\widehat{A}_0$ efficiency of DMU_0 will be preserved after the simultaneous proportionate change (5) of outputs and proportionate change (6) of inputs.*

Remark 9. *The size of the area* $\widehat{A}_0$ *is a measure of stability of efficiency for an efficient* DMU_0. *If for efficient* DMU_1 *and* DMU_2 *there holds* $\widehat{A}_1 > \widehat{A}_2$ *it can be said that* DMU_1 *is more stable than* DMU_2. *In other words* DMU_1 *is less sensitive to the simultaneous proportionate change of outputs and proportionate change of inputs preserving efficiency than* DMU_2.

3.4. Let us consider the simultaneous proportionate change (decrease) (7) of outputs and proportionate change (increase) (8) of inputs of a Pareto - Koopmans efficient DMU_0 with different coefficient of proportionality for each output and each input. We are interested in sufficient conditions for Pareto - Koopmans efficient DMU_0 to preserve efficiency after the simultaneous changes (7) of outputs and (8) of inputs.

Theorem 9. *Let us suppose that* DMU_0 *is Pareto - Koopmans efficient. Let for* p *in (54) with* $\alpha_t = y_{t0}(1-\widehat{\alpha}_t), t = 1,2,\dots,s$ *and* $\beta_t = x_{t0}(\widehat{\beta}_t - 1), t = 1,2,\dots,m$ *there hold*

$$1 + p = 1 - \sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1-\widehat{\alpha}_t) - \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1) > 0. \tag{62}$$

Then the conditions

$$\begin{aligned} \left[-\sum_{t=1}^{s} \omega_t y_{t0}(1-\widehat{\alpha}_t) - \sum_{t=1}^{m} \omega_{s+t} x_{t0}(\widehat{\beta}_t - 1) \right] \Gamma_{kj} \quad & + \\ + \left[\sum_{t=1}^{s} b_{kt}^{-1} y_{t0}(1-\widehat{\alpha}_t) + \sum_{t=1}^{m} b_{k,s+t}^{-1} x_{t0}(\widehat{\beta}_t - 1) \right] \bar{c}_j \quad & \geq \quad \bar{c}_j, \end{aligned} \tag{63}$$

$$j \quad \text{an index of nonbasic variables,}$$

are sufficient for DMU_0 *to preserve efficiency after the changes (7) of outputs and (8) of inputs.*

The proof is similar to the proof of *Theorem 2* in [16]. See also *Theorem 4* in [28].

3.5. Let us suppose that the set of efficient DMUs according to the additive model (48) is $E = \{1, 2, \dots, n_e\}$ and that the set of inefficient DMUs is $N = \{n_e + 1, n_e + 2, \dots, n\}$. Let us also suppose that the set of efficient DMUs corresponding to the optimal basic λ_j^* variables of the solution of (48), including $\lambda_0^* = \lambda_q^*$, is $EB = \{j_1, j_2, \dots, j_q, \dots, j_h\}$, $q \leq h \leq n_e$. Without loss of generality, in order to avoid cumbersome notation, let us suppose that $EB = \{1, 2, \dots, q, \dots, h\}$. We will also use the notation $SV = \{n+1, n+2, \dots, n+s+m\}$.

We are interested in variations of all data that preserve efficiency of DMU_0. Let us consider arbitrary changes of outputs of all DMUs:

$$\widehat{y}_{rj} = y_{rj} + \alpha_r > 0, \quad r = 1, 2, \dots, s, \; j = 1, 2, \dots, n \tag{64}$$

and arbitrary changes of inputs of all DMUs:

$$\widehat{x}_{ij} = x_{ij} + \beta_i > 0, \quad i = 1, 2, \dots, m, \; j = 1, 2, \dots, n. \tag{65}$$

(Here α_r and β_i are unrestricted in sign.)

The simultaneous change of outputs (64) and change of inputs (65) results in the following perturbations of the vectors of the matrix of the linear programming problem (48):

$$\begin{aligned} \widehat{P}_j &= P_j + [\alpha_1 \ \alpha_2 \ldots \alpha_s \ \beta_1 \ \beta_2 \ldots \beta_m \ 0\ ]^T, \ j = 1, 2, \ldots, n, \\ \widehat{P}_j &= P_j, \quad j \in SV. \end{aligned}$$

The optimal basis matrix B now assumes the form

$$\widehat{B} = B + \Delta B, \tag{66}$$

where

$$\Delta B = \begin{array}{c} \begin{matrix} & & & h & & & \\ & & & \downarrow & & & \end{matrix} \\ \begin{pmatrix} \alpha_1 & \alpha_1 & \cdots & \alpha_1 & 0 & \cdots & 0 \\ \alpha_2 & \alpha_2 & \cdots & \alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \alpha_s & \alpha_s & \cdots & \alpha_s & 0 & \cdots & 0 \\ -\beta_1 & -\beta_1 & \cdots & -\beta_1 & 0 & \cdots & 0 \\ -\beta_2 & -\beta_2 & \cdots & -\beta_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -\beta_m & -\beta_m & \cdots & -\beta_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array} = D. \tag{67}$$

Here h denotes the number of optimal basic λ_j^* variables, $j \in E$. We also have the following change of the right-hand side vector

$$\widehat{P}_0 = P_0 + [\alpha_1 \ \alpha_2 \ldots \alpha_s \ -\beta_1 - \beta_2 \ldots - \beta_m \ 0\ ]^T. \tag{68}$$

It is easy to show that

$$B^{-1}DB^{-1}D = pB^{-1}D, \tag{69}$$

where

$$p = \sum_{k=1}^{h} \gamma_k \tag{70}$$

with

$$\gamma_k = \sum_{t=1}^{s} b_{kt}^{-1} \alpha_t - \sum_{t=1}^{m} b_{k,s+t}^{-1} \beta_t, k = 1, 2, \ldots, s + m + 1. \tag{71}$$

Let us suppose that $p \neq -1$. Because of (69), it now follows from *Theorem 6* that the inverse of the perturbed optimal basis $\widehat{B}$ in (66) is given by

$$\begin{aligned} (\widehat{B})^{-1} &= (B + D)^{-1} \\ &= B^{-1}(I + \tau DB^{-1}) \\ &= (I + \tau B^{-1}D)B^{-1}, \end{aligned} \tag{72}$$

with

$$\tau = -\frac{1}{1 + p}. \tag{73}$$

Now we can prove the following result for general perturbations.

Theorem 10. *Sufficient conditions for an efficient DMU_0 to preserve efficiency after arbitrary changes of all data (64) and (65), are*

$$-\tau d(\sum_{k=1}^{h} \Gamma_{kj} - 1) \geq z_j - c_j, \quad j = 1, 2, \ldots, n, \text{ an index of nonbasic variables,} \tag{74}$$

and

$$-\tau d \sum_{k=1}^{h} \Gamma_{kj} \geq z_j - c_j, \quad j \in SV \text{ an index of nonbasic variables,} \tag{75}$$

where

$$d = \sum_{t=1}^{s} \omega_t \alpha_t - \sum_{t=1}^{m} \omega_{s+t} \beta_t. \tag{76}$$

For the proof and details see [27].

Remark 10. *Let us point out that in the case of positive perturbations $\alpha_{rj} > 0, \beta_{ij} > 0, r = 1, 2, \ldots, s, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, because of the translation invariance of the additive model (see Ali and Seiford [1]), the efficiency of DMU_0 will be preserved.*

3.6. Let us also consider decrease of outputs of efficient DMUs

$$\widehat{y}_{rj} = y_{rj} - \alpha_r > 0, \ \alpha_r \geq 0, \ r = 1, 2, \ldots, s, \quad j \in E \tag{77}$$

and increase of outputs of inefficient DMUs

$$\widehat{y}_{rj} = y_{rj} + \alpha_r, \ \alpha_r \geq 0, \ r = 1, 2, \ldots, s, \quad j \in N. \tag{78}$$

We will also consider increase of inputs of efficient DMUs

$$\widehat{x}_{ij} = x_{ij} + \beta_i, \ \beta_i \geq 0, \ i = 1, 2, \ldots, m, \quad j \in E, \tag{79}$$

and decrease of inputs of inefficient DMUs

$$\widehat{x}_{ij} = x_{ij} - \beta_i > 0, \ \beta_i \geq 0, \ i = 1, 2, \ldots, m, \quad j \in N. \tag{80}$$

The simultaneous change of outputs (77), (78) and change of inputs (79), (80) result in the following perturbation of the vectors of the matrix of the linear program (48):

$$\begin{aligned} \widehat{P}_j &= P_j + [-\alpha_1 - \alpha_2 \ldots - \alpha_s - \beta_1 - \beta_2 \ldots - \beta_m \ 0\,]^T, \quad j \in E, \\ \widehat{P}_j &= P_j + [\alpha_1 \ \alpha_2 \ \ldots \alpha_s \ \beta_1 \ \beta_2 \ \ldots \beta_m \ 0]^T, \quad j \in N, \\ \widehat{P}_j &= P_j, \quad j \in SV. \end{aligned} \tag{81}$$

It also means the following perturbation of the optimal basis matrix B:

$$\widehat{B} = B + \Delta B, \tag{82}$$

with

$$
\Delta B = \begin{array}{c} \begin{array}{ccccccc} & & & h & & & \\ & & & \downarrow & & & \end{array} \\ \begin{pmatrix} -\alpha_1 & -\alpha_1 & \cdots & -\alpha_1 & 0 & \cdots & 0 \\ -\alpha_2 & -\alpha_2 & \cdots & -\alpha_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -\alpha_s & -\alpha_s & \cdots & -\alpha_s & 0 & \cdots & 0 \\ -\beta_1 & -\beta_1 & \cdots & -\beta_1 & 0 & \cdots & 0 \\ -\beta_2 & -\beta_2 & \cdots & -\beta_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -\beta_m & -\beta_m & \cdots & -\beta_m & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \end{array}, \tag{83}
$$

where h is the number of optimal basic λ_j^* variables, $j \in E$. (Recall that N is the set of inefficient DMUs.) Finally, it means the following change of the right-hand side vector

$$
\widehat{P}_0 = P_0 + [-\alpha_1 - \alpha_2 \ldots - \alpha_s - \beta_1 - \beta_2 \ldots - \beta_m \ 0\ ]^T . \tag{84}
$$

Using the same notation as above for $p \neq 1$, but now with

$$
\gamma_k = \sum_{t=1}^{s} b_{kt}^{-1} \alpha_t + \sum_{t=1}^{m} b_{k,s+t}^{-1} \beta_t, \quad k = 1, 2, \ldots, s+m+1 \tag{85}
$$

and

$$
\tau = \frac{1}{1-p} \tag{86}
$$

we obtain the following result.

Theorem 11. *Sufficient conditions for an efficient DMU_0 to preserve its efficiency after the nonnegative changes of all data as in (77), (78), (79) and (80), are*

$$
-\tau d(\sum_{k=1}^{h} \Gamma_{kj} - 1) \geq z_j - c_j, \quad j \in E \ \textit{an index of nonbasic variables}, \tag{87}
$$

$$
-\tau d(\sum_{k=1}^{h} \Gamma_{kj} + 1) \geq z_j - c_j, \quad j \in N \ \textit{an index of nonbasic variables}, \tag{88}
$$

and

$$
-\tau d \sum_{k=1}^{h} \Gamma_{kj} \geq z_j - c_j, \quad j \in SV \ \textit{an index of nonbasic variables}, \tag{89}
$$

where

$$
d = \sum_{t=1}^{s} \omega_t \alpha_t + \sum_{t=1}^{m} \omega_{s+t} \beta_t. \tag{90}
$$

The proof is similar to the proof of *Theorem 1* in [27] (see *Theorem 10* above).

4. Summary and conclusions

Sensitivity analysis in DEA is studied in this paper for the CCR ratio model and for the additive model. The case of the additive changes of inputs and of outputs of an efficient DMU according to the CCR model preserving its efficiency is studied first. After that the cases of the proportionate changes of inputs or/and outputs with two coefficients of proportionality (one for inputs, the other for outputs) are studied. The case of the proportionate change of inputs and outputs with different coefficients of proportionality for each input and for each output is studied too. Sufficient conditions for an efficient DMU to preserve its efficiency after the corresponding changes in these cases are presented. Sensitivity analysis of the additive model is studied for the cases of additive and proportionate changes of inputs or/and outputs of an efficient DMU. The cases of arbitrary additive perturbations and of nonnegative additive perturbations of data of all DMUs preserving efficiency of an efficient DMU are studied too. Sufficiency conditions for preserving efficiency of an efficient DMU according to the additive model in these cases are described.

The problem of preserving efficiency of an efficient DMU according to the CCR and according to the Banker-Charnes-Cooper (BCC) model (see [2], [9], [3]) under additive (or proportionate changes) of all data is an open question. The case of the change of all data of an arbitrary subset of DMUs, preserving efficiency of an efficient DMU, is an open question too. The problem of preserving efficency of all efficient DMUs and inefficiency of all inefficient DMUs seems to be interesting too.

The question of solving the system of inequalities corresponding to the sufficiency conditions for preserving efficiency of an efficient DMU using PC is open too. It is important for applications of the results in sensitivity analysis in DEA to the real world problems.

References

[1] A. I. Ali, L. M. Seiford, *Translation invariance in data envelopment analysis*, Operations Research Letters **9**(1990), 403–405.

[2] R. D. Banker, A. Charnes, W. W. Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Management Science **30**(1984), 1078–1092.

[3] V. Boljunčić, *Data Envelopment Analysis*, M. Sc. Thesis, Department of Mathematics, University of Zagreb, 1994. (In Croatian)

[4] V. Boljunčić, *Sensitivity analysis in the additive model of data envelopment analysis*, International Journal of Systems Science **29**(1998), 219–222.

[5] A. Charnes, W. W. Cooper, *Structural sensitivity analysis in linear programming and an exact product form left inverse*, Naval Research Logistics Quarterly **15**(1968), 517–522.

[6] A. Charnes, W. W. Cooper, *Preface to topics in data envelopment analysis*, Annals of Operations Research **2**(1985), 59–94.

[7] A. Charnes, W. W. Cooper, B. Golany, L. M. Seiford, J. Stutz, *Foundations of data envelopment analysis for Pareto-Koopmans efficient empirical production functions*, Journal of Econometrics, **30**(1985), 91–107.

[8] A. Charnes, W. W. Cooper, A. Y. Lewin, R. C. Morey, J. Rousseau, *Sensitivity and stability analysis in DEA*, Annals of Operations Research **2**(1985), 139–156.

[9] A. Charnes, W. W. Cooper, A. Y. Lewin, L. M. Seiford (Eds.), *Data Envelopment Analysis: Theory, Methodology and Applications*, Kluwer Acad. Pub., Boston, 1994.

[10] A. Charnes, W. W. Cooper, M. Sears, S. Zlobec, *Efficiency evaluations in perturbed data envelopment analysis*, in: Parametric Optimization and Related Topics II, (J. Guddat, H. Th. Jongen, B. Kummer and F. Nožička, Eds.), Akademie-Verlag, Berlin, 1990, 38–49.

[11] A. Charnes, S. Haag, P. Jaska, J. Semple, *Sensitivity classifications in the additive model of data envelopment analysis*, International Journal of System Sciences **23**(1992), 789–798.

[12] A. Charnes, L. Neralić, *Sensitivity analysis in data envelopment analysis 1*, Glasnik Matematički Ser. III **24(44)**(1989), 211–226.

[13] A. Charnes, L. Neralić, *Sensitivity analysis in data envelopment analysis 2*, Glasnik Matematički Ser.III **24(44)**(1989), 449–463.

[14] A. Charnes, L. Neralić, *Sensitivity analysis of the additive model in data envelopment analysis*, European Journal of Operational Research **48**(1990), 332–341.

[15] A. Charnes, L. Neralić, *Sensitivity analysis of the simultaneous proportionate change of inputs and outputs in data envelopment analysis*, Paper presented at the 11th European Congress on Operational Research EURO XI, Aachen, Germany, July 16-19, 1991.

[16] A. Charnes, L. Neralić, *Sensitivity analysis of the simultaneous proportionate change of inputs and outputs in data envelopment analysis*, Research Report CCS 667, Center for Cybernetic Studies, The University of Texas at Austin, Austin, Texas, August 1991.

[17] A. Charnes, L. Neralić, *Sensitivity analysis in data envelopment analysis 3*, Glasnik Matematički Ser. III **27 (47)**(1992), 191–201.

[18] A. Charnes, L. Neralić, *Sensitivity analysis of the proportionate change of inputs (or outputs) in data envelopment analysis*, Glasnik Matematički Ser. III **27 (47)**(1992) 393–405.

[19] A. Charnes, L. Neralić, *Sensitivity analysis in data envelopment analysis for the case of discretionary inputs and outputs*, Paper presented at EURO XII/TIMS XXXI Joint International Conference Operational Research/Management Science, June 29 - July 1, 1992, Helsinki, Finland.

[20] A. CHARNES, L. NERALIĆ, *Sensitivity analysis in data envelopment analysis for the case of non-discretionary inputs and outputs*, Glasnik Matematički Ser. III **30 (50)**(1995), 359–371.

[21] A. CHARNES, S. ZLOBEC, *Stability of efficiency evaluations in data envelopment analysis*, Zeitschrift für Operations Research **33** (1989), 167–179.

[22] G. H. GOLUB, C. F. VAN LOAN, *Matrix Computation*, John Hopkins University Press, Baltimore, Maryland, 1983.

[23] L. NERALIĆ, *Sensitivity analysis of the proportionate change of outputs or/and inputs of the additive model in data envelopment analysis*, in: Proceedings of the 2nd Conference on Operational Research KOI '92, (V. Bahovec, Lj. Martić and L. Neralić, Eds.), Rovinj, 1992, 71–80. (In Croatian)

[24] L. NERALIĆ, *Sensitivity analysis in data envelopment analysis: A review*, in: Proceedings of the 2nd Slovenian Symposium on Operations Research SOR'94, (V. Rupnik and M. Bogataj, Eds.), Portorož, 1994, 29–42.

[25] L. NERALIĆ, *Sensitivity analysis of the proportionate change of outputs or/and inputs of the additive model in data envelopment analysis*, in: Proceedings of the International Conference "Restructuring Transitional Economies", (S. Sharma and P. Sikavica, Eds.), Part I, University of Zagreb, Faculty of Economics, Zagreb, 1995, 221–233.

[26] L. NERALIĆ, *Sensitivity analysis of the of the additive model in data envelopment analysis II*, in: Proceedings of the 6th International Conference on Operational Research KOI '96, (T. Hunjak, Lj. Martić and L. Neralić, Eds.), Rovinj, 1996, 17–22.

[27] L. NERALIĆ, *Sensitivity in data envelopment analysis for arbitrary perturbations of data*, Glasnik Matematički Ser. III **32 (52)**(1997), 315–335.

[28] L. NERALIĆ, T. R. SEXTON, *Sensitivity analysis in data envelopment analysis for the case of different coefficients of proportionality*, in: Parametric Optimization and Related Topics IV, (J. Guddat, H. Th. Jongen, F. Nožička and F. Twilt, Eds.), Proceedings of the International Conference on Parametric Optimization and Related Topics IV, Enschede (NL), June 6-9, 1995, Peter Lang Publ., Frankfurt am Main, 1996, 261–280.

[29] J. J. ROUSSEAU, J. H. SEMPLE, *Radii of classification preservation in data envelopment analysis: A case study of 'Program Follow-Through'*, Journal of the Operational Research Society **46**(1995), 943–957.

[30] R. G. THOMPSON, P. S. DHARMAPALA, R. M. THRALL, *DEA sensitivity analysis of efficiency measures with an application to Kansas farming and Illinois coal mining*, in: Data Envelopment Analysis: Theory, Methodology and Applications, (A. Charnes, W. W. Cooper, A. Y. Lewin and L. M. Seiford, Eds.), Kluwer Acad. Pub., Boston, 1995, 393–422.

[31] J. ZHU, *Robustness of the efficient DMUs in data envelopment analysis*, European Journal of Operational Research **90**(1996), 451–460.