Parameter estimation of diffusion models from discrete observations *

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Abstract. *A short review of diffusion parameter estimations methods from discrete observations is presented. The applicability of a new estimation method on inferences about a diffusion growth model is discussed.*

Key words: *diffusion process, stochastic differential equation, discrete observations, paremater estimation, diffusion growth model*

Sažetak. **Procjena parametara difuzijskih modela iz diskretnih opservacija.** *Prikazan je kratak pregled metoda procjene parametara difuzija iz diskretnih opservacija. Ujedno je razmatrana upotreba nove metode procjene na izvođenje zaključaka o jednom difuzijskom modelu rasta.*

Ključne riječi: *difuzijski proces, stohastička diferencijalna jednadžba, diskretne opservacije, procjena parametara, difuzijski model rasta*

1. Introduction

Diffusion processes have been used in modeling various phenomena, for example, noisy electrical signal, tumor growth (see e.g. [1, 7]) and interest rates (see e.g. [3]). Although paths of diffusion processes are continuous, in almost all cases (e.g. tumor size and interest rates) they could be only observed at discrete moments of time. This explains a large number of papers dealing with the problems of parameter estimation of diffusion models from discrete observations in the last twelve years.

This paper starts with the definition of the problem in §2. It continues with a review of the known method of estimation in §3. Finally, in §4 a new estimation method described in [8] is applied to the problems of parameter estimation of the diffusion growth models (see [8] and also the example in [9]) based on discretely observed paths of the growth process.

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2. The problem

Let $X = (X_t, t \geq 0)$ be a one-dimensional diffusion which satisfies stochastic differential equation (SDE) (see [14]) of the form

$$dX_t = \mu(X_t, \theta)\, dt + \nu(X_t, \theta)\, dW_t, \quad X_0 = x_0. \tag{1}$$

Here, $W = (W_t, t \geq 0)$ is a one-dimensional standard Brownian motion, μ and ν are $\mathbb{R}$-valued functions, $\mu(\cdot, \theta)$ is the drift function and $\nu(\cdot, \theta)$ is the diffusion coefficient function, θ is a parameter (or a vector of parameters) of the model, and x_0 is a number, a deterministic initial value of the process X.

The problem is to estimate the unknown parameter θ of X given a discrete observation $(X_{t_i}, 0 \leq i \leq n)$ of the trajectory $(X_t, t \geq 0)$ $(0 =: t_0 < t_1 < \cdots < t_n < +\infty,\ n \in \mathbb{N})$. We assume that θ belongs to the parameter space Θ which is an open space in the Euclidean space $\mathbb{R}^d$. Mostly, $\theta = (\vartheta, \sigma)$ when $\mu(\cdot, \theta) \equiv \mu(\cdot, \vartheta)$ and $\nu(\cdot, \theta) \equiv \nu(\cdot, \sigma)$. In that case, ϑ is called a drift parameter and σ is called a diffusion parameter.

Let $\Delta_n = \max_{1 \leq i \leq n}(t_i - t_{i-1})$. If $\Delta_n = t_i - t_{i-1} = t_n/n$ for all i, $1 \leq i \leq n$, and $n \in \mathbb{N}$, we simply use Δ instead of Δ_n.

3. The methods of estimation

Probably the most natural approach to estimation is the method of maximum likelihood (ML). For ergodic diffusions satisfying some additional conditions Dacunha-Castelle and Florens-Zmirnou [4] proved the consistency and asymptotic efficiency of the maximum likelihood estimator (MLE) of $\theta = (\vartheta, \sigma)$ when $n\Delta \to \infty$. The results are quite different for ϑ and σ. The necessary condition for the existence of a consistent estimator of the drift parameter ϑ is that $n\Delta_n \to \infty$. In ergodic case, it is asymptotic efficient with the speed $\sqrt{n\Delta_n}$. For σ, the condition $n\Delta_n = O(1)$ is needed.

Although MLE has the usual good properties, the transition density of the process X and so the likelihood function (LF) of the discretized process are explicitly known only in few special cases. Hence, other methods of estimations have had to be considered. It turns out that there exist roughly three classes of estimation methods.

One class consists of methods based on maximization of a discretized continuous-time log-likelihood function (LLF), obtained by replacing Lebesgue and Itô integrals with Riemman-Itô sums. The Euler type of approximation of integrals is used and discussed in [12, 6, 16]. Le Breton in [12] considered multidimensional diffusions with known diffusion coefficient function ν and drift function μ linear in unknown drift parameter ϑ. He proved that the sequence of random variables $(\frac{1}{\sqrt{\Delta_n}}\|\hat{\vartheta}_n - \hat{\vartheta}\|, n \in \mathbb{N})$ is bounded in probability when $\Delta_n \to 0$ with $t_n = T$ being constant. Here, $\hat{\vartheta}_n$ is a point of maximum of discretized LLF and $\hat{\vartheta}$ is a MLE of ϑ based on continuous observations of paths along time-interval $[0, T]$. Florens-Zmirnou [6] considered one-dimensional positive recurrent diffusions with diffusion coefficient functions of the form $\nu(x, \sigma) = \sigma$ and drift function of the general form $\mu(x, \vartheta)$. Under some additional assumptions on μ, he proved that both estimators, the estimators of drift and diffusion parameters, are consistent and asymptotic normal if $n\Delta_n \to \infty$ and $n\Delta_n^3 \to 0$. In case of multidimensional ergodic diffusion with $\nu(x,\sigma) = \sigma b(x)$, Yoshida [16] has got the same result. Moreover, Florens-Zmirnou [6] showed that the drift estimator has an asymptotic bias in case of $\Delta_n = \Delta$ being constant. More complex approaches in approximations of the integrals in LLF and score functions based on continuous observations are proposed in e.g. [11, 13]. In these papers the proposed estimators have been investigated by simulations.

Another class of estimation methods consists of methods based on martingale estimating functions that are usually obtained by compensating approximate continuous-time score function (see [2]). Bibby and Soerensen [2] showed that under some general conditions on SDE and for $\Delta_n = \Delta$ being constant, there exists a consistent and asymptotically efficient estimator of θ which solves martingale estimating equation.

Finally, the last class consists of methods based on a Gaussian approximation of the transition density of X. Namely, let $m_i(X_{t_i}, \theta)$ be an approximation of the conditional expectation $\mathbb{E}[X_{t_{i+1}} | X_{t_i}]$ and let $v_i(X_{t_i}, \theta)$ be an approximation of $\mathbb{E}[(X_{t_{i+1}} - m_i(X_{t_i}, \theta))^2 | X_{t_i}]$ $(0 \le i < n)$. Then Gaussian approximation of the transition density leads to the contrast function

$$-\frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{(X_{t_{i+1}} - m_i(X_{t_i},\theta))^2}{v_i(X_{t_i},\theta)} + \log v_i(X_{t_i},\theta)\right). \tag{2}$$

Kessler [10] proposed a method of this kind that gives an estimator of $\theta = (\vartheta, \sigma)$ which is consistent when $\Delta_n \to 0$ and $n\Delta_n \to \infty$. If in addition $n\Delta_n^p \to 0$ for an integer $p \ge 2$ then the estimators are asymptotically efficient. These hold for an ergodic diffusion X.

Two or all of these three approaches could lead to the same estimators (e.g. see [12, 6, 16]).

In [8] a new method of estimation has been proposed. It belongs to the last above mentioned class of methods and it is applied to the SDEs with $\nu(x,\sigma) = \sigma b(x)$ and $\mu(x,\vartheta)$ being an analytic function in ϑ. An estimator $\hat{\theta}_n = (\hat{\vartheta}_n, \hat{\sigma}_n)$ is obtained by maximization of the contrast function (2) with

$$m_i(X_{t_i},\theta) = X_{t_i} + \mu(X_{t_i}, \tau_\vartheta(h_i,\theta)h_i \text{ and } v_i(X_{t_i},\theta) = \tau_\sigma^2(h_i,\theta)b^2(X_{t_i})h_i \tag{3}$$

where $h_i = t_i - t_{i-1}$, $1 \le i \le n$. Here, $\tau = (\tau_\vartheta, \tau_\sigma)$ is a function such that for any $h > 0$, $\tau(h,\cdot)$ is a transformation of the parameter space $\bar{\Theta} := \Theta \times \mathbb{R}_+$. Θ is a connected open set in Euclidean space $\mathbb{R}^d$. It is assumed that $\vartheta \in \Theta$ and $\sigma \in \mathbb{R}_+$. This method is a generalization of some of above mentioned methods. Primarily it is designed for the cases when all observations could be taken only up to some maximal observation time T. Under the assumptions that MLE $\hat{\vartheta}$ based on continuous observations along $[0,T]$ is the unique point of maximization of LLF and the transformation τ has some regular properties as well as some additional assumptions on SDE are satisfied (see [8]), it turns out that $\hat{\sigma}_n$ is consistent and asymptotically normal and $\hat{\vartheta}_n$ converges to $\hat{\vartheta}$ in such way that the sequence $(\frac{1}{\sqrt{\Delta_n}}\|\hat{\vartheta}_n - \hat{\vartheta}\|, n \in \mathbb{N})$ is bounded in probability when $\Delta_n \to 0$ (again see [8]).

4. Estimation of a diffusion growth process

Let us consider the same diffusion growth model that has been discussed in [9] §3, i.e. we assume that X satisfies SDE

$$dX_t = (\alpha - \beta h(\gamma, X_t))X_t\,dt + \sigma X_t\,dW_t, \quad X_0 = x_0 > 0. \tag{4}$$

Here, $h(\gamma, x) = (x^\gamma - 1)/\gamma$ if $\gamma \neq 0$ and $h(\gamma, x) = \log x$ if $\gamma = 0$ ($\gamma \in \mathbb{R}$, $x > 0$). Drift parameter is $\vartheta = (\alpha, \beta, \gamma)$ and diffusion parameter is σ. The parameter space is

$$\bar{\Theta} = \{(\alpha, \beta, \gamma, \sigma) \in \mathbb{R}^3 \times \langle 0, +\infty\rangle \ : \ \beta > 0,\ \gamma(\alpha - \frac{\sigma^2}{2}) + \beta > 0\}. \tag{5}$$

Process X is ergodic (see [8]) and let p_0 be its stationary density function (again see [8]). Let

$$\ell_\infty(\vartheta) = \frac{1}{\sigma_0^2}\int_0^{+\infty}(\alpha - \beta h(\gamma, x))((\alpha_0 - \beta_0 h(\gamma_0, x)) - \frac{1}{2}(\alpha - \beta h(\gamma, x)))p_0(x)\,dx,$$

where $(\vartheta_0, \sigma_0) = (\alpha_0, \beta_0, \gamma_0, \sigma_0)$ is the true value of the unknown parameter which should be estimated. Let Θ_0 be a connected relatively compact set such that $\vartheta_0 \in \Theta_0$ and

$$\Theta_0 \subset \mathcal{C}\ell(\Theta_0) \subset \{\vartheta \in \Theta : (\gamma_0 \neq 0 \Rightarrow |\gamma - \gamma_0| < |\gamma_0|)\ \&\ D^2\ell_\infty(\vartheta) < 0\},$$

and let I be a bounded open interval in $\mathbb{R}_+$ such that $\mathcal{C}\ell(I) \subset \mathbb{R}_+$ and $\bar{\Theta}_0 := \Theta_0 \times I \subset \bar{\Theta}$.

The following theorems are proved in [8].

Let $(\hat{\vartheta}_T, T \geq 0)$ be a progressively measurable process from *Theorem 1* in [9] and let ℓ_T be LLF given by (7) in [9].

Theorem 1. *Almost surely there exists $T_0 > 0$ such that for all $T \geq T_0$, $\hat{\vartheta}_T \in \Theta_0$ and $\hat{\vartheta}_T$ is the unique point of the global maximum of the function $\vartheta \mapsto \ell_T(\vartheta)$ on $\mathcal{C}\ell(\Theta_0)$. Moreover, Hessian of ℓ_T at $\vartheta = \hat{\vartheta}_T$ is strictly negatively definite.*

From this theorem we can conclude that the method of estimation given by (3) could be applied. Namely, on an event of positive probability the growth model (4) satisfies conditions which are necessary for the estimation method (3) to be applied (see [8]) on this model and parameter space $\bar{\Theta}_0$.

Let $\hat{\theta}_{Tn} = (\hat{\vartheta}_{Tn}, \hat{\sigma}_{Tn})$ be an estimator obtained by the method (3) given a discrete observation of X along time interval $[0, T]$. Let $F_{Tn}(x)$, $x \in \mathbb{R}^d$, denote the distribution function of the random vector $\sqrt{T}(\hat{\vartheta}_{Tn} - \vartheta_0)$ and let $\Phi_{I_0}(x)$, $x \in \mathbb{R}^d$, denote the distribution function of normal distribution $\mathcal{N}(0, I_0^{-1})$ where I_0 is the matrix from *Theorem 1* in [9]. We will assume that for fixed T, $\Delta_n = \Delta_{Tn} \to 0$ when $n \to +\infty$.

Theorem 2.

$$\lim_{T\to+\infty}\overline{\lim_n}|F_{Tn}(x) - \Phi_{I_0}(x)| = 0, \ \ x \in \mathbb{R}^d.$$

This theorem and *Theorem 2* in [9] imply that a variant of *Theorem 2* [9] based on discrete observations could be proved (see [8]).

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