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Optimal design and hyperbolic problems*†

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Abstract. *Quite often practical problems of optimal design have no solution. This situation can be alleviated by relaxation, where one needs generalised materials which can mathematically be defined by using the theory of homogenisation. First mathematical results in this direction for general (nonperiodic) materials were obtained by Murat and Tartar.*

We present some results in optimal design where the equation of state is hyperbolic. The control function is related to the response of vibrating material under the given external force. As the problem under consideration has no solution, we consider its relaxation to H-closure of the original set of controls.

Key words: *optimal design, homogenisation, relaxation, H-convergence, stratified materials*

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1. Introduction

We consider the wave equation

$$\rho u'' - \mathsf{div}\,(\mathbf{A}\nabla u) = f$$

on $\Omega_T = \langle 0, T\rangle \times \Omega$, $\Omega \subseteq \mathbb{R}^d$ open and bounded, as the state equation for a problem of optimal shape design. The forcing term f, as well as the initial and boundary values, is assumed to be given in advance, which, together with the control $(\rho, \mathbf{A})$, uniquely determines the state function u.

We take the cost functional of the form

$$I(\rho, \mathbf{A}) = \int_{\Omega_T} F(t, \mathbf{x}, \rho(\mathbf{x}), \mathbf{A}(\mathbf{x}), u(t, \mathbf{x}))\, d\mathbf{x}\, dt, \tag{1}$$

which is the form usually connected with the name of Lagrange.

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Now the problem can be stated as:

Minimise $I(\rho, \mathbf{A})$ over the set $\mathcal{A}$ of admissible controls.

Let us see more closely how the control influences the cost functional.

Assume that we want to fill the set Ω with m given materials. Each material is characterised by two constant quantities ρ_i and $\mathbf{A}_i$ $(i = 1, \ldots, m)$ (mechanical properties of a particular material). Furthermore, for simplicity, suppose that all the materials are isotropic, i.e. $\mathbf{A}_i = \gamma_i \mathbf{I}$, where $\gamma_i \in [\alpha, \beta]$ and $\rho_i \in [\rho_-, \rho_+]$ for some positive ρ_- and α. If we denote by χ_i the characteristic function of the i-th material, the corresponding control function is uniquely given by

$$\begin{aligned} \rho &= \textstyle\sum_{i=1}^m \chi_i \rho_i \;, \\ \mathbf{A} &= \textstyle\sum_{i=1}^m \chi_i \gamma_i \mathbf{I} \;, \end{aligned} \tag{2}$$

while the set of admissible controls $\mathcal{A}$ is the set of all pairs $(\rho, \mathbf{A})$ obtained in this way.

Due to the special type of admissible controls we use, the above cost functional can be written in a slightly different form. For $(t, \mathbf{x}) \in \Omega_T$ and $\lambda \in \mathbb{R}$ we define

$$g_i(t, \mathbf{x}, \lambda) = F(t, \mathbf{x}, \rho_i, \gamma_i \mathbf{I}, \lambda) \;,$$

so that the cost functional reads

$$I(\rho, \mathbf{A}) = \int_{\Omega_T} \sum_{i=1}^m \chi_i(\mathbf{x}) g_i(t, \mathbf{x}, u(t, \mathbf{x}))\, d\mathbf{x}\, dt.$$

Both functions ρ and $\mathbf{A}$ are known if the characteristic functions are given. I is, in fact, a function of $\boldsymbol{\chi}$ alone, where we write $\boldsymbol{\chi}$ for the m-tuple $(\chi_1, \ldots, \chi_m)$.

Unfortunately, by writing the problem with $\boldsymbol{\chi}$ as the only argument for I, we would not be able to define a topology on $\mathcal{A}$ such that u depends continuously on $\boldsymbol{\chi}$ (except for the case of layered materials, see below), which means that I could not be continuous. This continuity is crucial in order to discuss the existence of solutions for the problem above, which is our goal.

It is well known that the problems of the above type in general (without very restrictive additional assumptions) have no solution. Nevertheless, as it will be shown below, it is possible to extend the functional I to a larger set $\mathcal{T}$, and to define a topology on $\mathcal{T}$ such that the extension is continuous. Due to the fact that $\mathcal{T}$ is a compact topological space, the relaxation amounts to closing the set $\mathcal{A}$ in $\mathcal{T}$, which is not a trivial task, as the topology on $\mathcal{T}$ is quite complicated.

2. Homogenisation of the diffusion equation

In this section we shall present some facts in the theory of homogenisation of the stationary diffusion equation. They were systematically presented for the first time in Tartar's *Cours Peccot* in 1977.

The homogeneous boundary value problem for diffusion equation

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}^1_0(\Omega) \end{cases} \tag{3}$$

has a unique solution (by Lax–Milgram lemma) if Ω is an open and bounded subset of $\mathbb{R}^d$, $f \in \mathrm{H}^{-1}(\Omega)$ and $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d))$ satisfying

$$\begin{aligned} &\mathbf{A}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \alpha |\boldsymbol{\xi}|^2 \,, \\ &\mathbf{A}(\mathbf{x})\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \tfrac{1}{\beta} |\mathbf{A}(\mathbf{x})\boldsymbol{\xi}|^2 \end{aligned} \tag{4}$$

for every $\boldsymbol{\xi} \in \mathbb{R}^d$ and almost every $\mathbf{x} \in \Omega$. We shall denote the set of all these matrix functions by $\mathfrak{M}(\alpha, \beta; \Omega)$. Moreover, if $\mathbf{A}(\mathbf{x})$ is a symmetric matrix for a.e. $\mathbf{x} \in \Omega$, the conditions (4) can be written in a simpler form as

$$\alpha \mathbf{I} \leq \mathbf{A}(\mathbf{x}) \leq \beta \mathbf{I} \,.$$

The main problem studied in the theory of homogenisation is the following: Given a sequence of coefficients $(\mathbf{A}_n)$ in $\mathfrak{M}(\alpha, \beta; \Omega)$ and the corresponding sequence of solutions (u_n) of (3) (for some fixed f), converging to some function u weakly in $\mathrm{H}^1(\Omega)$, what can we say about the equation this u satisfies? Is the equation of the same type? If yes, what is the corresponding $\mathbf{A}$?

Definition 1. *We say that a sequence $(\mathbf{A}_n)$ in $\mathfrak{M}(\alpha, \beta; \Omega)$ H-converges to $\mathbf{A}_\infty \in \mathfrak{M}(\alpha', \beta'; \Omega)$ if for every $f \in \mathrm{H}^{-1}(\Omega)$ the corresponding sequence of solutions (u_n) of (3) satisfies the following weak convergences*

$$\begin{aligned} u_n &\rightharpoonup u_\infty \quad \text{in} \quad \mathrm{H}^1_0(\Omega) \,, \\ \mathbf{A}_n \nabla u_n &\rightharpoonup \mathbf{A}_\infty u_\infty \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathbb{R}^d). \end{aligned}$$

The last convergence in particular implies that u_∞ is a solution of (3) with $\mathbf{A}_\infty$ instead of $\mathbf{A}$. It can be shown that this convergence comes from a weak topology (we shall call it the H-topology) on $\mathfrak{M}(\alpha, \beta; \Omega)$, which is metrisable. We shall just state some main results regarding H-convergence (for the proofs see [4], and [7] for further improvements).

Theorem 1. *The set $\mathfrak{M}(\alpha, \beta; \Omega)$ is compact in H-topology.*

One-dimensional case. For $\Omega \subseteq \mathbb{R}$, the sequence $(A_n) \in \mathfrak{M}(\alpha, \beta; \Omega)$ (in this case that will be a sequence of functions in $\mathrm{L}^\infty(\Omega; [\alpha, \beta])$) H-converges to A_∞ if and only if

$$\frac{1}{A_n} \overset{*}{\rightharpoonup} \frac{1}{A_\infty} \quad \text{in} \quad \mathrm{L}^\infty(\Omega).$$

This can be generalised to the case of layered materials in higher dimensions. Let $(\mathbf{A}_n)$ be a given sequence in $\mathfrak{M}(\alpha, \beta; \Omega)$ depending only on x_1 (we talk about layers perpendicular to the vector $\vec{e}_1$). Then we have: $\mathbf{A}_n \xrightarrow{H} \mathbf{A}_\infty$ if and only if $(i, j \geq 2)$

$$\frac{1}{(A_n)_{11}} \xrightarrow{*} \frac{1}{(A_\infty)_{11}} \quad \text{in} \quad \mathrm{L}^\infty(\Omega)$$

$$\frac{(A_n)_{1j}}{(A_n)_{11}} \xrightarrow{*} \frac{(A_\infty)_{1j}}{(A_\infty)_{11}} \quad \text{in} \quad \mathrm{L}^\infty(\Omega)$$

$$\frac{(A_n)_{i1}}{(A_n)_{11}} \xrightarrow{*} \frac{(A_\infty)_{i1}}{(A_\infty)_{11}} \quad \text{in} \quad \mathrm{L}^\infty(\Omega)$$

$$(A_n)_{ij} - \frac{(A_n)_{i1}(A_n)_{1j}}{(A_n)_{11}} \xrightarrow{*} (A_\infty)_{ij} - \frac{(A_\infty)_{i1}(A_\infty)_{1j}}{(A_\infty)_{11}} \quad \text{in} \quad \mathrm{L}^\infty(\Omega)\,.$$

These formulas allow us to compute the H-limit which is in general a difficult task. (In fact, there is one more situation where we know how to compute the H-limit explicitly: the periodic case.) The following theorem gives some pointwise estimates of the H-limit.

Theorem 2. *Let $(\mathbf{A}_n)$ be a sequence of symmetric matrices in $\mathfrak{M}(\alpha, \beta; \Omega)$ H-converging to $\mathbf{A}_\infty$. Then $\mathbf{A}_\infty$ is symmetric as well. Moreover, if*

$$\mathbf{A}_n \xrightarrow{*} \mathbf{A}_+ \quad \text{in} \quad \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})\,,$$
$$(\mathbf{A}_n)^{-1} \xrightarrow{*} (\mathbf{A}_-)^{-1} \quad \text{in} \quad \mathrm{L}^\infty(\Omega; \mathbb{R}^{d\times d})\,,$$

then

$$\mathbf{A}_- \leq \mathbf{A}_\infty \leq \mathbf{A}_+ \quad \text{(a.e. on } \Omega). \tag{5}$$

Mixtures of two isotropic materials. Let us consider the case where $\mathbf{A}_n = a_n\mathbf{I}$, $a_n = \chi_n\alpha + (1-\chi_n)\beta$ for $n \in \mathbb{N}$, while (χ_n) is a weakly $*$ convergent sequence of functions in $\mathrm{L}^\infty(\Omega; \{0,1\})$. According to the following *Theorem* (for proof see [5]), its limit is a function $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$.

Theorem 3. *Let K be a bounded subset of $\mathbb{R}^d$, and (v_n) a sequence in $\mathrm{L}^\infty(\Omega; \mathbb{R}^d)$ such that, for every $n \in \mathbb{N}$, $v_n(\mathbf{x}) \in K(\text{a.e. } \mathbf{x} \in \Omega)$. If v_∞ is a weak $*$ limit of the sequence (v_n), then $v_\infty(\mathbf{x}) \in \text{cl conv}K(\text{a.e. } \mathbf{x} \in \Omega)$. Conversely, for every function $v \in \mathrm{L}^\infty(\Omega, \mathbb{R}^d)$, such that $v(\mathbf{x}) \in \text{cl conv}K(\text{a.e. } \mathbf{x} \in \Omega)$, there exists a sequence (v_n) converging to v weakly $*$ in $\mathrm{L}^\infty(\Omega; \mathbb{R}^d)$, satisfying $v_n(\mathbf{x}) \in K(\text{a.e. } \mathbf{x} \in \Omega)$, for every $n \in \mathbb{N}$.*

We define two functions (for $\vartheta \in [0,1]$):

$$a_+(\vartheta) = \vartheta\alpha + (1-\vartheta)\beta,$$
$$\frac{1}{a_-(\vartheta)} = \frac{\vartheta}{\alpha} + \frac{1-\vartheta}{\beta}\,.$$

Now we have the convergence

$$a_n \xrightarrow{*} a_+ \circ \theta \quad \text{in} \quad \mathrm{L}^\infty(\Omega)\,, \tag{6}$$

as well as (this can be deduced from the main theorem for Young measures, see [5])

$$\frac{1}{a_n} \xrightarrow{*} \frac{1}{a_- \circ \theta} \quad \text{in} \quad \mathrm{L}^\infty(\Omega).$$

If we assume the convergence $\mathbf{A}_n \xrightarrow{H} \mathbf{A}_\infty$, by *Theorem 2* we have that

$$(a_- \circ \theta)\,\mathbf{I} \leq \mathbf{A}_\infty \leq (a_+ \circ \theta)\,\mathbf{I} \quad \text{a.e. on } \Omega,$$

or, in other words, that every eigenvalue of $\mathbf{A}_\infty(\mathbf{x})$ lies between $a_-(\theta(\mathbf{x}))$ and $a_+(\theta(\mathbf{x}))$ for a.e. $\mathbf{x} \in \Omega$. The following *Theorem* (see [3]) gives more precise estimates in this special case. According to the second part of the *Theorem*, these bounds are the best possible.

Theorem 4. *Let $(\mathbf{A}_n)$ be a sequence as above, H-converging to $\mathbf{A}_\infty$ with the corresponding sequence (a_n) satisfying (6). If we denote the eigenvalues of $\mathbf{A}_\infty(\mathbf{x})$ by $\mu_1(\mathbf{x}), \dots, \mu_d(\mathbf{x})$, then we have*

$$\begin{cases} a_- \circ \theta \le \mu_j \le a_+ \circ \theta \,, \quad j = 1, \dots, d, \\ \sum_{j=1}^d \frac{1}{\mu_j - \alpha} \le \frac{1}{(a_-\circ\theta) - \alpha} + \frac{d-1}{(a_+\circ\theta)-\alpha} \,, \\ \sum_{j=1}^d \frac{1}{\beta - \mu_j} \le \frac{1}{\beta - (a_-\circ\theta)} + \frac{d-1}{\beta - (a_+\circ\theta)} \,. \end{cases} \tag{7}$$

Conversely, if $\mathbf{A}_*$ is any matrix whose eigenvalues satisfy (7) for some function $\theta \in \mathrm{L}^\infty(\Omega; [0,1])$, then there exists a sequence of functions (a_n) which take only the values α and β, satisfying (6), as well as the convergence $a_n\mathbf{I} \xrightarrow{H} \mathbf{A}_*$.

Remark 1. *Characterisation of the H-limit in the case of mixtures of two anisotropic materials is known too (see [2]), but there are no results about the case of a mixture of more than two materials (either isotropic or anisotropic).*

3. Homogenisation of the wave equation

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, and $T > 0$. By V we shall denote the space $\mathrm{H}^1_0(\Omega)$, while $H = \mathrm{L}^2(\Omega)$. For given symmetric $\mathbf{A} \in \mathfrak{M}(\alpha, \beta; \Omega)$ and $\rho \in \mathrm{L}^\infty(\Omega; [\rho_-, \rho_+])$, as well as $v \in V$, $w \in H$ and $f \in \mathrm{L}^2(\Omega_T)$, we consider the initial-boundary problem

$$\begin{cases} \rho u'' - \mathsf{div}\,(\mathbf{A}\nabla u) &= f \\ u(0,\cdot) &= v \\ \rho u'(0,\cdot) &= w \,, \end{cases} \tag{8}$$

with boundary conditions prescribed by the requirements $u \in \mathrm{L}^2([0,T]; V)$ and $u' \in \mathrm{L}^2([0,T]; V)$. This problem has a unique solution satisfying $u'' \in \mathrm{L}^2([0,T]; V')$ as well. From the imbedding theorem we easily get the following regularity:

$$u \in \mathrm{C}([0,T]; V) \quad \text{and} \quad u' \in \mathrm{C}([0,T]; H) \,,$$

together with the estimate

$$(\forall\, t \in [0,T]) \quad \|u(t)\|_V + \|u'(t)\|_H \le C \left(\|v\|_V + \|w\|_H + \|f\|_{\mathrm{L}^2(\Omega_T)} \right) \,,$$

where the constant C depends only on the numbers α, β, ρ_- and ρ_+ (for details v. [1, XVIII §5]).

Let us now consider a sequence of such problems (8), and show that the limit of their solutions satisfies an analogous equation, following the ideas of Tartar [6].

Theorem 5. *Assume that* (ρ_n) *and* $(\mathbf{A}_n)$ *are sequences in* $\mathrm{L}^\infty(\Omega;[\rho_-,\rho_+])$ *and* $\mathfrak{M}(\alpha,\beta;\Omega)$*, respectively, such that*

$$\rho_n \xrightarrow{\mathrm{L}^\infty(\Omega)*} \rho_\infty \qquad \text{and} \qquad \mathbf{A}_n \xrightarrow{H} \mathbf{A}_\infty \ .$$

Suppose that each $\mathbf{A}_n$ *is a symmetric matrix a.e. on* Ω *(which then implies the symmetry of the* H*-limit* $\mathbf{A}_\infty$ *as well). Let* u_n *be a solution of the initial boundary value problem*

$$\left.\begin{aligned} \rho_n u_n'' - \mathsf{div}\,(\mathbf{A}_n \nabla u_n) &= f \\ u_n(0,\cdot) &= v_n \\ \rho_n u_n'(0,\cdot) &= w_n \ , \end{aligned}\right\} \tag{9}$$

with the boundary conditions given by $u_n \in \mathrm{L}^2([0,T];V)$*,* $u_n' \in \mathrm{L}^2([0,T];V)$*, where we assume that* $v_n \rightharpoonup v_\infty$ *in* V*, and* $w_n \rightharpoonup w_\infty$ *in* H*; the forcing term* f *we take from the space* $\mathrm{L}^2(\Omega_T)$*. Then we have*

$$u_n \rightharpoonup u_\infty \quad \text{in } \mathrm{H}^1(\Omega_T),$$

where u_∞ *is the solution of problem (9) for* $n=\infty$*.*

Proof. Let us take an arbitrary test function $\phi \in C_c^\infty(\langle 0,T\rangle)$ and for $n \in \mathbb{N}\cup\{\infty\}$ define

$$U_n(\mathbf{x}) := \int_0^T u_n(t,\mathbf{x})\phi(t)\,dt \ .$$

By testing the convergence of (u_n) on the functions of the form $\phi \boxtimes \psi$ (tensor product), it can easily be seen that

$$U_n \xrightarrow{V} U_\infty \ .$$

Thus, the question is reduced to the one regarding the equation for U_∞.

Multiplying the equation (9_1) by ϕ and integrating in t we get

$$\rho_n \int_0^T u_n''\phi\,dt - \mathsf{div}\left(\mathbf{A}_n \nabla \int_0^T u_n\phi\,dt\right) = \int_0^T f\phi\,dt \ ,$$

or

$$-\mathsf{div}\,(\mathbf{A}_n \nabla U_n) = g_n \ ,$$

where the function g_n is given by (ϕ has a compact support)

$$\int_0^T f\phi\,dt + \rho_n \int_0^T u_n'\phi'\,dt \ .$$

Note that $g_n \to g_\infty$ strongly in the space V'.

From the standard properties of H-convergence it follows that

$$\mathbf{A}_n \nabla U_n \xrightarrow{\mathrm{L}^2(\Omega)} \mathbf{A}_\infty \nabla U_\infty \ .$$

As the functions of the form $\phi \boxtimes \psi$ are dense in $\mathrm{L}^2(\langle 0,T\rangle \times \Omega)$, we get that

$$\mathbf{A}_n \nabla u_n \xrightarrow{\mathrm{L}^2(\langle 0,T\rangle\times\Omega)} \mathbf{A}_\infty \nabla u_\infty \ ,$$

thus u_∞ satisfies the equation (9_1), with ∞ instead of n. The corresponding boundary conditions follow from the regularity of the solution. □

4. Optimal shape design for the wave equation

We shall now apply some of the results described in last two sections to the optimal shape design problem stated in the *Introduction*. For simplicity, from now on we write it in the following form

$$\begin{cases} J(\boldsymbol{\chi},\mathbf{A}) = \int_{\Omega_T} \sum_{i=1}^m \chi_i(\mathbf{x}) g_i(t,\mathbf{x},u(t,\mathbf{x}))\, d\mathbf{x}\, dt \to \inf \\ (\boldsymbol{\chi},\mathbf{A}) \in \mathcal{A}\,, \end{cases} \tag{10}$$

where the function u is determined by (8) with ρ given by (2_1), while

$$\mathcal{A} = \left\{ (\boldsymbol{\chi},\mathbf{A}) : \boldsymbol{\chi} \in \mathrm{L}^\infty(\Omega;\{0,1\}^m),\ \sum_{i=1}^m \chi_i = 1,\ \mathbf{A} = \sum_{i=1}^m \chi_i \gamma_i \mathbf{I} \quad \text{a.e. on } \Omega \right\}.$$

As stated in the *Introduction*, although J, in fact, depends just on $\boldsymbol{\chi}$, we are forced to consider $\mathbf{A}$ as an additional argument in order to get the continuity of J.

The functional J can well be extended to the set

$$\mathcal{T} = \left\{ (\boldsymbol{\chi},\mathbf{A}) \in \mathrm{L}^\infty(\Omega;[0,1]^m) \times \mathfrak{M}(\alpha,\beta;\Omega) : \sum_{i=1}^m \chi_i = 1 \text{ a.e. on } \Omega \right\}.$$

We shall consider the following product topology on $\mathcal{T}$: L^∞ weak $*$ for $\boldsymbol{\chi}$ and H-topology for $\mathbf{A}$. As a simple consequence of *Theorem 5* we have

Corollary 1. *The mapping $(\boldsymbol{\chi},\mathbf{A}) \mapsto u$ is continuous from $\mathcal{T}$ to $\mathrm{H}^1(\Omega_T)$ weak.*

Compliance. The cost functional of great interest is the compliance (the work done by external force)

$$J_C(\rho,\mathbf{A}) = \int_{\Omega_T} f(t,\mathbf{x}) u(t,\mathbf{x})\, d\mathbf{x}\, dt.$$

The compliance depends only on the state function. The mapping $(\rho,\mathbf{A}) \mapsto u$ is continuous from the considered topology to $\mathrm{H}^1(\Omega_T)$ weak, so as $f \in \mathrm{L}^2(\Omega_T)$ we can conclude that J_C is continuous.

Theorem 6. *Let g_i, $i = 1,\ldots,m$, be Carathéodory's functions (i.e. measurable in $t,\mathbf{x}$ and continuous in λ) satisfying*

$$|g_i(t,\mathbf{x},\lambda)| \le k_i(t,\mathbf{x}) + c_i|\lambda|^q \qquad \text{for } \lambda \in \mathbb{R},\ \text{a.e. } (t,\mathbf{x}) \in \Omega_T, \tag{11}$$

with $k_i \in \mathrm{L}(\Omega_T)$, $c_i \ge 0$ and some $q \in [2,q^*\rangle$, where

$$q^* = \begin{cases} \infty, & d = 1 \\ \frac{2d+2}{d-1}, & d > 1. \end{cases}$$

Then the cost functional J is continuous on $\mathcal{T}$.

Proof. Because of metrisability of the topology on $\mathcal{T}$, it is enough to consider sequential continuity only. Let $(\boldsymbol{\chi}_n,\mathbf{A}_n)$ be a sequence in $\mathcal{T}$ such that $\boldsymbol{\chi}_n \xrightarrow{*} \boldsymbol{\chi}$ and $\mathbf{A}_n \xrightarrow{H} \mathbf{A}$. Then $\rho_n \xrightarrow{*} \rho$ as well and, according to *Theorem 5*, the sequence of corresponding solutions (u_n) of (9) converges to the solution u of (8) weakly in $\mathrm{H}^1(\Omega_T)$. The Sobolev imbedding theorem gives

$$\mathrm{H}^1(\Omega_T) \hookrightarrow \mathrm{L}^q(\Omega_T) \quad \text{for } q \in [2, q^*\rangle.$$

These inclusions are compact, so we know that $u_n \to u$ in $\mathrm{L}^q(\Omega_T)$ for indices q listed above. Using the estimates (11) on g_i the following convergences hold

$$g_i(\cdot,\cdot,u_n) \longrightarrow g_i(\cdot,\cdot,u) \quad \text{in } \mathrm{L}^1(\Omega_T),\ i = 1,\dots,m\ ;$$

and finally (because of the weak $*$ convergence of $(\boldsymbol{\chi}_n)$)

$$J(\boldsymbol{\chi}_n, \mathbf{A}_n) \longrightarrow J(\boldsymbol{\chi}, \mathbf{A}).$$

□

If the set $\mathcal{A}$ is closed in $\mathcal{T}$, it is also a compact set and J will have a minimum. Unfortunately, in the light of the characterisation of weak $*$ limits described in *Theorem 3*, we see that it can not be closed in such topology on $\mathcal{T}$, so we cannot establish the existence of a solution to our problem.

Relaxation is theoretically simple: it consists of closing the set $\mathcal{A}$ in $\mathcal{T}$, but there is a problem with the lack of precise characterisation of H-closure, except for the case $m = 2$. Let us concentrate on that case.

Mixtures of two isotropic materials. For simplicity, take $\gamma_1 = \alpha$ and $\gamma_2 = \beta$, $\chi_1 = \chi$ so that χ_2 becomes $1 - \chi$. With these definitions, we can simply use the results of *Theorem 4* and write a relaxation as

$$\begin{cases} J(\chi, \mathbf{A}) \to \inf \\ (\chi, \mathbf{A}) \in \{(\theta, \mathbf{A}) \in \mathrm{L}^\infty(\Omega; [0,1]) \times \mathfrak{M}(\alpha, \beta; \Omega) : \sigma(\mathbf{A}(\mathbf{x})) \text{ satisfies (7) for a.e. } \mathbf{x} \in \Omega\} \end{cases}.$$

Layered materials. The case of layered materials is quite different. Suppose that a sequence of characteristic functions (χ_n), depending only on x_1, weakly $*$ converges to $\boldsymbol{\theta}$ in $\mathrm{L}^\infty(\Omega; [0,1]^m)$. Define

$$a_+ = \textstyle\sum_{i=1}^m \theta_i \gamma_i\ ,$$
$$\frac{1}{a_-} = \textstyle\sum_{i=1}^m \frac{\theta_i}{\gamma_i}$$

almost everywhere on Ω. Using the results for layered materials given above, we see that the H-limit of the sequence $(\mathbf{A}_n)$ defined by (2) is a matrix function

$$\mathbf{A}_\infty = \mathrm{diag}(a_-, a_+, a_+, \dots, a_+), \quad \text{a.e. on } \Omega. \tag{12}$$

It is important to notice that $\mathbf{A}_\infty$ is computed from $\boldsymbol{\theta}$, so we have the following relaxation

$$\begin{cases} J(\boldsymbol{\chi}) = \int_{\Omega_T} \sum_{i=1}^m \chi_i(\mathbf{x}) g_i(t, \mathbf{x}, u(t, \mathbf{x}))\, d\mathbf{x}\, dt \to \inf \\ \boldsymbol{\chi} \in \mathrm{L}^\infty(\Omega; [0,1]^m)\ , \quad \sum_{i=1}^m \chi_i = 1\ ; \end{cases}$$

where u is calculated from (8) with $\mathbf{A}$ defined by (12) and ρ by (2_1).

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