Optimal design for plates and relaxation[*][†]

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Abstract. *The optimal design problem for a plate of variable thickness assuming the Kirchhoff model for pure bending of symmetric plates is studied. It is well known that this problem has no solution. The relaxation procedure is thus necessary, leading naturally to homogenisation.*

Following the ideas of Tartar, a new procedure is proposed, giving the known result of Muñoz and Pedregal. Some applications of the proposed method for more general materials are presented.

Key words: *optimal design, homogenisation, relaxation, H-convergence, Young measures*

AMS subject classifications: 49J20, 35B27, 73V25

1. Introduction

We are interested in the behaviour of a thin solid elastic plate made of a linearly elastic material. The plate is assumed to be symmetric with respect to the horizontal midplane. If we denote the central section of the plate in its unperturbed configuration by Ω, which is a bounded region in $\mathbb{R}^2$, we can describe the plate as the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega \quad \& \quad |x_3| \leq h(x_1, x_2)\}\,.$$

We limit our investigation to the Kirchhoff model for pure bending of symmetric plates under transverse loads. According to that theory, the vertical displacement u satisfies a fourth–order elliptic equation of the form

$$\mathsf{div}\,\mathsf{div}\,(\mathsf{M}\nabla\nabla u) = f \qquad \text{in} \qquad \Omega\,, \tag{1}$$

where $f \in L^2(\Omega)$ is the vertical load on the plate, while M is a tensor–valued function given by

$$\mathsf{M}(\mathbf{x}) := \frac{2}{3}h^3(\mathbf{x})\mathsf{B}\,, \tag{2}$$

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where B is a constant tensor of the fourth rank which is material dependent. The tensor B (and hence M) is assumed to have the following symmetry:

$$B_{ijkl} = B_{jikl} = B_{ijlk}\,. \tag{3}$$

More precisely, tensor B can be understood as a linear operator on the space of all symmetric 2×2 real matrices, hereafter denoted by Sym, and has nine entries.

For simplicity, we shall discuss only plates that are clamped at the edges

$$u = \nabla_{\mathbf{n}} u = 0 \qquad \text{on} \qquad \partial\Omega\,, \tag{4}$$

meaning that u belongs to the Sobolev space $\mathrm{H}^2_0(\Omega)$.

In order to insure the well–posedness of the boundary value problem (1,4), we assume that M belongs to the space $\mathfrak{M}_2(\alpha, \beta; \Omega)$ defined as

$$\Big\{\mathsf{M} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}; \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})\ \mathsf{M}(\mathbf{x})\mathbf{S}\cdot\mathbf{S} \geq \alpha\mathbf{S}\cdot\mathbf{S} \\ \&\ \mathsf{M}^{-1}(\mathbf{x})\mathbf{S}\cdot\mathbf{S} \geq \tfrac{1}{\beta}\mathbf{S}\cdot\mathbf{S}\ (\text{a.e. } \mathbf{x})\Big\}\,.$$

Indeed, for $\mathsf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$, the bilinear form

$$a(u, v) := \int_\Omega \mathsf{M}\nabla\nabla u \cdot \nabla\nabla v\, d\mathbf{x} \tag{5}$$

is elliptic and bounded on $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$, therefore by using the Lax–Milgram Lemma, we establish the existence and uniqueness for the above problem.

The boundary value problem (4) is variational, which means that its solution u is the unique minimiser of the functional

$$\mathcal{E}(v) := \frac{1}{2}\int_\Omega \mathsf{M}(\mathbf{x})\nabla\nabla v \cdot \nabla\nabla v\, d\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\, d\mathbf{x}$$

over the Sobolev space $\mathrm{H}^2_0(\Omega)$. The *compliance* L is the work done by external load f:

$$L = \int_\Omega f(\mathbf{x})u(\mathbf{x})\, d\mathbf{x} = \int_\Omega \mathsf{M}(\mathbf{x})\nabla\nabla u \cdot \nabla\nabla u\, d\mathbf{x}\,.$$

For a given load f, we think of L as a function of the (half)thickness h. It represents an overall measure of the rigidity or flexibility of the plate under f, therefore it seems natural to study the problem of optimisation for minimal compliance, i.e. minimising $L(h)$ among all admissible plates with the prescribed volume.

Throughout this paper we shall entirely restrict our attention to the case where (half)thickness h depends on the variable x_1 alone, and x_1 belongs to an interval I, the projection of Ω to x_1–axis. For the given $h_{\min}, h_{\max}$ and V, we define the set of admissible functions by

$$\mathcal{H} := \left\{h \in \mathrm{L}^\infty(I; Q) : \int_\Omega h = V\right\},$$

where Q stands for the segment $[h_{\min}, h_{\max}]$. Naturally, the above values should be prescribed in a consistent way: $0 < h_{\min}\mathsf{vol}\,\Omega < V < h_{\max}\mathsf{vol}\,\Omega < \infty$.

It is well known that the problem of minimising L over $\mathcal{H}$ generally has no solution, due to rapid oscillations of minimising sequences [4]. Therefore, we perform a relaxation of the problem, introducing a set of generalised thicknesses. The basic tool to show the equality of infima for the original and relaxed problem is the following lemma (cf. Lemma 1.1 in [5]):

Lemma 1. *Let* (M^n) *be a sequence in* $\mathfrak{M}_2(\alpha,\beta;\Omega)$*. Furthermore, assume that*

$$\begin{array}{rcl}
\frac{1}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & \frac{1}{M^\infty_{1111}}\,,\\
\frac{M^n_{1112}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & \frac{M^\infty_{1112}}{M^\infty_{1111}}\\
\frac{M^n_{1122}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & \frac{M^\infty_{1122}}{M^\infty_{1111}}\,,\\
\frac{M^n_{1211}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & \frac{M^\infty_{1211}}{M^\infty_{1111}}\,,\\
\frac{M^n_{2211}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & \frac{M^\infty_{2211}}{M^\infty_{1111}}\,,\\
M^n_{1212}-\frac{M^n_{1211}M^n_{1112}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & M^\infty_{1212}-\frac{M^\infty_{1211}M^\infty_{1112}}{M^\infty_{1111}}\,,\\
M^n_{1222}-\frac{M^n_{1211}M^n_{1122}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & M^\infty_{1222}-\frac{M^\infty_{1211}M^\infty_{1122}}{M^\infty_{1111}}\,,\\
M^n_{2212}-\frac{M^n_{2211}M^n_{1112}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & M^\infty_{2212}-\frac{M^\infty_{2211}M^\infty_{1112}}{M^\infty_{1111}}\,,\\
M^n_{2222}-\frac{M^n_{2211}M^n_{1122}}{M^n_{1111}} & \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} & M^\infty_{2222}-\frac{M^\infty_{2211}M^\infty_{1122}}{M^\infty_{1111}}\,.
\end{array} \tag{6}$$

If u_n*, for each* $n\in\mathbb{N}\cup\{\infty\}$*, is the solution of problem (1,4) corresponding to* M^n*, then* $u_n \longrightarrow u_\infty$ *in* $\mathrm{H}^2_0(\Omega)$*.*

Following the same ideas as in [5], we define the set of generalised (half)thicknesses to be the set of all Young measures associated to sequences in $\mathcal{H}$:

$$\overline{\mathcal{H}} := \left\{\nu := (\nu_x)_{x\in I} : \mathrm{supp}\nu_x \in Q \text{ (a.e. } x), \int_I\int_Q \lambda\, d\nu_x(\lambda)dx = V\right\}.$$

The crucial part in defining a compliance on $\overline{\mathcal{H}}$ is making the use of *Lemma 1.* Let (h_n) be a minimising sequence for L in $\mathcal{H}$. According to (2), nonconstant expressions on the right-hand side of (6) can be written as follows:

$$\begin{array}{rcl}
\frac{1}{M^n_{1111}} & = & \frac{3}{2B_{1111}}h_n^{-3}\,,\\
M^n_{1212}-\frac{M^n_{1211}M^n_{1112}}{M^n_{1111}} & = & \frac{2}{3}h_n^3\left(B_{1212}-\frac{B_{1211}B_{1112}}{B_{1111}}\right)\,,\\
M^n_{1222}-\frac{M^n_{1211}M^n_{1122}}{M^n_{1111}} & = & \frac{2}{3}h_n^3\left(B_{1222}-\frac{B_{1211}B_{1122}}{B_{1111}}\right)\,,\\
M^n_{2212}-\frac{M^n_{2211}M^n_{1112}}{M^n_{1111}} & = & \frac{2}{3}h_n^3\left(B_{2212}-\frac{B_{2211}B_{1112}}{B_{1111}}\right)\,,\\
M^n_{2222}-\frac{M^n_{2211}M^n_{1122}}{M^n_{1111}} & = & \frac{2}{3}h_n^3\left(B_{2222}-\frac{B_{2211}B_{1122}}{B_{1111}}\right)\,.
\end{array}$$

The above weak limits can clearly be expressed through the moments of order ± 3 of the Young measure ν, corresponding to the sequence (h_n). Therefore, if we set

$$\begin{aligned} m(x) &:= \int_Q \lambda^3 \, d\nu_x(\lambda)\,, \\ c^{-1}(x) &:= \int_Q \lambda^{-3} \, d\nu_x(\lambda)\,, \end{aligned}$$

and define the components of tensor M^∞ by using m and c instead of h^3 and h^{-3} respectively, then by *Lemma 1*, the solutions u_n to the problem (1, 4), corresponding to tensors M^n, will converge weakly to the solution u_∞ corresponding to M^∞.

After defining the compliance $\overline{L}$ for elements in $\overline{\mathcal{H}}$ by

$$\overline{L}(\nu) := \int_\Omega f u \, d\mathbf{x}\,,$$

where u is the solution to the boundary value problem (1, 4), with tensor M depending on ν through functions m and c as described above, we have the following result:

Theorem 1. *The pair* $(\overline{L}, \overline{\mathcal{H}})$ *is a relaxation for* $(L, \mathcal{H})$.

Our research was inspired by [5], where the authors obtained the result for a narrower class of orthotropic materials, i.e. those with non–vanishing coefficients M_{1111}, M_{2222} and

$$M_{1122} = M_{2211}\,, \qquad M_{1212} = M_{2112} = M_{2121}\,.$$

After taking into account our improvement, the remainder of proof of the above theorem follows along the same lines as in [5].

2. Homogenisation and H–convergence

In the development of the theory of homogenisation for second order operators, the rôle played by $\mathsf{div} - \mathsf{rot}$ lemma [7] is crucial. For the equation of plate, the following compensated compactness result plays the same rôle (for the proof see [1]):

Lemma 2. *Let the following convergences be valid*

$$\begin{aligned} w^n &\xrightarrow{\mathrm{H}^2_{loc}(\Omega)} w^\infty\,, \\ \mathbf{D}^n &\xrightarrow{\mathrm{L}^2_{loc}(\Omega; M_{2\times 2})} \mathbf{D}^\infty\,, \end{aligned}$$

with an additional assumption that the sequence $(\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{loc}(\Omega)$.

Then we have that

$$\mathbf{E}^n \cdot \mathbf{D}^n \longrightarrow \mathbf{E}^\infty \cdot \mathbf{D}^\infty$$

vaguely (i.e. weakly–$*$ in the space of Radon measures), where we denote $\mathbf{E}^n := \nabla\nabla w^n$ *(and analogously for ∞ instead of n).*

Next, we define a notion of H-convergence adapted to the problem under investigation. We say that a sequence of tensor functions (M^n) in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathsf{M}^\infty \in \mathfrak{M}_2(\alpha,\beta;\Omega)$, if for any $f \in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions (u_n) of the problems

$$\text{Find } u_n \in \mathrm{H}^2_0(\Omega) \text{ such that } \big(\forall v \in \mathrm{H}^2_0(\Omega)\big) \quad \int_\Omega \mathsf{M}^n \nabla\nabla u_n \cdot \nabla\nabla v \, d\mathbf{x} = \langle f, v\rangle \tag{7}$$

converges to a limit u_∞ weakly in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathsf{M}^n\nabla\nabla u_n)$ converges to $\mathsf{M}^\infty\nabla\nabla u_\infty$ weakly in the space $\mathrm{L}^2(\Omega;\mathrm{Sym})$.

Note that u_∞ is then necessarily a solution of (7), where we write ∞ instead of n. Besides, if the sequence (M^n) is contained in the set $\mathfrak{M}_2(\alpha,\beta;\Omega)$, the statement of the definition does not imply that the limit is in the same set, but we have to know that in advance. The first step in the direction of weakening the conditions of the above definition, or, which amounts to the same, to prove an analogous theorem as for the compactness of H-convergence in the case of second order elliptic problems (as in [6]), is the theorem on G-convergence.

For the bilinear form a defined by (5), there is an operator $A \in \mathcal{L}(\mathrm{H}^2_0(\Omega);\mathrm{H}^{-2}(\Omega))$ defined by the formula

$$Au := \mathsf{div}\,\mathsf{div}\,(\mathsf{M}\nabla\nabla u) \tag{8}$$

which represents the bilinear form in the sense that the following holds true:

$$\big(\forall u, v \in \mathrm{H}^2_0(\Omega)\big) \qquad a(u,v) = {}_{\mathrm{H}^{-2}(\Omega)}\langle\, Au, v\,\rangle_{\mathrm{H}^2_0(\Omega)} \; .$$

As the form a is bounded and coercive, it follows that the operator A is bounded and invertible, and that its inverse A^{-1} is continuous as well. Thus we are able to apply the standard results valid for the G-convergence of operators (see e.g. [8]), which gives us appropriate compactness (moreover, for symmetric tensors M^n we have the compactness with the same constants α and β). Of course, it remains to be proven that the operator being the G-limit of the operators of the form (8) is of the same form itself.

Lemma 3. *Let V be a real, reflexive and separable Banach space. If the operator $A \in \mathcal{L}(V;V')$ is coercive (i.e. $(\exists\alpha > 0)$ $(\forall u \in V)$ $\langle Au, u\rangle \geq \alpha\|u\|^2_V$), then the equation $Au = f$ has the unique solution $u \in V$ for any $f \in V'$, and the inequality*

$$\|u\|_V = \|A^{-1}f\|_V \leq \frac{1}{\alpha}\|f\|_{V'}$$

holds.

Let a sequence (M^n) of tensor functions in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ be given; we define the operators $A_n : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2(\Omega)$ by formula (8). The following uniform estimates are true:

$$\|A_n\| \leq \beta \qquad \text{and} \qquad \big(\forall u \in \mathrm{H}^2_0(\Omega)\big) \quad \langle A_n u, u\rangle \geq \alpha\|u\|_{\mathrm{H}^2_0(\Omega)} \; .$$

Our goal is to prove the compactness of the set $\mathfrak{M}_2(\alpha,\beta;\Omega)$; as the first step we achieve, besides the G-convergence of the operators, the abstract convergence of the sequence $(\mathsf{M}^n\nabla\nabla u_n)$ as well. The limit will be identified in the next step.

Lemma 4. *There is a subsequence (M^{n_k}) of the above sequence, and the operators $A_\infty \in \mathcal{L}(\mathrm{H}^2_0(\Omega);\mathrm{H}^{-2}(\Omega))$ and $R \in \mathcal{L}(\mathrm{H}^{-2}(\Omega);\mathrm{L}^2(\Omega;\mathrm{Sym}))$ such that*

$$A_{n_k} \xrightarrow{G} A_\infty$$

(i.e. $A_{n_k}^{-1} \rightharpoonup A_\infty^{-1}$ weakly in the sense of operators), and that for arbitrary $f \in$ $\mathrm{H}^{-2}(\Omega)$ we have

$$\mathsf{M}^{n_k}\nabla\nabla u_{n_k} \xrightharpoonup{\mathrm{L}^2(\Omega;\mathrm{Sym})} Rf\,,$$

where (u_{n_k}) is a sequence of solutions of problems (7), with the given f.

The compactness is given precisely by the following theorem (for the details of the proof see [1]):

Theorem 2. *Let (M^n) be a sequence in $\mathfrak{M}_2(\alpha,\beta;\Omega)$. Then there is a subsequence (M^{n_k}) and a tensor function $\mathsf{M}^\infty \in \mathfrak{M}_2(\alpha,\beta;\Omega)$ such that*

$$\mathsf{M}^{n_k} \xrightharpoonup{H} \mathsf{M}^\infty\,.$$

In the proof of the theorem there are two more statements to be shown:

1. The operator A_∞ is of the same form as operators A_n (cf. *Lemma 4*), in the sense that there is a tensor M^∞ such that $A_\infty u = \mathsf{div}\,\mathsf{div}\,(\mathsf{M}^\infty\nabla\nabla u)$,
2. The tensor M^∞ is an element of the set $\mathfrak{M}_2(\alpha,\beta;\Omega)$.

In order to prove (a), we can use the method of oscillating test functions [7], while for (b) we use the following lemma:

Lemma 5. *Assume that $\mathsf{M}^\infty \in \mathrm{L}^2(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$. Furthermore, let the operator C in $\mathrm{L}(\mathrm{H}_0^2(\Omega),\mathrm{L}^2(\Omega;\mathrm{Sym}))$, defined by the formula $Cv := \mathsf{M}^\infty\nabla\nabla v$, be such that the following estimate holds*

$$\|C\|_{\mathcal{L}(\mathrm{H}_0^2(\Omega),\mathrm{L}^2(\Omega;\mathrm{Sym}))} \le \gamma\,.$$

Then $\mathsf{M}^\infty \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$ and we have

$$|\mathsf{M}^\infty(\mathbf{x})|_{\mathcal{L}(\mathrm{Sym};\mathrm{Sym})} \le \gamma \qquad (\text{a.e. } \mathbf{x}\in\Omega)\,.$$

3. Effective properties of a layered plate

The goal of this section is to prove *Lemma 1*, which enables us to generalise the result of Muñoz and Pedregal [5] to a more general setting. In fact, *Lemma 1* is a simple consequence of the following

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^2$ be an open and bounded set, and (M^n) a sequence of tensor valued functions in $\mathfrak{M}_2(\alpha,\beta;\Omega)$, such that for each n, M^n depends on x_1 only. Then M^n H–converges to M^∞ if and only if (6) holds.*

In order to prove the necessity part of the above theorem, we shall require the following definition:

The sequence (u_n) in $\mathrm{L}^2(\Omega)$ *does not oscillate* in variable x_1 if the following is satisfied:

1. $u_n \rightharpoonup u_\infty$ in $\mathrm{L}^2(\Omega)$,

2. For each sequence of L^∞ functions (f_n), depending on x_1 only, such that f_n weakly–$*$ converges to a function f_∞, the product $f_n u_n$ converges to $f_\infty u_\infty$ weakly in $\mathrm{L}^2(\Omega)$.

The following Lemma is a consequence of *Lemma 2* and this definition.

Lemma 6. *Let* $(\mathbf{D}^n)$ *be a sequence in* $\mathrm{L}^2(\Omega;\mathbb{R}^d)$ *which weakly converges to* $\mathbf{D}^\infty$*. If the sequence* $(\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n)$ *is contained in a precompact set in* $\mathrm{H}^{-2}_{loc}(\Omega)$*, then* D^n_{11} *does not oscillate in* x_1*.*

Proof. One should notice that $\mathbf{E}^n := f_n \mathsf{e}_1 \otimes \mathsf{e}_1$ is a gradient of a gradient of some scalar function (which depends on x_1 only); therefore the statement follows directly from *Lemma 2*. □

Let us assume that M^n H–converges to M^∞. According to the definition given in the previous section, for any $f \in \mathrm{H}^{-2}(\Omega)$, the solutions u_n of boundary value problem (1, 4) satisfy

$$u_n \xrightarrow{\mathrm{H}^2_0(\Omega)} u_\infty\,,$$
$$\mathsf{M}^n \nabla\nabla u_n \xrightarrow{\mathrm{L}^2(\Omega;\mathrm{Sym})} \mathsf{M}^\infty \nabla\nabla u_\infty\,.$$

In order to employ *Lemma 2*, for $n \in \mathbb{N} \cup \{\infty\}$ let us denote

$$\mathbf{E}^n := \nabla\nabla u_n \qquad \text{and} \qquad \mathbf{D}^n := \mathsf{M}^n \mathbf{E}^n\,. \tag{9}$$

As $\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n = f$, according to *Lemma 6*, D^n_{11} does not oscillate in x_1. On the other hand, let us define $\tilde{\mathbf{D}}^n$ by

$$\tilde{D}^n_{ij} := \begin{cases} 0\,, & \text{for } i = j = 1 \\ g^n_{ij}\,, & \text{otherwise}\,, \end{cases}$$

where g^n_{ij} are L^∞ functions which depend on x_1 only. Let us further assume that for each pair of indices (i,j) the function g^n_{ij} converges to g^∞_{ij} weakly–$*$ in L^∞. Since $\tilde{\mathbf{D}}^n$ clearly satisfies $\mathsf{div}\,\mathsf{div}\,\tilde{\mathbf{D}}^n = 0$, according to *Lemma 2* we have that $\tilde{\mathbf{D}}^n \cdot \mathbf{E}^n \rightharpoonup \tilde{\mathbf{D}}^\infty \cdot \mathbf{E}^\infty$ in $\mathcal{D}'$. As the sequence $(\tilde{\mathbf{D}}^n \cdot \mathbf{E}^n)$ is also bounded in $\mathrm{L}^2(\Omega)$, a simple uniqueness argument shows us that the above convergence is valid in this space as well. Convenient choices of functions g^n_{ij} lead to a conclusion that for each pair of indices $(i,j) \neq (1,1)$ the component E^n_{ij} does not oscillate in x_1.

This procedure enables us to extract *good* (nonoscillating) and *bad* (oscillating) components from matrices $\mathbf{E}^n$ i $\mathbf{D}^n$

$$G^n_{ij} := \begin{cases} D^n_{11}\,, & \text{for } i = j = 1 \\ E^n_{ij}\,, & \text{otherwise} \end{cases} \qquad \text{and} \qquad O^n_{ij} := \begin{cases} E^n_{11}\,, & \text{for } i = j = 1 \\ D^n_{ij}\,, & \text{otherwise} \end{cases}\,.$$

Combining this with (9) we obtain

$$\mathbf{O}^n = \mathsf{K}^n \mathbf{G}^n\,,$$

where $\mathsf{K}^n := \Psi(\mathsf{M}^n)$. Reasonably brief calculation shows that the components of tensor K^n are (up to a constant) exactly the terms on the left-hand side in (6). On the other hand, since M^n belongs to the space $\mathfrak{M}_2(\alpha,\beta;\Omega)$, we conclude that

$\alpha \le M^n_{1111}(\mathbf{x}) \le \beta$ (a.e. $\mathbf{x} \in \Omega$), which clearly implies that the sequence (K^n) is bounded in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym};\mathrm{Sym}))$, and hence has a cluster point in weak–$*$ topology.

Using the fact that $(\mathbf{G}^n)$ does not oscillate in x_1, and (K^n) depends on x_1 only, after taking a subsequence, we obtain

$$\mathbf{O}^\infty = \mathsf{K}^\infty \mathbf{G}^\infty .$$

By using this equality it is easy to read off that $\mathsf{K}^\infty := \Psi(\mathsf{M}^\infty)$, or in other words

$$\Psi(\mathsf{M}^{n_k}) \xrightarrow{\mathrm{L}^\infty(\Omega)*} \Psi(\mathsf{M}^\infty) ,$$

which is exactly what we wanted to prove. Of course, each cluster point of the sequence K^n must satisfy the above convergence result, thus it holds for the entire sequence $(\Psi(\mathsf{M}^n))$, rather than for a subsequence. This furnishes the proof of the *only if* part of the theorem.

As for the *if* part, we are in fact able to prove a slightly stronger result.

Theorem 4. *With notation as in the proof of* Theorem 3*, assume that the sequence* (u_n) *converges weakly in* $\mathrm{H}^2(\Omega)$ *to* u_∞*, and that the sequence* $(\mathsf{div}\,\mathsf{div}\,\mathbf{D}^n)$ *is contained in a precompact set in* $\mathrm{H}^{-2}_{loc}(\Omega)$*. In addition to this, assume that*

$$\Psi(\mathsf{M}^n) \xrightarrow{\;\star\;} \Psi(\mathsf{M}^\infty) \quad \text{in } \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym};\mathrm{Sym})) ,$$

where, for each n*,* M^n *is a tensor valued function which satisfies* $\alpha \le M^n_{1111}(\mathbf{x}) \le \beta$ (a.e. $\mathbf{x} \in \Omega$)*. Then*

$$\mathbf{D}^n \xrightarrow{\mathrm{L}^2(\Omega;\mathbb{R}^2)} \mathbf{D}^\infty .$$

Proof of this theorem uses similar arguments as the one demonstrated above. However it heavily relies on the use of oscillating test functions [7] and some explicit constructions, which makes it technical; therefore, we chose to omit it.

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