ON RECTANGULAR INVERSE SYSTEMS OF TOPOLOGICAL SPACES

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Abstract. For every cofinite inverse system of compact Hausdorff spaces $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, there exists a cofinite inverse system of compact polyhedra $\boldsymbol{Z} = (Z_\lambda^\tau, r_{\lambda\lambda'}^{\tau\tau'}, \Lambda \times T)$ and there are mappings $u_\lambda^\tau \colon X_\lambda \to Z_\lambda^\tau$, $(\lambda, \tau) \in \Lambda \times T$, such that $u_\lambda^\tau p_{\lambda\lambda'} = r_{\lambda\lambda'}^{\tau\tau} u_{\lambda'}^\tau$, for $\lambda \le \lambda'$, and $r_\lambda^{\tau\tau'} u_\lambda^{\tau'} = u_\lambda^\tau$, for $\tau \le \tau'$. Moreover, for every $\lambda \in \Lambda$, the mapping $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda = (Z_\lambda^\tau, r_{\lambda\lambda}^{\tau\tau'}, T)$, given by the mappings u_λ^τ, $\tau \in T$, is a limit of $\boldsymbol{Z}_\lambda$. If mappings $p_\lambda \colon X \to X_\lambda$ form a limit $\boldsymbol{p} \colon X \to \boldsymbol{X}$, then the mappings $v_\lambda^\tau = u_\lambda^\tau p_\lambda \colon X \to Z_\lambda^\tau$ form a limit $\boldsymbol{v} \colon X \to \boldsymbol{Z}$. An analogous result holds for cofinite inverse systems of topological spaces and ANR-resolutions (polyhedral resolutions).

1. Introduction

By a rectangular or double inverse system we mean an inverse system $\boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, whose index set $N = \Lambda \times T$ is the product of two directed sets Λ and T. By definition, $\Lambda \times T$ is ordered by the product ordering, i.e., $(\lambda, \tau) \le (\lambda', \tau')$, provided $\lambda \le \lambda'$ and $\tau \le \tau'$. If $\nu = (\lambda, \tau) \in \Lambda \times T$, we denote $Z_\nu = Z_{(\lambda,\tau)}$ by Z_λ^τ. Similarly, if $\nu = (\lambda, \tau) \le (\lambda', \tau') = \nu'$, we denote $r_{\nu\nu'} = r_{(\lambda,\tau)(\lambda',\tau')}$ by $r_{\lambda\lambda'}^{\tau\tau'}$. If $\tau = \tau'$ or $\lambda = \lambda'$, we use shorter notation $r_{\lambda\lambda'}^\tau$ and $r_\lambda^{\tau\tau'}$, respectively. We will refer to $r_{\lambda\lambda'}^\tau$ and $r_\lambda^{\tau\tau'}$ as to horizontal and vertical bonding mappings of $\boldsymbol{Z}$, respectively. Clearly, if $(\lambda, \tau) \le (\lambda', \tau')$, then $(\lambda, \tau) \le (\lambda', \tau) \le (\lambda', \tau')$ and $(\lambda, \tau) \le (\lambda, \tau') \le (\lambda', \tau')$. Consequently,

$$(1.1) \qquad r_{\lambda\lambda'}^\tau r_{\lambda'}^{\tau\tau'} = r_{\lambda\lambda'}^{\tau\tau'} = r_\lambda^{\tau\tau'} r_{\lambda\lambda'}^{\tau'},$$

which shows that the horizontal and the vertical bonding mappings determine all bonding mappings in a rectangular system. For any fixed $\lambda \in \Lambda$, $\boldsymbol{Z}_\lambda = (Z_\lambda^\tau, r_\lambda^{\tau\tau'}, T)$ is an inverse system, called the vertical system of $\boldsymbol{Z}$ at level λ. The mappings $r_{\lambda\lambda'}^\tau \colon Z_{\lambda'}^\tau \to Z_\lambda^\tau$, $\tau \in T$, form a level mapping $\boldsymbol{r}_{\lambda\lambda'} \colon \boldsymbol{Z}_{\lambda'} \to \boldsymbol{Z}_\lambda$. For $\lambda \leq \lambda' \leq \lambda''$, $\boldsymbol{r}_{\lambda\lambda'}\boldsymbol{r}_{\lambda'\lambda''} = \boldsymbol{r}_{\lambda\lambda''}$. Similarly, for any fixed $\tau \in T$, $\boldsymbol{Z}^\tau = (Z_\lambda^\tau, r_{\lambda\lambda'}^\tau, \Lambda)$ is an inverse system, called the horizontal system of $\boldsymbol{Z}$ at level τ. The mappings $r_\lambda^{\tau\tau'} \colon Z_\lambda^{\tau'} \to Z_\lambda^\tau$, $\lambda \in \Lambda$, form a level mapping $\boldsymbol{r}^{\tau\tau'} \colon \boldsymbol{Z}^{\tau'} \to \boldsymbol{Z}^\tau$. For $\tau \leq \tau' \leq \tau''$, $\boldsymbol{r}^{\tau\tau'}\boldsymbol{r}^{\tau'\tau''} = \boldsymbol{r}^{\tau\tau''}$.

If $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is an inverse system of spaces and $u_\lambda^\tau \colon X_\lambda \to Z_\lambda^\tau$ are mappings such that

$$(1.2) \qquad\qquad u_\lambda^\tau p_{\lambda\lambda'} = r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau, \ \lambda \leq \lambda',$$

then the mappings u_λ^τ, $\lambda \in \Lambda$, form a level mapping $\boldsymbol{u}^\tau \colon \boldsymbol{X} \to \boldsymbol{Z}^\tau$. If

$$(1.3) \qquad\qquad r_\lambda^{\tau\tau'} u_\lambda^{\tau'} = u_\lambda^\tau, \ \tau \leq \tau',$$

then the mappings $u_\lambda^\tau \colon X_\lambda \to Z_\lambda^\tau$, $\tau \in T$, form a mapping $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$. If both (1.2) and (1.3) hold, then $\boldsymbol{u}_\lambda p_{\lambda\lambda'} = \boldsymbol{r}_{\lambda\lambda'}\boldsymbol{u}_{\lambda'}$ and $\boldsymbol{r}^{\tau\tau'}\boldsymbol{u}^{\tau'} = \boldsymbol{r}^\tau$.

Assuming (1.2) and (1.3), every mapping $\boldsymbol{p} \colon X \to \boldsymbol{X}$, which consists of mappings $p_\lambda \colon X \to X_\lambda$, induces a mapping $\boldsymbol{v} \colon X \to \boldsymbol{Z}$, which consists of mappings $v_\lambda^\tau \colon X \to Z_\lambda^\tau$, where

$$(1.4) \qquad\qquad v_\lambda^\tau = u_\lambda^\tau p_\lambda.$$

To see that the mappings v_λ^τ do form a mapping of X to $\boldsymbol{Z}$, one has to show that, for $(\lambda, \tau) \leq (\lambda', \tau')$,

$$(1.5) \qquad\qquad r_{\lambda\lambda'}^{\tau\tau'} v_{\lambda'}^{\tau'} = v_\lambda^\tau.$$

Because of (1.1), the verification of (1.5) reduces to the verification of $r_{\lambda\lambda'}^{\tau'} v_{\lambda'}^{\tau'} = v_\lambda^{\tau'}$ and $r_\lambda^{\tau\tau'} v_\lambda^{\tau'} = v_\lambda^\tau$. However, by (1.4) and (1.2), $r_{\lambda\lambda'}^{\tau'} v_{\lambda'}^{\tau'} = r_{\lambda\lambda'}^{\tau'} u_{\lambda'}^{\tau'} p_{\lambda'} = u_\lambda^{\tau'} p_{\lambda\lambda'} p_{\lambda'} = u_\lambda^{\tau'} p_\lambda = v_\lambda^{\tau'}$. Similarly, by (1.4) and (1.3), $r_\lambda^{\tau\tau'} v_\lambda^{\tau'} = r_\lambda^{\tau\tau'} u_\lambda^{\tau'} p_\lambda = u_\lambda^\tau p_\lambda = v_\lambda^\tau$.

The purpose of the present paper is to prove the following results.

THEOREM 1.1. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a cofinite inverse system of compact Hausdorff spaces. Then there exists a cofinite rectangular inverse system of compact polyhedra* $\boldsymbol{Z} = (Z_\lambda^\tau, r_{\lambda\lambda'}^{\tau\tau'}, \Lambda \times T)$ *and there exist limits* $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$ *of vertical systems of* $\boldsymbol{Z}$, $\lambda \in \Lambda$, *such that* $\boldsymbol{u}_\lambda p_{\lambda\lambda'} = \boldsymbol{r}_{\lambda\lambda'}\boldsymbol{u}_{\lambda'}$, *for* $\lambda \leq \lambda'$.

REMARK 1.2. For a cofinite directed set Λ, denote by $\mathcal{C}_\Lambda$ the category, whose objects are inverse systems of compact Hausdorff spaces indexed by Λ and whose morphisms are level mappings between such systems. Then the horizontal systems $\boldsymbol{Z}^\tau$, $\tau \in T$, of a cofinite rectangular system $\boldsymbol{Z}$ of compact Hausdorff spaces and the level mappings $\boldsymbol{r}^{\tau\tau'} \colon \boldsymbol{Z}^{\tau'} \to \boldsymbol{Z}^\tau$, $\tau \leq \tau'$, between

horizontal systems of $\boldsymbol{Z}$ form an inverse system $\boldsymbol{Z}^*$ in $\mathcal{C}_\Lambda$. By Theorem 1.1, the mappings $\boldsymbol{u}^\tau\colon \boldsymbol{X} \to \boldsymbol{Z}^\tau$, $\tau \in T$, form a limit $\boldsymbol{u}^*\colon \boldsymbol{X} \to \boldsymbol{Z}^*$ of $\boldsymbol{Z}^*$ in $\mathcal{C}_\Lambda$.

COROLLARY 1.3. *Let* $\boldsymbol{X} = (X_n, p_{nn'}, \mathbb{N})$ *be an inverse sequence of metric compacta. Then there exists a rectangular inverse system of compact polyhedra* $\boldsymbol{Z} = (Z_n^m, r_{nn'}^{mm'}, \mathbb{N} \times \mathbb{N})$ *and there exist limits* $\boldsymbol{u}_n\colon X_n \to \boldsymbol{Z}_n$ *of vertical sequences of* $\boldsymbol{Z}$, $n \in \mathbb{N}$, *such that* $\boldsymbol{u}_n p_{nn'} = \boldsymbol{r}_{nn'} \boldsymbol{u}_{n'}$, *for* $n \leq n'$.

REMARK 1.4. If $\boldsymbol{X} = (X_n, p_{nn'}, \mathbb{N})$ is an inverse sequence of metric compacta, $\boldsymbol{Z}_n = (Z_n^m, r_n^{mm'}, \mathbb{N})$, $n \in \mathbb{N}$, are inverse sequences of compact polyhedra and $\boldsymbol{u}_n = (u_n^m)\colon X_n \to \boldsymbol{Z}_n$ are limits, then it is not always possible to define horizontal mappings $r_{nn'}^m\colon Z_{n'}^m \to Z_n^m$ and obtain a rectangular inverse system $\boldsymbol{Z} = (Z_n^m, r_{nn'}^{mm'}, \mathbb{N} \times \mathbb{N})$ such that $\boldsymbol{u}_n p_{nn'} = \boldsymbol{r}_{nn'} \boldsymbol{u}_{n'}$, for $n \leq n'$. A simple example is obtained as follows (see [8], pages 363-364). Let $X_1 = I = [0, 1]$, let $X_2 = C$ be the Cantor set and let $p_{12}\colon C \to [0, 1]$ be the well-known Cantor mapping. Let $\boldsymbol{Z}_1$ consists of copies of I with identities as bonding mappings and let $\boldsymbol{u}_1\colon X_1 \to \boldsymbol{Z}_1$ be given also by identity mappings. Let $\boldsymbol{u}_2\colon X_2 \to \boldsymbol{Z}_2$ be the standard expansion of C into an inverse sequence of finite sets. Then there is no level mapping $\boldsymbol{r}_{12}\colon \boldsymbol{Z}_2 \to \boldsymbol{Z}_1$ such that $\boldsymbol{u}_1 p_{12} = \boldsymbol{r}_{12} \boldsymbol{u}_2$. Indeed, p_{12} is surjective and thus, $p_{12} = r_{12}^m u_2^m$ would imply that also $r_{12}^m\colon Z_2^m \to I$ is surjective, which is not the case, because $r_{12}^m(Z_2^m)$ is a finite set.

THEOREM 1.5. *Let* $\boldsymbol{X}, \boldsymbol{Z}$ *and* $\boldsymbol{u}_\lambda$, $\lambda \in \Lambda$, *have all the properties stated in Theorem 1.1. If* $\boldsymbol{p} = (p_\lambda)\colon X \to \boldsymbol{X}$ *is a limit of* $\boldsymbol{X}$, *then the induced mapping* $\boldsymbol{v}\colon X \to \boldsymbol{Z}$ *is a limit of* $\boldsymbol{Z}$.

The following corollary is an immediate consequence of Theorems 1.1 and 1.5.

COROLLARY 1.6. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a cofinite inverse system of compact Hausdorff spaces with limit space* X. *Then there exists a cofinite rectangular inverse system of compact polyhedra* $\boldsymbol{Z} = (Z_\lambda^\tau, r_{\lambda\lambda'}^{\tau\tau'}, \Lambda \times T)$ *such that, for every* $\lambda \in \Lambda$, *the limit space of the vertical system* $\boldsymbol{Z}_\lambda$ *is* X_λ *and the limit space of* $\boldsymbol{Z}$ *is* X.

THEOREM 1.7. *Let* $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ *be a cofinite inverse system of topological spaces. Then there exists a cofinite rectangular inverse system of ANRs for metric spaces* (*of polyhedra*) $\boldsymbol{Z} = (Z_\lambda^\tau, r_{\lambda\lambda'}^{\tau\tau'}, \Lambda \times T)$ *and there exist resolutions* $\boldsymbol{u}_\lambda\colon X_\lambda \to \boldsymbol{Z}_\lambda$, $\lambda \in \Lambda$, *such that* $\boldsymbol{u}_\lambda p_{\lambda\lambda'} = \boldsymbol{r}_{\lambda\lambda'} \boldsymbol{u}_{\lambda'}$, *for* $\lambda \leq \lambda'$.

REMARK 1.8. For a cofinite directed set Λ, denote by $\mathcal{S}_\Lambda$ the category, whose objects are inverse systems of topological spaces indexed by Λ and whose morphisms are level mappings between such systems. Then the horizontal systems $\boldsymbol{Z}^\tau$, $\tau \in T$, of a cofinite rectangular system $\boldsymbol{Z}$ of topological spaces and the level mappings $\boldsymbol{r}^{\tau\tau'}\colon \boldsymbol{Z}^{\tau'} \to \boldsymbol{Z}^\tau$, $\tau \leq \tau'$, between horizontal systems of $\boldsymbol{Z}$ form an inverse system $\boldsymbol{Z}^*$ in $\mathcal{S}_\Lambda$. Moreover, the mappings

$\boldsymbol{u}^\tau\colon \boldsymbol{X} \to \boldsymbol{Z}^\tau$, $\tau \in T$, form a mapping $\boldsymbol{u}^*\colon \boldsymbol{X} \to \boldsymbol{Z}^*$ in $\mathcal{S}_\Lambda$. By Theorem 1.7, $\boldsymbol{u}^*$ can be viewed as a resolution of $\boldsymbol{X}$.

THEOREM 1.9. *Let $\boldsymbol{X}, \boldsymbol{Z}$ and $\boldsymbol{u}_\lambda$, $\lambda \in \Lambda$, have all the properties stated in Theorem 1.7. If $\boldsymbol{p}\colon X \to \boldsymbol{X}$ is a resolution of X, then the induced mapping $\boldsymbol{v}\colon X \to \boldsymbol{Z}$ is also a resolution of X.*

Corollary 1.6 was conjectured at the 1998 Dubrovnik topology conference by Yu.T. Lisitsa (see [1]). The result was also quoted in Lisitsa's recent paper on strong shape of compact Hausdorff spaces ([2], Theorem 1). For the proof he refers to [6]. However, in [6] (and also in [7]) only a weaker assertion was proved, because the systems $\boldsymbol{Z}$ constructed in those papers were not rectangular. More precisely, the index set of $\boldsymbol{Z}$ was the disjoint union $\coprod_{\lambda \in \Lambda} T_\lambda$ of a family of (possibly different) cofinite sets T_λ. A weaker version of Corollary 1.6 appears in [3] as Theorem 4. In that version strict commutativity of some of the diagrams is replaced by commutativity up to homotopy. Theorems 1.7 and 1.9 generalize Theorems 1.1 and 1.5 to arbitrary topological spaces. It is well known that in non-compact situations inverse limits must be replaced by resolutions (see [5] and [4]).

The author wishes to thank the referee for suggesting that methods developed in this paper and a result of N. Uglešić [7] could be used to obtain the polyhedral version of Theorem 1.7.

2. SOME FACTS ON LIMITS AND RESOLUTIONS

In this section we recall some facts concerning inverse systems of topological spaces. We consider systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, indexed by directed ordered sets Λ. If the index set Λ is cofinite, i.e., every element has only finitely many predecessors, we speak of a cofinite inverse system. A mapping $\boldsymbol{f} = (f, f_\mu)\colon \boldsymbol{X} \to \boldsymbol{Y}$ between inverse systems $\boldsymbol{X}$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ consists of a function $f\colon M \to \Lambda$ and of mappings $f_\mu\colon X_{f(\mu)} \to Y_\mu$, $\mu \in M$, such that, for $\mu \le \mu'$, there exists a $\lambda \ge f(\mu), f(\mu')$ such that

$$(2.1) \qquad f_\mu p_{f(\mu)\lambda} = q_{\mu\mu'} f_{\mu'} p_{f(\mu')\lambda}.$$

A mapping $\boldsymbol{f} = (f, f_\mu)$ is called special if f is increasing and

$$(2.2) \qquad f_\mu p_{f(\mu)f(\mu')} = q_{\mu\mu'} f_{\mu'}.$$

A special mappig, where f is the identity function, is called a level mapping and is denoted by $\boldsymbol{f} = (f_\lambda)$. Composition $\boldsymbol{h} = \boldsymbol{g}\boldsymbol{f}$ of mappings $\boldsymbol{f}$ and $\boldsymbol{g} = (g, g_\mu)\colon \boldsymbol{Y} \to \boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$ consists of the function $h = fg\colon M \to \Lambda$ and of the mappings $h_\nu = g_\nu f_{g(\nu)}\colon X_{h(\nu)} \to Z_\nu$. The identity mapping $\boldsymbol{1_Y}\colon \boldsymbol{Y} \to \boldsymbol{Y}$ consists of the identity function $1_M\colon M \to M$ and of the identity mappings $1_\mu\colon Y_\mu \to Y_\mu$.

A mapping $\boldsymbol{f} = (f_\mu)\colon X \to \boldsymbol{Y}$ of a space X to a system $\boldsymbol{Y}$ consists of mappings $f_\mu\colon X \to Y_\mu$, $\mu \in M$, such that $q_{\mu\mu'} f_{\mu'} = f_\mu$, for $\mu \le \mu'$. A limit of

a system $\boldsymbol{Y}$ is a mapping $\boldsymbol{q} = (q_\mu)\colon Y \to \boldsymbol{Y}$, which has the following universal property: for any mapping $\boldsymbol{f}\colon X \to \boldsymbol{Y}$, there exists a unique mapping $f\colon X \to Y$ such that $\boldsymbol{q}f = \boldsymbol{f}$. Limits always exist and are unique up to natural isomorphism. If in a system $\boldsymbol{Y}$ all terms Y_μ are compact Hausdorff spaces, then so is the limit space Y. Every compact Hausdorff space is the limit space of an inverse system of compact polyhedra.

Resolutions of a space X are mappings $\boldsymbol{p} = (p_\lambda)\colon X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, which satisfy two additional conditions (see [5], I,6.1, or [4], II,6):

(B1) For every normal (numerable) covering $\mathcal{U}$ of X, there is a $\lambda \in \Lambda$ and there is a normal covering $\mathcal{U}_\lambda$ of X_λ such that the covering $p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$.

(B2) For every $\lambda \in \Lambda$ and every normal covering $\mathcal{U}_\lambda$ of X_λ, there is a $\lambda' \geq \lambda$ such that

$$(2.3) \qquad p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{St}(p_\lambda(X), \mathcal{U}_\lambda).$$

If all X_λ are normal spaces, condition (B2) can be replaced by the equivalent condition:

(B2)′ For every $\lambda \in \Lambda$ and every open neighborhood U of the closure $\overline{p_\lambda(X)}$ in X_λ, there is a $\lambda' \geq \lambda$ such that

$$(2.4) \qquad p_{\lambda\lambda'}(X_{\lambda'}) \subseteq U.$$

It is well known that, for X_λ Tychonoff and X topologically complete (e.g., for X paracompact), every resolution $\boldsymbol{p}\colon X \to \boldsymbol{X}$ is an inverse limit. Furthermore, if X and X_λ are compact Hausdorff spaces, also the converse holds, i.e., if $\boldsymbol{p}\colon X \to \boldsymbol{X}$ is an inverse limit, then $\boldsymbol{p}$ is a resolution. It is also known that every topological space X admits an ANR-resolution (a polyhedral resolution) $\boldsymbol{p}\colon X \to \boldsymbol{X}$, i.e., a resolution such that all X_λ are ANR's for metric spaces (are polyhedra with the CW-topology). For the proofs of these results see, e.g., [5] or [4].

The following cofinality lemma will prove useful in the next section.

LEMMA 2.1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be inverse systems of topological spaces and let $\boldsymbol{f} = (f, f_\mu)\colon \boldsymbol{X} \to \boldsymbol{Y}$ be a special mapping of systems such that the function $f\colon M \to \Lambda$ is cofinal and every $f_\mu\colon X_{f(\mu)} \to Y_\mu$, $\mu \in M$, is a homeomorphism. If $\boldsymbol{p}\colon X \to \boldsymbol{X}$ is a resolution, then the composition $\boldsymbol{q} = \boldsymbol{f}\boldsymbol{p}\colon X \to \boldsymbol{Y}$ is also a resolution.*

PROOF. If $\boldsymbol{p}$ consists of mappings $p_\lambda\colon X \to X_\lambda$, $\lambda \in \Lambda$, $\boldsymbol{q}$ consists of mappings $q_\mu = f_\mu p_{f(\mu)}\colon X \to Y_\mu$, $\mu \in M$. Let $\mathcal{U}$ be a normal covering of X. Since $\boldsymbol{p}$ is a resolution, there exist a $\lambda \in \Lambda$ and a normal covering $\mathcal{U}_\lambda$ of X_λ such that $(p_\lambda)^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$. Since f is cofinal, there exists a $\mu \in M$ such that $\lambda \leq f(\mu)$. Since $f_\mu\colon X_{f(\mu)} \to Y_\mu$ is a homeomorphism, there exists a normal covering $\mathcal{V}_\mu$ of Y_μ such that $f_\mu^{-1}(\mathcal{V}_\mu) = p_{\lambda f(\mu)}^{-1}(\mathcal{U}_\lambda)$. Consequently,

$q_\mu^{-1}(\mathcal{V}_\mu) = p_{f(\mu)}^{-1}f_\mu^{-1}(\mathcal{V}_\mu) = p_{f(\mu)}^{-1}p_{\lambda f(\mu)}^{-1}(\mathcal{U}_\lambda) = p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$, which is property (B1) for $\boldsymbol{q}$.

To verify property (B2), consider $\mu \in M$ and a normal covering $\mathcal{V}_\mu$ of Y_μ. Then consider the normal covering $f_\mu^{-1}(\mathcal{V}_\mu)$ of $X_{f(\mu)}$. By property (B2) for $\boldsymbol{p}$, there exists an index $\lambda \geq f(\mu)$ such that $p_{f(\mu)\lambda}(X_\lambda) \subseteq \mathrm{St}(p_{f(\mu)}(X), f_\mu^{-1}(\mathcal{V}_\mu))$. Since f is inceasing and cofinal, there exists a $\mu' \geq \mu$ such that $f(\mu') \geq \lambda$. Moreover, $q_{\mu\mu'}f_{\mu'} = f_\mu p_{f(\mu)f(\mu')}$, because $\boldsymbol{f}$ is a special mapping. Since $f_{\mu'}$ is a homeomorphism, $Y_{\mu'} = f_{\mu'}(X_{f(\mu')})$. Therefore, taking into account the fact that $q_\mu = f_\mu p_{f(\mu)}$, one concludes that

$$(2.5) \quad \begin{aligned} q_{\mu\mu'}(Y_{\mu'}) &= q_{\mu\mu'}f_{\mu'}(X_{f(\mu')}) = f_\mu p_{f(\mu)f(\mu')}(X_{f(\mu')}) \\ &= f_\mu p_{f(\mu)\lambda}p_{\lambda f(\mu')}(X_{f(\mu')}) \subseteq f_\mu p_{f(\mu)\lambda}(X_\lambda) \\ &\subseteq f_\mu(\mathrm{St}(p_{f(\mu)}(X), f_\mu^{-1}(\mathcal{V}_\mu))) = \mathrm{St}(q_\mu(X), \mathcal{V}_\mu), \end{aligned}$$

which establishes property (B2) for $\boldsymbol{q}$. □

3. Proof of Theorem 1.7 in the ANR case

The proof consists of the proof of Theorem 2 in [6], enriched by additional arguments. We give here the entire proof divided in eight steps, referring only for some details in some of the steps to the corresponding parts of [6].

STEP 1. Consider the family $(X_\lambda, \lambda \in \Lambda)$ of all terms of $\boldsymbol{X}$. For every $\lambda \in \Lambda$, choose an ANR-resolution $\boldsymbol{e}_\lambda = (e_\lambda^\delta)\colon X_\lambda \to \boldsymbol{Y}_\lambda = (Y_\lambda^\delta, e_\lambda^{\delta\delta'}, \Delta_\lambda)$. Let $M = \prod_{\lambda\in\Lambda} \Delta_\lambda$ be the product of the ordered sets Δ_λ, i.e., elements of M are functions μ with domain Λ and values $\mu(\lambda) \in \Delta_\lambda$, for all $\lambda \in \Lambda$. Put $\mu \leq \mu'$ whenever $\mu(\lambda) \leq \mu'(\lambda)$, for all $\lambda \in \Lambda$. Note that M is a directed ordered set. Now put $X_\lambda^\mu = Y_\lambda^{\mu(\lambda)}$, $p_\lambda^{\mu\mu'} = e_\lambda^{\mu(\lambda)\mu'(\lambda)}$ and $p_\lambda^\mu = e_\lambda^{\mu(\lambda)}$. Then $\boldsymbol{X}_\lambda = (X_\lambda^\mu, p_\lambda^{\mu\mu'}, M)$ is an inverse system and the mappings $p_\lambda^\mu\colon X_\lambda \to X_\lambda^\mu$ form a mapping $\boldsymbol{p}_\lambda\colon X_\lambda \to \boldsymbol{X}_\lambda$. Let $\boldsymbol{f}\colon \boldsymbol{Y}_\lambda \to \boldsymbol{X}_\lambda$ be the special mapping give by the increasing function $f\colon M \to \Delta_\lambda$, where $f(\mu) = \mu(\lambda)$, and by the identity mappings $f^\mu\colon Y_\lambda^{f(\mu)} = Y_\lambda^{\mu(\lambda)} \to Y_\lambda^{\mu(\lambda)} = X_\lambda^\mu$. Note that f is cofinal, because, for any $\delta \in \Delta_\lambda$, any function μ such that $\mu(\lambda) = \delta$ and $\mu(\lambda') \in \Delta_{\lambda'}$, for $\lambda' \in \Lambda$, belongs to M and has the property that $f(\mu) = \delta$. Since $\boldsymbol{e}_\lambda\colon X_\lambda \to \boldsymbol{Y}_\lambda$ is a resolution, Lemma 2.1 implies that also $\boldsymbol{f}\boldsymbol{e}_\lambda\colon Y_\lambda \to \boldsymbol{X}_\lambda$ is a resolution. However, $\boldsymbol{f}\boldsymbol{e}_\lambda$ consists of mappings $f^\mu e_\lambda^{f(\mu)} = e_\lambda^{\mu(\lambda)} = p_\lambda^\mu$, which shows that $\boldsymbol{f}\boldsymbol{e}_\lambda = \boldsymbol{p}_\lambda$ and thus, $\boldsymbol{p}_\lambda$ is a resolution. Note that all systems $\boldsymbol{X}_\lambda$, $\lambda \in \Lambda$, are indexed by the same set M.

STEP 2. For $(\lambda, \mu) \in \Lambda \times M$, consider the product $Y_\lambda^\mu = \prod_{\zeta\leq\lambda} X_\zeta^\mu$. Since Λ is cofinite, every set $\{\zeta \in \Lambda | \zeta \leq \lambda\}$ is finite and therefore, Y_λ^μ is an ANR. Also consider the mappings $q_\lambda^{\mu\mu'} = \prod_{\zeta\leq\lambda} p_\zeta^{\mu\mu'}\colon Y_\lambda^{\mu'} \to Y_\lambda^\mu$, defined for $\mu \leq \mu'$. Then $\boldsymbol{Y}_\lambda = (Y_\lambda^\mu, q_\lambda^{\mu\mu'}, M)$ is an inverse system. Moreover, the mappings $p_\zeta^\mu p_{\zeta\lambda}\colon X_\lambda \to X_\lambda^\mu$, $\zeta \leq \lambda$, into factors of Y_λ^μ, determine mappings $q_\lambda^\mu\colon X_\lambda \to Y_\lambda^\mu$, which form a mapping $\boldsymbol{q}_\lambda\colon X_\lambda \to \boldsymbol{Y}_\lambda$. We claim that $\boldsymbol{q}_\lambda$

has property (B1) (see Lemma 5 in [6]). Indeed, If $\mathcal{U}$ is a normal covering of X_λ, by property (B1) for $\boldsymbol{p}_\lambda$, there exists an index $\mu \in M$ and an open covering $\mathcal{V}$ of X_λ^μ such that $(p_\lambda^\mu)^{-1}(\mathcal{V})$ refines $\mathcal{U}$. Consider the open covering $\mathcal{W}$ of Y_λ^μ, consisting of the sets $W = (\prod_{\zeta < \lambda} X_\zeta^\mu) \times V$, where $V \in \mathcal{V}$. Clearly, $(q_\lambda^\mu)^{-1}(\mathcal{W}) = (p_\lambda^\mu)^{-1}(\mathcal{V})$ and thus, it refines $\mathcal{U}$.

STEP 3. Order $\Lambda \times M$ by putting $\nu = (\lambda, \mu) \leq (\lambda', \mu') = \nu'$, whenever $\lambda \leq \lambda'$ and $\mu \leq \mu'$. Define a rectangular inverse system of ANR's $\boldsymbol{Y} = (Y_\nu, q_{\nu\nu'}, \Lambda \times M)$ as follows. Put $Y_\nu = Y_\lambda^\mu$ and for $\nu \leq \nu'$, let $q_{\nu\nu'} \colon Y_{\lambda'}^{\mu'} \to Y_\lambda^\mu$ be the composition of two mappings. The first one is the natural projection $Y_{\lambda'}^{\mu'} = \prod_{\zeta \leq \lambda'} X_\zeta^{\mu'} \to \prod_{\zeta \leq \lambda} X_\zeta^{\mu'} = Y_\lambda^{\mu'}$ and the second one is the mapping $q_\lambda^{\mu\mu'} \colon Y_\lambda^{\mu'} \to Y_\lambda^\mu$. Note that

(3.1) $$q_{\nu\nu'} q_{\lambda'}^{\mu'} = q_\lambda^\mu p_{\lambda\lambda'}.$$

This is so because, for $\zeta \leq \lambda$, the ζ-coordinate mapping of the left side of (3.1) equals $p_\zeta^{\mu\mu'} p_\zeta^{\mu'} p_{\zeta\lambda'} = p_\zeta^\mu p_{\zeta\lambda'}$, while the ζ-coordinate mapping of the right side of (3.1) equals $p_\zeta^\mu p_{\zeta\lambda} p_{\lambda\lambda'} = p_\zeta^\mu p_{\zeta\lambda'}$. Also note that, for $\lambda = \lambda'$, $q_{\nu\nu'} = q_\lambda^{\mu\mu'}$.

STEP 4. For $\lambda \in \Lambda$, let Γ_λ be the set of all pairs $\gamma = (\mu, G)$, where $\mu \in M$ and G is an open neighborhood of the closure $\overline{q_\lambda^\mu(X_\lambda)}$ in Y_λ^μ. Since Y_λ^μ is an ANR, so is G. Order Γ_λ by putting $\gamma \leq \gamma' = (\mu', G')$, whenever $\mu \leq \mu'$ and $q_\lambda^{\mu\mu'}(G') \subseteq G$. Put $S_\lambda^\gamma = G$ and define $s_\lambda^{\gamma\gamma'} \colon S_\lambda^{\gamma'} \to S_\lambda^\gamma$, $\gamma \leq \gamma'$, as the restriction $q_\lambda^{\mu\mu'}|G' \colon G' \to G$ of $q_\lambda^{\mu\mu'}$. Then $\boldsymbol{S}_\lambda = (S_\lambda^\gamma, s_\lambda^{\gamma\gamma'}, \Gamma_\lambda)$ is an inverse system of ANRs. The restrictions of the mappings $q_\lambda^\mu \colon X_\lambda \to Y_\lambda^\mu$ to the codomains G are mappings $s_\lambda^\gamma \colon X_\lambda \to S_\lambda^\gamma$, which form a mapping $\boldsymbol{s}_\lambda \colon X_\lambda \to \boldsymbol{S}_\lambda$. Moreover, $\boldsymbol{s}_\lambda$ is a resolution (see Lemma 2 in [6] or Lemma 6.24 of [4]). Indeed, if $\mathcal{U}$ is a normal covering of X_λ, then (B1) for $\boldsymbol{q}_\lambda$ yields a $\mu \in M$ and a normal covering $\mathcal{V}$ of Y_λ^μ such that $(q_\lambda^\mu)^{-1}(\mathcal{V})$ refines $\mathcal{U}$. However, the pair $\gamma = (\mu, Y_\lambda^\mu)$ belongs to Γ_λ, $S_\lambda^\gamma = Y_\lambda^\mu$ and $s_\lambda^\gamma = q_\lambda^\mu$. Therefore, $\mathcal{V}$ is an open covering of S_λ^γ and $(s_\lambda^\gamma)^{-1}(\mathcal{V})$ refines $\mathcal{U}$, which proves (B1) for $\boldsymbol{s}_\lambda$. Now assume that $\gamma = (\mu, G) \in \Gamma_\lambda$ and U is an open neighborhood of the closure of $s_\lambda^\gamma(X_\lambda)$ in $S_\lambda^\gamma = G$. Clearly, this closure coincides with the closure of $q_\lambda^\mu(X_\lambda)$ in Y_λ^μ. Therefore, $\gamma' = (\mu, U)$ also belongs to Γ_λ and $\gamma \leq \gamma'$, because $q_\lambda^{\mu\mu}(U) = U \subseteq G$. However, $Y_\lambda^{\gamma'} = U$ and $s_\lambda^{\gamma\gamma'}(Y_\lambda^{\gamma'}) = q_\lambda^{\mu\mu}(U) = U$, which shows that $\boldsymbol{s}_\lambda$ also has property (B2)$'$.

STEP 5. Consider the disjoint union $\Gamma = \coprod_{\lambda \in \Lambda} \Gamma_\lambda$ and order it by putting $\gamma = (\mu, G) \leq (\mu', G') = \gamma'$, for $\gamma \in \Gamma_\lambda$, $\gamma' \in \Gamma_{\lambda'}$, whenever $\nu = (\lambda, \mu) \leq (\lambda', \mu') = \nu'$ in $\Lambda \times M$ and $q_{\nu\nu'}(G') \subseteq G$. Note that the restriction of this ordering to Γ_λ coincides with the previously defined ordering on Γ_λ. Also note that for every $\gamma \in \Gamma_\lambda$ and every $\lambda' \geq \lambda$, there exists a $\gamma' \in \Gamma_{\lambda'}$ such that $\gamma \leq \gamma'$ in Γ. Indeed, if $\gamma = (\mu, G)$, then G is an open neighborhood of $\overline{q_\lambda^\mu(X_\lambda)}$ in Y_λ^μ and $\nu' = (\lambda', \mu) \geq \nu = (\lambda, \mu)$. Since, by (3.1), $q_\lambda^\mu p_{\lambda\lambda'} = q_{\nu\nu'} q_{\lambda'}^\mu$,

we see that $q_{\nu\nu'}(\overline{q^\mu_{\lambda'}(X_{\lambda'})}) \subseteq \overline{q_{\nu\nu'}(q^\mu_{\lambda'}(X_{\lambda'}))} = \overline{q^\mu_\lambda(p_{\lambda\lambda'}(X_{\lambda'}))} \subseteq \overline{q^\mu_\lambda(X_\lambda)} \subseteq G$.
Consequently, there exists an open neighborhood G' of $\overline{q^\mu_{\lambda'}(X_{\lambda'})}$ in $Y^{\mu'}_{\lambda'}$ such
that $q_{\nu\nu'}(G') \subseteq G$. This means that $\gamma' = (\mu, G') \in \Gamma_{\lambda'}$ has the property
that $\gamma \leq \gamma'$ in Γ. It is now easy to see that Γ is directed. Indeed, if $\gamma_i = (\mu_i, G_i) \in \Gamma_{\lambda_i}$, for $i = 1, 2$, and we choose $\lambda' \geq \lambda_1, \lambda_2$, then there exist
elements $\gamma'_1, \gamma'_2 \in \Gamma_{\lambda'}$ such that $\gamma_i \leq \gamma'_i$, $i = 1, 2$. Since $\Gamma_{\lambda'}$ is directed, there
exists an element $\gamma' \in \Gamma_{\lambda'}$ such that $\gamma'_i \leq \gamma'$, $i = 1, 2$, and thus, $\gamma_i \leq \gamma'$,
$i = 1, 2$.

We now define an inverse system $\boldsymbol{S} = (S_\gamma, s_{\gamma\gamma'}, \Gamma)$, consisting of ANRs,
as follows. If $\gamma = (\mu, G) \in \Gamma_\lambda$, we put $S_\gamma = S^\gamma_\lambda = G$ and if $\gamma' = (\mu', G') \in \Gamma_{\lambda'}$,
$\gamma \leq \gamma'$, we define $s_{\gamma\gamma'} \colon S_{\gamma'} \to S_\gamma$ as the restriction $q_{\nu\nu'}|G' \colon G' \to G$ of
$q_{\nu\nu'} \colon Y^{\mu'}_{\lambda'} \to Y^\mu_\lambda$. Note that, for $\lambda = \lambda'$, $s_{\gamma\gamma'} = s^{\gamma\gamma'}_\lambda$, because $s_{\gamma\gamma'}$ is the
restriction of $q_{\nu\nu'}$ to the domain G' and the codomain G, while $s^{\gamma\gamma'}_\lambda$ is the
corresponding restriction of $q^{\mu\mu'}_\lambda$. However, for $\lambda = \lambda'$, $q_{\nu\nu'} = q^{\mu\mu'}_\lambda$. Also note
that (3.1) implies

$$(3.2) \qquad\qquad s_{\gamma\gamma'} s^{\gamma'}_{\lambda'} = s^\gamma_\lambda p_{\lambda\lambda'}.$$

In the special case when $\lambda = \lambda'$, (3.2) becomes $s_{\gamma\gamma'} s^{\gamma'}_\lambda = s^\gamma_\lambda$.

STEP 6. Let $\tilde{T}$ be the set of all functions $\tau \colon \Lambda \to \Gamma$ such that $\tau(\lambda) \in \Gamma_\lambda$,
for $\lambda \in \Lambda$. Order $\tilde{T}$ by putting $\tau \leq \tau'$, whenever $\tau(\lambda) \leq \tau'(\lambda)$, for all $\lambda \in \Lambda$.
The subset $T \subseteq \tilde{T}$, consisting of all increasing functions τ, is cofinal in $\tilde{T}$, i.e.,
for an arbitrary $\tau_0 \in \tilde{T}$, there exists an increasing function $\tau \colon \Lambda \to \Gamma$ such
that $\tau(\lambda) \in \Gamma_\lambda$ and $\tau_0(\lambda) \leq \tau(\lambda)$, for all $\lambda \in \Lambda$. Indeed, we can construct such
a function τ, defining $\tau(\lambda)$, by induction on the number of predecessors of λ.
This number is finite, because Λ is cofinite. If the number of predecessors of
λ is 1, we put $\tau(\lambda) = \tau_0(\lambda)$. Now assume that $\lambda_1, \ldots, \lambda_n$ are all predecessors
of λ and that we have already defined $\tau(\lambda_i) \in \Gamma_{\lambda_i}$ so that $\tau_0(\lambda_i) \leq \tau(\lambda_i)$,
for $i \in \{1, \ldots, n\}$. Since $\lambda_i \leq \lambda$, by Step 5, there are elements $\gamma_i \in \Gamma_\lambda$ such
that $\tau(\lambda_i) \leq \gamma_i$ in Γ. Now put $\tau(\lambda) = \gamma$, where $\gamma \in \Gamma_\lambda$ and $\gamma \geq \gamma_i$, for
$i \in \{1, \ldots, n\}$, and $\gamma \geq \tau_0(\lambda)$. Such a γ exists because Γ_λ is directed. It is
now easy to conclude that the set T is directed. Indeed, $\tilde{T}$ is directed, because
the sets Γ_λ are directed. Therefore, if $\tau_1, \tau_2 \in T \subseteq \tilde{T}$, there exists a function
$\tau_0 \in \tilde{T}$ such that $\tau_1, \tau_2 \leq \tau_0$. By cofinality of T in $\tilde{T}$, there is a function $\tau \in T$
such that $\tau_0 \leq \tau$ and thus, $\tau_1, \tau_2 \leq \tau$.

We now define an inverse system $\boldsymbol{Z} = (Z^\tau_\lambda, r^{\tau\tau'}_{\lambda\lambda'}, \Lambda \times T)$, whose index
set $\Lambda \times T$ is the product of directed sets Λ and T. For $(\lambda, \tau) \in \Lambda \times T$,
put $Z^\tau_\lambda = S_{\tau(\lambda)}$ and for $(\lambda, \tau) \leq (\lambda', \tau')$, put $r^{\tau\tau'}_{\lambda\lambda'} = s_{\tau(\lambda)\tau'(\lambda')}$. Note that
$\tau(\lambda) \leq \tau'(\lambda)$, because $\tau \leq \tau'$, and $\tau'(\lambda) \leq \tau'(\lambda')$, because τ' is an increasing
function. Consequently, $\tau(\lambda) \leq \tau'(\lambda')$ and the mapping $s_{\tau(\lambda)\tau'(\lambda')}$ is well
defined. It readily follows that (1.1) is fulfilled and thus, $\boldsymbol{Z}$ is a rectangular
inverse system of ANRs.

STEP 7. For a fixed $\lambda \in \Lambda$, consider the system $\boldsymbol{S}_\lambda = (S_\lambda^\gamma, s_\lambda^{\gamma\gamma'}, \Gamma_\lambda)$ and the resolution $\boldsymbol{s}_\lambda \colon X_\lambda \to \boldsymbol{S}_\lambda$. Also consider the function $f \colon T \to \Gamma_\lambda$, defined by $f(\tau) = \tau(\lambda)$. This function is increasing and cofinal. Indeed, $\tau \le \tau'$ implies $\tau(\lambda) \le \tau'(\lambda)$ and for an arbitrary element $\gamma \in \Gamma_\lambda$, there exists a function $\tau_0 \in \tilde{T}$ such that $\tau_0(\lambda) = \gamma$. Consequently, by Step 5, there also exists a $\tau \in T$ such that $\tau \ge \tau_0$ and thus, $f(\tau) = \tau(\lambda) \ge \tau_0(\lambda) = \gamma$. Now consider the special mapping $\boldsymbol{f} = (f, f^\tau) \colon \boldsymbol{S}_\lambda \to \boldsymbol{Z}_\lambda$ to the vertical system $\boldsymbol{Z}_\lambda = (Z_\lambda^\tau, r_\lambda^{\tau\tau'}, T)$ of $\boldsymbol{Z}$ at level λ, where $f^\tau \colon S_\lambda^{f(\tau)} = S_\lambda^{\tau(\lambda)} \to S_\lambda^{\tau(\lambda)} = S_{\tau(\lambda)} = Z_\lambda^\tau$ is the identity mapping. That $\boldsymbol{f}$ is indeed a special mapping follows from $f^\tau s_\lambda^{f(\tau)f(\tau')} = s_\lambda^{\tau(\lambda)\tau'(\lambda)} = s_{\tau(\lambda)\tau'(\lambda)} = r_\lambda^{\tau\tau'} = r_\lambda^{\tau\tau'} f^{\tau'}$, for $\tau \le \tau'$. The composition $\boldsymbol{u}_\lambda = \boldsymbol{f}\boldsymbol{s}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$ consists of mappings $u_\lambda^\tau = f^\tau s_\lambda^{f(\tau)} = s_\lambda^{\tau(\lambda)} \colon X_\lambda \to S_\lambda^{\tau(\lambda)} = Z_\lambda^\tau$. An application of Lemma 2.1 shows that $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$ is a resolution. The mappings u_λ^τ satisfy condition (1.3), because $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$ is a mapping. They also satisfy condition (1.2), because it assumes the form $s_\lambda^{\tau(\lambda)} p_{\lambda\lambda'} = s_{\tau(\lambda)\tau(\lambda')} s_{\lambda'}^{\tau(\lambda')}$, which is a special case of (3.2).

We have thus, constructed all objects and proved all properties required by Theorem 1.7 with one exception, $\Lambda \times T$ need not be cofinite. We will obtain this last property by applying to the already obtained construction a known procedure which converts arbitrary systems to cofinite systems (see Lemma 6.31 of [4]).

STEP 8. Let $\boldsymbol{Z}^\star = (Z_\lambda^{\star\alpha}, r_{\lambda\lambda'}^{\star\alpha\alpha'}, \Lambda \times T^\star)$ be a new rectangular system defined as follows. $T^\star$ is the set of all finite subsets α of T (endowed with the ordering inherited from T) such that α has a terminal element $\overline{\alpha} \in T$. Because of anti-symmetry, $\overline{\alpha}$ is uniquely determined by α. The ordering $\le$ of $T^\star$ is given by the inclusion $\subseteq$. Clearly, $T^\star$ is cofinite. Note that $\alpha_1 \le \alpha_2$ implies $\overline{\alpha}_1 \le \overline{\alpha}_2$. Put $Z_\lambda^{\star\alpha} = Z_\lambda^{\overline{\alpha}}$ and $r_{\lambda\lambda'}^{\star\alpha\alpha'} = r_{\lambda\lambda'}^{\overline{\alpha}\overline{\alpha}'}$. Also define mappings $u_\lambda^{\star\alpha} \colon X_\lambda \to Z_\lambda^{\star\alpha} = Z_\lambda^{\overline{\alpha}}$ by putting $u_\lambda^{\star\alpha} = u_\lambda^{\overline{\alpha}}$. Note that the analogues of (1.2) and (1.3) hold. Indeed, (1.2) assumes the form $u_\lambda^{\star\alpha} p_{\lambda\lambda'} = r_{\lambda\lambda'}^{\star\alpha} u_{\lambda'}^{\star\alpha}$, which is equivalent to $u_\lambda^{\overline{\alpha}} p_{\lambda\lambda'} = r_{\lambda\lambda'}^{\overline{\alpha}} u_{\lambda'}^{\overline{\alpha}}$. Similarly, (1.3) assumes the form $r_\lambda^{\star\alpha\alpha'} u_\lambda^{\star\alpha'} = u_\lambda^{\star\alpha}$, which is equivalent to $r_\lambda^{\overline{\alpha}\overline{\alpha}'} r_\lambda^{\overline{\alpha}'} = u_\lambda^{\overline{\alpha}}$.

It remains to prove that the mapping $\boldsymbol{u}_\lambda^\star \colon X_\lambda \to \boldsymbol{Z}_\lambda^\star = (Z_\lambda^{\star\alpha}, r_\lambda^{\star\alpha\alpha'}, T^\star)$, formed by the mappings $u_\lambda^{\star\alpha}$, is a resolution. Consider the special mapping $\boldsymbol{f} = (f, f^\alpha) \colon \boldsymbol{Z}_\lambda \to \boldsymbol{Z}_\lambda^\star$, where $f \colon T^\star \to T$ is given by $f(\alpha) = \overline{\alpha}$ and $f^\alpha \colon Z_\lambda^{f(\alpha)} = Z_\lambda^{\overline{\alpha}} \to Z_\lambda^{\overline{\alpha}} = Z_\lambda^{\star\alpha}$ is the identity mapping. That $\boldsymbol{f} \colon \boldsymbol{Z}_\lambda \to \boldsymbol{Z}_\lambda^\star$ is a special mapping follows from the fact that f increases and $f^\alpha r_{\lambda\lambda'}^{\overline{\alpha}\overline{\alpha}'} = r_{\lambda\lambda'}^{\overline{\alpha}\overline{\alpha}'} = r_{\lambda\lambda'}^{\star\alpha\alpha'} f^{\alpha'}$. Moreover, f is cofinal, because, for every $\tau \in T$, the element $\alpha = \{\tau\} \in T^\star$ has the property that $f(\alpha) = \overline{\{\tau\}} = \tau$. Note that the composition $\boldsymbol{f}\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda^\star$ consists of mappings $f^\alpha u^{\overline{\alpha}} = u_\lambda^{\overline{\alpha}} = u_\lambda^{\star\alpha}$ and thus, coincides with $\boldsymbol{u}_\lambda^\star$. Since $\boldsymbol{u}_\lambda \colon X_\lambda \to \boldsymbol{Z}_\lambda$ is a resolution, an application of Lemma 2.1 yields the desired conclusion that also $\boldsymbol{u}_\lambda^\star$ is a resolution. $\square$

4. PROOF OF THEOREM 1.7 IN THE POLYHEDRAL CASE

The proof of Theorem 1.7 in the ANR case does not apply to the polyhedral case, because the Cartesian product of two (infinite) polyhedra need not be a polyhedron (CW-topology). Therefore, we give a different proof. It consists of a result of N. Uglešić (see[7], Theorem 2), here stated as Proposition 4.1, and of additional arguments, which we will describe in details and which are similar to some arguments used in the ANR case.

PROPOSITION 4.1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a cofinite inverse system of topological spaces. Then there exist cofinite polyhedral resolutions $\boldsymbol{q}_\lambda \colon X_\lambda \to \boldsymbol{Y}_\lambda$, $\lambda \in \Lambda$, and there exist mappings $\boldsymbol{v}_{\lambda\lambda'} \colon \boldsymbol{Y}_{\lambda'} \to \boldsymbol{Y}_\lambda$, $\lambda \le \lambda'$, such that $\boldsymbol{q}_\lambda p_{\lambda\lambda'} = \boldsymbol{v}_{\lambda\lambda'} \boldsymbol{q}_{\lambda'}$. Moreover, $\boldsymbol{v}_{\lambda\lambda'} \boldsymbol{v}_{\lambda'\lambda''} = \boldsymbol{v}_{\lambda\lambda''}$, for $\lambda \le \lambda' \le \lambda''$, and $\boldsymbol{v}_{\lambda\lambda} = \mathrm{id}$.*

The assertion that the polyhedral resolutions $\boldsymbol{q}_\lambda \colon X_\lambda \to \boldsymbol{Y}_\lambda$ are cofinite is not included in the statement of Theorem 2 of [7]. However, this follows from the proof of that theorem (see e.g., Remark 2 and a remark in the proof of Lemma 2 of [7]). Let $\boldsymbol{Y}_\lambda = (Y_\lambda^\mu, q_\lambda^{\mu\mu'}, M_\lambda)$ and let $\boldsymbol{q}_\lambda$ be given by mappings $q_\lambda^\mu \colon X_\lambda \to Y_\lambda^\mu$, $\mu \in M_\lambda$. Let the mappings $\boldsymbol{v}_{\lambda\lambda'}$ be given by functions $v_{\lambda\lambda'} \colon M_\lambda \to M_{\lambda'}$ and by mappings $v_{\lambda\lambda'}^\mu \colon Y_{\lambda'}^{v_{\lambda\lambda'}(\mu)} \to Y_\lambda^\mu$ such that, for $\mu, \mu' \in M_\lambda$, $\mu \le \mu'$, there exists an index $\nu \in M_{\lambda'}$, $\nu \ge v_{\lambda\lambda'}(\mu), v_{\lambda\lambda'}(\mu')$, for which

(4.1) $$v_{\lambda\lambda'}^\mu q_{\lambda'}^{v_{\lambda\lambda'}(\mu)\nu} = q_\lambda^{\mu\mu'} v_{\lambda\lambda'}^{\mu'} q_{\lambda'}^{v_{\lambda\lambda'}(\mu')\nu}.$$

Note that whenever (4.1) holds for some ν, then it also holds for every $\nu' \ge \nu$. To see this it suffices to compose (4.1) on the right with $q_{\lambda'}^{\nu\nu'}$.

Following the proof of Theorem 3 in [7], we now consider the coproduct $M = \coprod_{\lambda\in\Lambda} M_\lambda$ on which we define an ordering $\le$ as follows. If $\mu \in M_\lambda$, $\mu' \in M_{\lambda'}$, we put $\mu \le \mu'$ provided $\lambda \le \lambda'$ and for every $\mu^* \in M_\lambda$, $\mu^* \le \mu$, one has $v_{\lambda\lambda'}(\mu^*) \le \mu'$ (in particular, $v_{\lambda\lambda'}(\mu) \le \mu'$) and

(4.2) $$v_{\lambda\lambda'}^{\mu^*} q_{\lambda'}^{v_{\lambda\lambda'}(\mu^*)\mu'} = q_\lambda^{\mu^*\mu} v_{\lambda\lambda'}^\mu q_{\lambda'}^{v_{\lambda\lambda'}(\mu)\mu'}.$$

If $\lambda = \lambda'$, $\boldsymbol{v}_{\lambda\lambda'}$ is the identity mapping and thus, condition $v_{\lambda\lambda'}(\mu^*) \le \mu'$ becomes $\mu^* \le \mu'$ (in particular, $\mu \le \mu'$), while (4.1) assumes the form $q_\lambda^{\mu^*\mu'} = q_\lambda^{\mu^*\mu} q_\lambda^{\mu\mu'}$, which always holds. Consequently, the new ordering, restricted to M_λ, coincides with the original ordering on M_λ. This implies antisymmetry of $\le$ on M. Indeed, if $\mu \in M_\lambda$, $\mu' \in M_{\lambda'}$, $\mu \le \mu'$ and $\mu' \le \mu$, then $\lambda \le \lambda'$ and $\lambda' \le \lambda$ and thus, $\lambda = \lambda'$. Since $\le$ on M_λ is antisymmetric, it follows that $\mu = \mu'$. To prove transitivity of $\le$, assume that $\mu \le \mu'$, $\mu' \le \mu''$ and $\mu^* \le \mu$. Then the first inequality implies $v_{\lambda\lambda'}(\mu^*) \le \mu'$ and (4.2). Since also $v_{\lambda\lambda'}(\mu) \le \mu'$, the second inequality shows that $v_{\lambda'\lambda''}(v_{\lambda\lambda'}(\mu^*)) \le \mu''$,

(4.3) $$v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\mu^*)} q_{\lambda''}^{v_{\lambda'\lambda''}(v_{\lambda\lambda'}(\mu^*))\mu''} = q_{\lambda'}^{v_{\lambda\lambda'}(\mu^*)\mu'} v_{\lambda'\lambda''}^{\mu'} q_{\lambda''}^{v_{\lambda'\lambda''}(\mu')\mu''},$$

(4.4) $$v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\mu)}q_{\lambda''}^{v_{\lambda'\lambda''}(v_{\lambda\lambda'}(\mu))\mu''} = q_{\lambda'}^{v_{\lambda\lambda'}(\mu)\mu'}v_{\lambda'\lambda''}^{\mu'}q_{\lambda''}^{v_{\lambda'\lambda''}(\mu')\mu''}.$$

Note that $\boldsymbol{v}_{\lambda\lambda'}\boldsymbol{v}_{\lambda'\lambda''} = \boldsymbol{v}_{\lambda\lambda''}$ implies $v_{\lambda'\lambda''}v_{\lambda\lambda'} = v_{\lambda\lambda''}$, $v_{\lambda\lambda'}^{\mu^*}v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\mu^*)} = v_{\lambda\lambda''}^{\mu^*}$ and $v_{\lambda\lambda'}^{\mu}v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\mu)} = v_{\lambda\lambda''}^{\mu}$. Therefore, $v_{\lambda\lambda''}(\mu^*) \le \mu''$ and, by (4.3), (4.2) and (4.4),

(4.5) $$v_{\lambda\lambda''}^{\mu^*}q_{\lambda''}^{v_{\lambda\lambda''}(\mu^*)\mu''} = q_{\lambda}^{\mu^*\mu}v_{\lambda\lambda''}^{\mu}q_{\lambda''}^{v_{\lambda\lambda''}(\mu)\mu''}$$

and thus, $\mu \le \mu''$.

In the sequel we will need the following property of M: If $\mu \in M_\lambda$ and $\lambda \le \lambda'$, then there exists a $\mu' \in M_{\lambda'}$ such that $\mu \le \mu'$. Indeed, since M_λ is cofinite, the set of all predecessors of μ in M_λ is a finite set $\{\mu_1^*, \ldots \mu_k^*\}$. By property (4.1) of $\boldsymbol{v}_{\lambda\lambda'}$, for every $1 \le i \le k$, there is a $\nu_i \in M_{\lambda'}$ such that, for $1 \le i \le k$, $\gamma_{\lambda\lambda'}(\mu_i^*), \gamma_{\lambda\lambda'}(\mu) \le \nu_i$ and

(4.6) $$v_{\lambda\lambda'}^{\mu_i^*}q_{\lambda'}^{\gamma_{\lambda\lambda'}(\mu_i^*)\nu_i} = q_{\lambda}^{\mu_i^*\mu}v_{\lambda\lambda'}^{\mu}q_{\lambda'}^{\gamma_{\lambda\lambda'}(\mu)\nu_i}.$$

Choosing for μ' an element of $M_{\lambda'}$ such that $\nu_1, \ldots, \nu_k \le \mu'$, one concludes that, for all predecessors μ_i^* of μ, $v_{\lambda\lambda'}(\mu_i^*) \le \mu'$ and also

(4.7) $$v_{\lambda\lambda'}^{\mu_i^*}q_{\lambda'}^{v_{\lambda\lambda'}(\mu_i^*)\mu'} = q_{\lambda}^{\mu_i^*\mu}v_{\lambda\lambda'}^{\mu}q_{\lambda'}^{v_{\lambda\lambda'}(\mu)\mu'}.$$

By definition, this means that $\mu \le \mu'$.

We now define a new ordered set T as follows. Let $\tilde{T}$ be the set of all functions $\tilde{\tau}: \Lambda \to M$ such that $\tilde{\tau}(\lambda) \in M_\lambda$. Order $\tilde{T}$ by putting $\tilde{\tau} \le \tilde{\tau}'$ provided $\tilde{\tau}(\lambda) \le \tilde{\tau}'(\lambda)$, for all $\lambda \in \Lambda$. Clearly, $\tilde{T}$ is directed. Let T be the subset of $\tilde{T}$ consisting of all increasing functions $\tau: \Lambda \to M$.

For any function $\tilde{\tau} \in \tilde{T}$, there exists a function $\tau \in T$ such that $\tilde{\tau} \le \tau$. We define the values $\tau(\lambda)$, using induction by the number of predecessors of λ. If this number is 1, we put $\tau(\lambda) = \tilde{\tau}(\lambda)$. If we have already defined values $\tau(\lambda_i) \in M_{\lambda_i}$, $\tau(\lambda_i) \le \tilde{\tau}(\lambda_i)$, for all predecessors $\lambda_1, \lambda_2, \ldots, \lambda_k$ of λ, different from λ, we choose (using the above stated property of M) elements $\mu_i \in M_\lambda$ such that $\tau(\lambda_i) \le \mu_i$. Then we take for $\tau(\lambda)$ any element $\mu \in M_\lambda$ such that $\mu \ge \mu_1, \ldots, \mu_k, \tilde{\tau}(\lambda)$. By the transitivity of $\le$, $\tau(\lambda_i) \le \tau(\lambda)$ and $\tilde{\tau}(\lambda) \le \tau(\lambda)$. The first property implies that τ is an increasing functions from $\tilde{T}$, i.e., $\tau \in T$, and the second property implies that $\tilde{\tau} \le \tau$. It is now easy to see that T is a directed set. Indeed, let $\tau_1, \tau_2 \in T \subseteq \tilde{T}$. Clearly, there exists a function $\tilde{\tau} \in \tilde{T}$ such that $\tau_1, \tau_2 \le \tilde{\tau}$. If we choose $\tau \in T$ such that $\tilde{\tau} \le \tau$, then $\tau_1, \tau_2 \le \tau$.

We will now define a rectangular system of polyhedra $\boldsymbol{Z} = (Z_\lambda^\tau, r_{\lambda\lambda'}^{\tau\tau'}, \Lambda \times T)$. For $\lambda \in \Lambda$, $\tau \in T$, put $Z_\lambda^\tau = Y_\lambda^{\tau(\lambda)}$. For $\lambda \le \lambda'$, $\tau \in T$, we define $r_{\lambda\lambda'}^\tau: Z_{\lambda'}^\tau \to Z_\lambda^\tau$ by putting

(4.8) $$r_{\lambda\lambda'}^\tau = v_{\lambda\lambda'}^{\tau(\lambda)}q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')},$$

i.e., by taking for $r_{\lambda\lambda'}^\tau$ the composition of the mappings $q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')}: Z_{\lambda'}^\tau = Y_{\lambda'}^{\tau(\lambda')} \to Y_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))}$ and $v_{\lambda\lambda'}^{\tau(\lambda)}: Y_{\lambda'}^{\gamma_{\lambda\lambda'}(\tau(\lambda))} \to Y_\lambda^{\tau(\lambda)} = Z_\lambda^\tau$. Note that $r_{\lambda\lambda'}^\tau$

is well defined because τ is an increasing function and thus, $\lambda \leq \lambda'$ implies $\tau(\lambda) \leq \tau(\lambda')$, $\tau(\lambda) \in M_\lambda$ and $\tau(\lambda') \in M_{\lambda'}$. However, by the definition of the ordering $\leq$ in M, the latter inequality implies $v_{\lambda\lambda'}(\tau(\lambda)) \leq \tau(\lambda')$. For $\lambda \in \Lambda$ and $\tau \leq \tau'$ we define $r_\lambda^{\tau\tau'}\colon Z_\lambda^{\tau'} \to Z_\lambda^\tau$, by putting

$$(4.9) \qquad r_\lambda^{\tau\tau'} = q_\lambda^{\tau(\lambda)\tau'(\lambda)}.$$

Let us show that, for $\lambda \leq \lambda' \leq \lambda''$, $\tau \in T$, one has

$$(4.10) \qquad r_{\lambda\lambda'}^\tau r_{\lambda'\lambda''}^\tau = r_{\lambda\lambda''}^\tau.$$

Indeed, by (4.8),

$$(4.11) \qquad r_{\lambda\lambda'}^\tau r_{\lambda'\lambda''}^\tau = v_{\lambda\lambda'}^{\tau(\lambda)} q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')} v_{\lambda'\lambda''}^{\tau(\lambda')} q_{\lambda''}^{v_{\lambda'\lambda''}(\tau(\lambda'))\tau(\lambda'')}.$$

Moreover, since $v_{\lambda\lambda'}(\tau(\lambda)) \leq \tau(\lambda')$ and $\tau(\lambda') \leq \tau(\lambda'')$, (4.2) yields

$$(4.12) \qquad q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')} v_{\lambda'\lambda''}^{\tau(\lambda')} q_{\lambda''}^{v_{\lambda'\lambda''}(\tau(\lambda'))\tau(\lambda'')} = v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\tau(\lambda))} q_{\lambda''}^{v_{\lambda\lambda''}(\tau(\lambda))\tau(\lambda'')}.$$

Consequently,
$$(4.13)$$
$$r_{\lambda\lambda'}^\tau r_{\lambda'\lambda''}^\tau = v_{\lambda\lambda'}^{\tau(\lambda)} v_{\lambda'\lambda''}^{v_{\lambda\lambda'}(\tau(\lambda))} q_{\lambda''}^{v_{\lambda\lambda''}(\tau(\lambda))\tau(\lambda'')} = v_{\lambda\lambda''}^{\tau(\lambda)} q_{\lambda''}^{v_{\lambda\lambda''}(\tau(\lambda))\tau(\lambda'')} = r_{\lambda\lambda''}^\tau.$$

The verification that $\lambda \in \Lambda$ and $\tau \leq \tau' \leq \tau''$ yields

$$(4.14) \qquad r_\lambda^{\tau\tau'} r_\lambda^{\tau'\tau''} = r_\lambda^{\tau\tau''}$$

is straightforward. In order to complete the proof that $\boldsymbol{Z}$ is a rectangular inverse system, it remains to show that

$$(4.15) \qquad r_{\lambda\lambda'}^\tau r_{\lambda'}^{\tau\tau'} = r_\lambda^{\tau\tau'} r_{\lambda\lambda'}^{\tau'}, \ \lambda \leq \lambda', \ \tau \leq \tau',$$

First note that $\tau \leq \tau'$ implies $\tau(\lambda) \leq \tau'(\lambda)$. Therefore, by $\tau'(\lambda) \leq \tau'(\lambda')$ and by the definition of the ordering $\leq$,

$$(4.16) \qquad v_{\lambda\lambda'}^{\tau(\lambda)} q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau'(\lambda')} = q_\lambda^{\tau(\lambda)\tau'(\lambda)} v_{\lambda\lambda'}^{\tau'(\lambda)} q_{\lambda'}^{v_{\lambda\lambda'}(\tau'(\lambda))\tau'(\lambda')}.$$

(4.16) yields (4.15) if one takes into account that $r_{\lambda\lambda'}^\tau = v_{\lambda\lambda'}^{\tau(\lambda)} q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')}$, $r_{\lambda\lambda'}^{\tau'} = v_{\lambda\lambda'}^{\tau'(\lambda)} q_{\lambda'}^{v_{\lambda\lambda'}(\tau'(\lambda))\tau'(\lambda')}$ and $q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau'(\lambda')} = q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')} q_{\lambda'}^{\tau(\lambda')\tau'(\lambda')}$.

For a fixed $\lambda \in \Lambda$, consider the resolution $\boldsymbol{q}_\lambda\colon X_\lambda \to \boldsymbol{Y}_\lambda$. Also consider the function $f\colon T \to M_\lambda$, defined by $f(\tau) = \tau(\lambda)$. This function is increasing, because $\tau \leq \tau'$ implies $\tau(\lambda) \leq \tau'(\lambda)$. Moreover, f is cofinal. Indeed, for an arbitrary element $\mu \in M_\lambda$, we first define a function $\tilde\tau \in \tilde{T}$, by putting $\tilde\tau(\lambda) = \mu$ and by taking for $\tilde\tau(\lambda')$, $\lambda' \neq \lambda$, any value in $M_{\lambda'}$. Then we choose for τ a function from T such that $\tilde\tau \leq \tau$. Clearly, $f(\tau) = \tau(\lambda) \geq \tilde\tau(\lambda) = \mu$.

Now consider the special mapping $\boldsymbol{f} = (f, f^\tau)\colon \boldsymbol{Y}_\lambda \to \boldsymbol{Z}_\lambda = (Z_\lambda^\tau, r_\lambda^{\tau\tau'}, T)$, where $f^\tau\colon Y_\lambda^{f(\tau)} = Y_\lambda^{\tau(\lambda)} \to Y_\lambda^{\tau(\lambda)} = Z_\lambda^\tau$ is the identity mapping. That $\boldsymbol{f}$ is indeed a special mapping follows from the equality $r_\lambda^{\tau\tau'} f^{\tau'} = q_\lambda^{\tau(\lambda)\tau'(\lambda)} = f^\tau q_\lambda^{\tau(\lambda)\tau'(\lambda)}$. The composition $\boldsymbol{u}_\lambda = \boldsymbol{f}\boldsymbol{q}_\lambda\colon X_\lambda \to \boldsymbol{Z}_\lambda$ consists of mappings

$u_\lambda^\tau = f^\tau q_\lambda^{f(\tau)} = q_\lambda^{\tau(\lambda)}\colon X_\lambda \to Y_\lambda^{\tau(\lambda)} = Z_\lambda^\tau$. An application of Lemma 2.1 shows that $\boldsymbol{u}_\lambda\colon X_\lambda \to \boldsymbol{Z}_\lambda$ is a resolution.

Now let us prove that $\boldsymbol{u}_\lambda p_{\lambda\lambda'} = \boldsymbol{r}_{\lambda\lambda'}\boldsymbol{u}_{\lambda'}$, i.e., $u_\lambda^\tau p_{\lambda\lambda'} = r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau$. Indeed, $\boldsymbol{q}_\lambda p_{\lambda\lambda'} = \boldsymbol{v}_{\lambda\lambda'}\boldsymbol{q}_{\lambda'}$ implies $q_\lambda^{\tau(\lambda)}p_{\lambda\lambda'} = v_{\lambda\lambda'}^{\tau(\lambda)}q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))}$. Since

$$\text{(4.17)} \qquad q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))} = q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')}q_{\lambda'}^{\tau(\lambda')}$$

and $r_{\lambda\lambda'}^\tau$ is given by (4.8), we see that $u_\lambda^\tau p_{\lambda\lambda'} = q_\lambda^{\tau(\lambda)}p_{\lambda\lambda'} = v_{\lambda\lambda'}^{\tau(\lambda)}q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))} = v_{\lambda\lambda'}^{\tau(\lambda)}q_{\lambda'}^{v_{\lambda\lambda'}(\tau(\lambda))\tau(\lambda')}q_{\lambda'}^{\tau(\lambda')} = r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau$.

In this way we have proved all the assertions of Theorem 1.7 in the polyhedral case, except for the cofiniteness of T. To obtain also this property, it suffices to repeat Step 8 from the proof of Theorem 1.7 in the ANR case. ☐

5. Proofs of other results

PROOF OF THEOREM 1.9. We need to prove that $\boldsymbol{v}$ is a resolution of X, i.e., it has properties (B1) and (B2). To establish (B1), assume that $\mathcal{U}$ is a normal covering of X. Then, by property (B1) for $\boldsymbol{p}$, there exists a $\lambda \in \Lambda$ and a normal covering $\mathcal{U}_\lambda$ of X_λ such that $p_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$. Using (B1) for $\boldsymbol{u}_\lambda$, we conclude that there exist a $\tau \in T$ and a normal covering $\mathcal{U}_\lambda^\tau$ of Z_λ^τ such that $(u_\lambda^\tau)^{-1}(\mathcal{U}_\lambda^\tau)$ refines $\mathcal{U}_\lambda$. Since $v_\lambda^\tau = u_\lambda^\tau p_\lambda$, it follows that $(v_\lambda^\tau)^{-1}(\mathcal{U}_\lambda^\tau)$ refines $\mathcal{U}$.

To establish (B2) for $\boldsymbol{v}$, assume that $\mathcal{U}_\lambda^\tau$ is a normal covering of Z_λ^τ and let $\mathcal{V}_\lambda^\tau$ be a normal star-refinement of $\mathcal{U}_\lambda^\tau$. Consider the normal covering $\mathcal{V}_\lambda = (u_\lambda^\tau)^{-1}(\mathcal{V}_\lambda^\tau)$ of X_λ. By (B2) for $\boldsymbol{p}$, there exists a $\lambda' \geq \lambda$ such that

$$\text{(5.1)} \qquad p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{St}\,(\overline{p_\lambda(X)}, \mathcal{V}_\lambda).$$

We claim that

$$\text{(5.2)} \qquad u_\lambda^\tau p_{\lambda\lambda'}(X_{\lambda'}) \subseteq \mathrm{St}\,(\overline{v_\lambda^\tau(X)}, \mathcal{V}_\lambda^\tau).$$

Indeed, by (5.1), for every $x \in X_{\lambda'}$, there exists a member V_λ of $\mathcal{V}_\lambda$ and a point $y \in V_\lambda \cap \overline{p_\lambda(X)}$ such that $p_{\lambda\lambda'}(x) \in V_\lambda$. Moreover, there exists a member $V \in \mathcal{V}_\lambda^\tau$ such that $(u_\lambda^\tau)^{-1}(V) = V_\lambda$. Consequently, $u_\lambda^\tau p_{\lambda\lambda'}(x) \in V$, $u_\lambda^\tau(y) \in V$ and $u_\lambda^\tau(y) \in u_\lambda^\tau(\overline{p_\lambda(X)}) \subseteq \overline{u_\lambda^\tau p_\lambda(X)} = \overline{v_\lambda^\tau(X)}$. The last two relations show that $V \subseteq \mathrm{St}\,(\overline{v_\lambda^\tau(X)}, \mathcal{V}_\lambda^\tau)$ and thus, $u_\lambda^\tau p_{\lambda\lambda'}(x) \in \mathrm{St}\,(\overline{v_\lambda^\tau(X)}, \mathcal{V}_\lambda^\tau)$, which establishes (5.2). By (1.2), $u_\lambda^\tau p_{\lambda\lambda'} = r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau$ and thus, (5.2) becomes

$$\text{(5.3)} \qquad r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau(X_{\lambda'}) \subseteq \mathrm{St}\,(\overline{v_\lambda^\tau(X)}, \mathcal{V}_\lambda^\tau).$$

Now consider the normal covering $\mathcal{V}_{\lambda'}^\tau = (r_{\lambda\lambda'}^\tau)^{-1}(\mathcal{V}_\lambda^\tau)$. We claim that

$$\text{(5.4)} \qquad r_{\lambda\lambda'}^\tau(\mathrm{St}\,(\overline{u_{\lambda'}^\tau(X_{\lambda'})}, \mathcal{V}_{\lambda'}^\tau) \subseteq \mathrm{St}\,(\overline{v_\lambda^\tau(X)}, \mathcal{U}_\lambda^\tau).$$

Indeed, if $y \in \mathrm{St}\,(\overline{u_{\lambda'}^\tau(X_{\lambda'})}, \mathcal{V}_{\lambda'}^\tau)$, then there exists a member V' of $\mathcal{V}_{\lambda'}^\tau$ such that $y \in V'$ and $V' \cap \overline{u_{\lambda'}^\tau(X_{\lambda'})} \neq \emptyset$. Since V' is open, it follows that also $V' \cap u_{\lambda'}^\tau(X_{\lambda'}) \neq \emptyset$ and thus, $r_{\lambda\lambda'}^\tau(V') \cap r_{\lambda\lambda'}^\tau u_{\lambda'}^\tau(X_{\lambda'}) \neq \emptyset$. By the definition

of $\mathcal{V}_{\lambda'}^{\tau}$, there exists a member W of $\mathcal{V}_{\lambda}^{\tau}$ such that $V' = (r_{\lambda\lambda'}^{\tau})^{-1}(W)$ and thus, $W \cap r_{\lambda\lambda'}^{\tau} u_{\lambda'}^{\tau}(X_{\lambda'}) \neq \emptyset$. Now (5.3) implies that $W \cap \mathrm{St}\,(\overline{v_{\lambda}^{\tau}(X)}, \mathcal{V}_{\lambda}^{\tau}) \neq \emptyset$. Consequently, there is a member W' of $\mathcal{V}_{\lambda}^{\tau}$ such that $W \cap W' \neq \emptyset$ and $W' \cap \overline{v_{\lambda}^{\tau}(X)} \neq \emptyset$. Since $\mathcal{V}_{\lambda}^{\tau}$ is a star-refinement of $\mathcal{U}_{\lambda}^{\tau}$, there exists a member U of $\mathcal{U}_{\lambda}^{\tau}$ such that $W \cup W' \subseteq U$. Clearly, $U \cap \overline{v_{\lambda}^{\tau}(X)} \neq \emptyset$ and thus, $r_{\lambda\lambda'}^{\tau}(y) \in r_{\lambda\lambda'}^{\tau}(V') \subseteq W \subseteq U \subseteq \mathrm{St}\,(\overline{v_{\lambda}^{\tau}(X)}, \mathcal{U}_{\lambda}^{\tau})$, which establishes (5.4).

By property (B2) for the resolution $\boldsymbol{u}_{\lambda'} \colon X_{\lambda'} \to \boldsymbol{Z}_{\lambda'}$, there exists a $\tau' \geq \tau$ such that $r_{\lambda'}^{\tau\tau'}(Z_{\lambda'}^{\tau'}) \subseteq \mathrm{St}\,(\overline{u_{\lambda'}^{\tau}(X_{\lambda'})}, \mathcal{V}_{\lambda'}^{\tau})$ and thus, by (5.4), $r_{\lambda\lambda'}^{\tau\tau'}(Z_{\lambda'}^{\tau'}) = r_{\lambda\lambda'}^{\tau} r_{\lambda'}^{\tau\tau'}(Z_{\lambda'}^{\tau'}) \subseteq r_{\lambda\lambda'}^{\tau}(\mathrm{St}\,(\overline{u_{\lambda'}^{\tau}(X_{\lambda'})}, \mathcal{V}_{\lambda'}^{\tau})) \subseteq \mathrm{St}\,(\overline{v_{\lambda}^{\tau}(X)}, \mathcal{U}_{\lambda}^{\tau})$, which shows that $\boldsymbol{v}$ also has property (B2). □

PROOF OF THEOREMS 1.1 AND 1.5. A proof of Theorem 1.1 is obtained by a small variation of the proof of Theorem 1.7 in the ANR case. All spaces which appear in the proof are compact Hausdorff spaces. Therefore, resolutions and limits coincide. Whenever in the proof of Theorem 1.7 appears an inverse systems of ANRs, at the corresponding place in the proof of Theorem 1.1 we have a system of compact polyhedra (which are always compact ANRs). In Step 4, instead of considering open neighborhoods G of $\overline{q_{\lambda}^{\mu}(X_{\lambda})} = q_{\lambda}^{\mu}(X_{\lambda})$ in Y_{λ}^{μ}, we consider neighborhoods of $q_{\lambda}^{\mu}(X_{\lambda})$ in Y_{λ}^{μ}, which are compact polyhedra. Another proof is obtained from the proof of Theorem 1.7 in the polyhedral case. Indeed, in the proof of Proposition 4.1 (see [7]) all members Y_{λ}^{μ} of $\boldsymbol{Y}_{\lambda}$ are either nerves of open coverings of X_{λ} or subcomplexes of their subdivisions. In the compact case, it suffices to consider finite open coverings. Consequently, one can assume that all Y_{λ}^{μ}, hence also all $Z_{\lambda}^{\tau} = Y_{\lambda}^{\tau(\lambda)}$, are compact polyhedra.

Theorem 1.5 immediately follows from Theorem 1.9. Indeed, if $\boldsymbol{p}$ is a limit, then it is also a resolution. Consequently, $\boldsymbol{v}$ is also a resolution. However, since we are in the compact situation, this resolution is actually a limit. □

The proof of Corollary 1.3 uses the following elementary lemma on directed sets.

LEMMA 5.1. *In a directed set M every countable subset $M' \subseteq M$ is contained in a countable directed subset $M_1 \subseteq M$. More precisely, if $M' = \{\mu_1, \mu_2, \ldots\}$, then there exists an increasing sequence $m_1 \leq m_2 \leq \ldots$ of elements of M such that $\mu_i \leq m_i$, for $i \in \mathbb{N}$, and thus the set $M_1 = M' \cup \{m_1, m_2, \ldots\}$ is a countable directed subset of M, which contains the set M' and the set $\{m_1 \leq m_2 \leq \ldots\}$ is cofinal in M_1.*

PROOF. We define the sequence m_i by induction, beginning with $m_1 = \mu_1$. Assume that we have already defined $m_1 \leq m_2 \leq \ldots \leq m_i$ in such a way that $\mu_j \leq m_j$, for $1 \leq j \leq i$. We take for m_{i+1} an element of M such that $m_{i+1} \geq m_i, \mu_{i+1}$. □

PROOF OF COROLLARY 1.3. Let $\boldsymbol{X} = (X_n, p_{nn'}, \mathbb{N})$ be an inverse sequence of metric compacta. By Theorem 1.1, there exists a rectangular inverse system of compact polyhedra $\boldsymbol{Z} = (Z_n^\mu, r_{nn'}^{\mu\mu'}, \mathbb{N} \times M)$ and there exist limits $\boldsymbol{u}_n = (u_n^\mu) \colon X_n \to \boldsymbol{Z}_n$ of the vertical systems $\boldsymbol{Z}_n = (Z_n^\mu, r_n^{\mu\mu'}, M)$ such that $u_n^\mu p_{nn'} = r_{nn'}^\mu u_{n'}^\mu$, for $n \leq n'$ and $\mu \in M$. Since X_n is a metrizable compactum, there exists a countable collection of open coverings $\{\mathcal{U}_n^i | i \in \mathbb{N}\}$ of X_n, which form a basis of open coverings of X_n, i.e., for every open covering $\mathcal{U}_n$ of X_n, there exists an $i \in \mathbb{N}$ such that $\mathcal{U}_n^i$ refines $\mathcal{U}_n$. By property (B1) of $\boldsymbol{u}_n$, for every $i \in \mathbb{N}$, there is a $\mu_n^i \in M$ and there is an open covering $\mathcal{V}_n^i$ of $Z_n^{\mu_n^i}$ such that $(u_n^{\mu_n^i})^{-1}(\mathcal{V}_n^i)$ refines $\mathcal{U}_n^i$. Clearly, $M' = \{\mu_n^i | (n,i) \in \mathbb{N} \times \mathbb{N}\}$ is a countable subset of M such that for every directed subset $M'' \subseteq M$, which contains M', and every $n \in \mathbb{N}$, the restriction $\boldsymbol{u}_n | M'' \colon X_n \to \boldsymbol{Z}_n | M''$ has property (B1). By Lemma 5.1, there exists a countable directed subset $M_1 \subseteq M$ such that $M' \subseteq M_1$.

We will now define by induction a sequence of countable directed subsets $M_1 \subseteq M_2 \subseteq \ldots$ of M such that, for every $\mu \in M_k$, every $n \in \mathbb{N}$ and every open neighborhood V_n^μ of $r_n^\mu(X_n)$ in Z_n^μ, there is a $\mu' \in M_{k+1}$, $\mu' \geq \mu$, such that $r_n^{\mu\mu'}(Z_n^{\mu'}) \subseteq V_n^\mu$. The initial term M_1 has already been constructed. Now assume that we have constructed the terms $M_1, \ldots, M_k$, $k \geq 1$. To construct M_{k+1} associate with every $n \in \mathbb{N}$ and every $\mu \in M_k$ a basis $\{V_n^{\mu i} | i \in \mathbb{N}\}$ of open neighborhoods of $u_n^\mu(X_n)$ in Z_n^μ. By property (B2) of $\boldsymbol{u}_n$, there are indices $\mu_n^i \geq \mu$ in M such that $r_n^{\mu\mu_n^i}(Z_n^{\mu_n^i}) \subseteq V_n^{\mu i}$. Clearly, $M_k \cup \{\mu_n^i | (n,i) \in \mathbb{N} \times \mathbb{N}\}$ is a countable subset of M. Therefore, by Lemma 5.1, there exists a countable directed subset $M_{k+1} \subseteq M$, which contains $M_k \cup \{\mu_n^i | (n,i) \in \mathbb{N} \times \mathbb{N}\}$. Now let $n \in \mathbb{N}$, $\mu \in M_k$ and let V_n^μ be an open neighborhood of $u_n^\mu(X_n)$ in Z_n^μ. There exists an $i \in \mathbb{N}$ such that $V_n^{\mu i} \subseteq V_n^\mu$. Clearly, $\mu' = \mu_n^i \in M_{k+1}$, $\mu' \geq \mu$ and $r_n^{\mu\mu'}(X_n^{\mu'}) \subseteq V_n^{\mu i} \subseteq V_n^\mu$. Clearly, the union $M_0 = M_1 \cup M_2 \cup \ldots$ is a countable directed subset of M such that, for every $n \in \mathbb{N}$, the restrictions $\boldsymbol{u}_n | M_0 \colon X_n \to \boldsymbol{Z}_n | M_0$ have property (B2). Since $M_0 \supseteq M'$, the restrictions $\boldsymbol{u}_n | M_0$ also have property (B1). Consequently, the rectangular system of compact polyhedra $\boldsymbol{Z} | (\mathbb{N} \times M_0)$ has the property that the mappings $\boldsymbol{u}_n | M_0 \colon X_n \to \boldsymbol{Z}_n | M_0$ are limits. Since M_0 is countable, the second statement of Lemma 5.1, applied to $M' = M_0$, yields an increasing sequence $m_1 \leq m_2 \leq \ldots$ of elements of M_0, which forms a cofinal subset N_0 of M_0. Therefore, restricting $\boldsymbol{Z}$ to $\mathbb{N} \times N_0$, we obtain a new rectangular system where the mappings $\boldsymbol{u}_n | N_0 \colon X_n \to \boldsymbol{Z}_n$ are limits of its vertical systems. If N_0 is infinite, there is no loss of generality in assuming that $N_0 = \mathbb{N}$. If N_0 is finite, we can repeat its last row infinitely many times and thus, again assume that $N_0 = \mathbb{N}$. $\square$

References

[1] Yu.T. Lisitsa, *Second structure theorem for strong homology*, Geometric Topology, Dubrovnik 1998, Abstracts, p. 33.

[2] Yu.T. Lisitsa, *Basic theorems of strong shape theory of compact Hausdorff spaces*, Vestnik Ross. Univ. Druzhby Narodov, Ser. Matematika 2000, No. **6(1)**, 94–113 (in Russian).

[3] Yu.T. Lisitsa, *Second structure theorem for strong homology*, Topology Appl. **113** (2001), 107–134.

[4] S. Mardešić, Strong shape and homology, Springer Monographs in Mathematics, Springer, Berlin 2000.

[5] S. Mardešić and J. Segal, Shape theory, North-Holland, Amsterdam 1982.

[6] S. Mardešić and N. Uglešić, *On iterated inverse limits*, Topology Appl. **120** (2002), 157–167.

[7] N. Uglešić, *Iterated resolutions*, Glasnik Mat. **35(55)** (2000), 245–259.

[8] T. Watanabe, *Approximative expansions of maps into inverse systems*, Banach Center Publ., Warsaw, **18** (1986), pp. 363–370.

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