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STATISTICAL (T) RATES OF CONVERGENCE

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Abstract. The basis for comparing rates of convergence of two null sequences is that "$x = (x_n)$ converges $(stat\ T)$ faster than $z = (z_n)$ provided that (x_n/z_n) is T-statistically convergent to zero" where $T = (t_{mn})$ is a mean. In this paper we extend the previously known results either on the ordinary convergence or statistical rates of convergence of two null sequences.We also consider lacunary statistical rates of convergence.

1. Introduction

Bajraktarevic [1, 2] and Miller [19, 23] studied rates of convergence of families of null sequences. The relationship between rates of convergence and summability methods may be found in [9, 19, 20, 21, 22, 23]. Recently Fridy, Miller and Orhan [16] have considered statistical rates of convergence and extended results from some of the above mentioned papers. In this paper, using a mean $T = (t_{mn})$, we study statistical (T) rates of convergence and show that statistical speed of convergence strongly depends on T. We also extend some results in [16]. The final section of the paper concerns lacunary statistical rates of convergence.

If K is a subset of the positive integers $\mathbb{N}$, K_n will denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ will denote the cardinality of K_n. The natural density of K ([8]) is given by $\delta(K) := \lim_n n^{-1} |K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set $K := K(\varepsilon) := \{k \in \mathbb{N} : \ |x_k - L| \geq \varepsilon\}$ has natural density zero [7, 10, 11, 12, 18].

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In this case, we write $st-\lim x = L$. Hence x is statistically convergent to L iff $(C_1\chi_{K(\varepsilon)})_n \to 0$ (as $n\to\infty$, for every $\varepsilon>0$), where C_1 is the Cesàro mean of order one and χ_K is the characteristic function of the set K.

Statistical convergence can be generalized by using a regular nonnegative summability matrix T in place of C_1 (see, e.g., [3, 4, 5, 6, 8, 15, 17]). Regular nonnegative summability matrices turn out to be too general for our purposes here, instead we use the concept of a mean.

A matrix $T=(t_{mn})$ is called a mean if $t_{mn}>0$ when $n\le m$, $t_{mn}=0$ if $n>m$, $\sum_{n=1}^{\infty} t_{mn}=1$ for all m and $\lim_m t_{mn}=0$ for each n.

Recall that the set $K\subseteq\mathbb{N}$ has $T-$density if $\delta_T(K):=\lim_m \sum_{n\in K} t_{mn}$ exists ([8]). The sequence $x=(x_n)$ converges (stat T) to L means that for each $\varepsilon>0$ we have

$$\lim_m \sum_{n=1}^{m}[t_{mn} : |x_n-L|\ge\varepsilon]=0. \tag{1.1}$$

So (1.1) is equivalent to the fact that $\delta_T(\{n\in\mathbb{N} : |x_n-L|<\varepsilon\})=1$, for every $\varepsilon>0$.

We say that a property holds for $T-$ almost all n if the set $\{k\in\mathbb{N} : P(k)$ is false$\}$ has $T-$density zero.

2. Statistical (T) Rates of Convergence

If z and x are two nonvanishing null sequences (i.e., $x_n\ne 0$ for all n and $\lim x_n=0$) then we say that z converges (*stat* T) faster than x provided that z/x converges (*stat* T) to zero.

The following example shows that statistical (T) speed of convergence strongly depends on T.

Example 2.1. Let $x=(1/n)$ and $y=(y_n)$ where

$$y_n=\begin{cases}\dfrac{1}{n^2}, & \text{if } n \text{ is odd},\\[2mm] \dfrac{1}{\sqrt{n}}, & \text{if } n \text{ is even}.\end{cases}$$

Define the means T_1 and T_2 as follows:

$T_1=(t^{(1)}_{mn})$ satisfies $\sum_{n=1}^{m}[t^{(1)}_{mn} : n \text{ even}]=1-\frac{1}{m}$, for all m, and all of the non-zero terms in the last summand are equal. Also $\sum_{n=1}^{m}[t^{(1)}_{mn} : n \text{ odd}]=\frac{1}{m}$, for all m, and all of the non-zero terms in the last summand are equal. $T_2=(t^{(2)}_{mn})$ is the same as T_1 with the roles of even and odd reversed. Then x converges (*stat* T_1) faster than y, but y converges (*stat* T_2) faster than x.

The last example suggests the following theorem.

THEOREM 2.2. *If x and y are nonvanishing null sequences and P_1 and P_2 are disjoint infinite subsets of $\mathbb{N}$ satisfying $\lim_{n\in P_1}(x_n/y_n) = 0$ and $\lim_{n\in P_2}(y_n/x_n) = 0$, then there exist means T_1 and T_2 such that x converges $(stat\,T_1)$ faster than y and y converges $(stat\,T_2)$ faster than x.*

PROOF. There exists an $m_0 \in \mathbb{N}$ such that both P_1 and P_2 contain elements smaller than m_0 . Set $T_1 = (t^{(1)}_{mn})$, $T_2 = (t^{(2)}_{mn})$; two means, defined as follows.

For $m \geq m_0$, $\sum_{n=1}^{\infty}[t^{(1)}_{mn} : n \in P_1 \ , n \leq m] = 1 - \frac{1}{m}$ with all the terms in this summand taken to be equal and $\sum_{n=1}^{\infty}[t^{(1)}_{mn} : n \in P_2 \ , n \leq m] = \frac{1}{m}$ with all terms equal. Let $t^{(1)}_{nm} = \frac{1}{m}$ if $n \leq m$.

Define $T_2 = (t^{(2)}_{mn})$ as we defined $T_1$with the roles of P_1 and P_2 reversed. Then x converges $(stat\,T_1)$ faster than y and y converges $(stat\,T_2)$ faster than x.

From the last result we see that if $P \subseteq \mathbb{N}$ is infinite and $\lim_{n\in P}(x_n/y_n) = 0$ where $x = (x_n)$ and $y = (y_n)$ are two nonvanishing null sequences then there exists a mean T such that x converges $(stat\,T)$ faster than y. ∎

We now consider the converse.

THEOREM 2.3. *If x and y are nonvanishing null sequence and T is a mean and x converges $(stat\,T)$ faster than y then there exists an infinite set P, $P \subseteq \mathbb{N}$ such that $\lim_{n\in P}(x_n/y_n) = 0$.*

PROOF. By Theorem 1 in [17], there exists an infinite set P, $P \subseteq \mathbb{N}$, such that $\delta_T(P) = 1$ and $\lim_{n\in P}(x_n/y_n) = 0$. ∎

The following theorem is an analog of Theorem 1 in [16].

THEOREM 2.4. *Let $\mathcal{A}$ be a collection of nonvanishing null sequences and let T be a mean. There exists a nonvanishing null sequence z that converges (stat T) faster than each x in $\mathcal{A}$ if and only if there exists a sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of subcollections of $\mathcal{A}$ such that*

(i) *each x in $\mathcal{A}$ is in $T-$ almost all $\mathcal{A}_n$, i.e.,*

$$\lim_n \sum_{k=1}^{\infty}[t_{nk} : x \in \mathcal{A}_k] = 1,$$

(ii) *for each n,*

$$y_n = \inf\{|x_n| : x \in \mathcal{A}_n\} > 0.$$

PROOF. (i) Necessity. Suppose $\mathcal{A}$ is a collection of nonvanishing null sequences and z is nonvanishing null sequence that converges (stat T) faster than each x in $\mathcal{A}$. Define $\mathcal{A}_n := \{x \in \mathcal{A} : |x_n| > |z_n|\}$. Then $\mathcal{A}_n \subseteq \mathcal{A}$, for each n and each x in $\mathcal{A}$ is in $T-$ almost all $\mathcal{A}_n$ since z converging (stat

T) faster than x implies $\delta_T(\{n \in \mathbb{N} : \left|\frac{z_n}{x_n} - 0\right| < 1\}) = 1$ or $\delta_T(\{n \in \mathbb{N} : |z_n| < |x_n|\}) = 1$, which says $\delta_T(\{k : x \in \mathcal{A}_k\}) = 1$. Also, if $\mathcal{A}_n \neq \varnothing$ then $y_n = \inf\{|x_n| : x \in \mathcal{A}_n\} \geq |z_n| > 0$, and if $\mathcal{A}_n = \varnothing$ then

$$y_n = \inf \varnothing = \infty > 0.$$

(ii) Sufficiency. Suppose $\mathcal{A}$ is a collection of nonvanishing null sequences and $\{\mathcal{A}_n\}_{n=1}^{\infty}$ is a sequence of subcollections of $\mathcal{A}$ that satisfies (i) and (ii). Define

$$z_n = \begin{cases} \min(y_n t_n, t_n), & \text{if } \mathcal{A}_n \neq \varnothing, \\ t_n, & \text{if } \mathcal{A}_n = \varnothing, \end{cases}$$

where $t_n = \min(t_{n1}, t_{n2}, ..., t_{nn})$. Notice that $0 < t_n \leq \frac{1}{n}$. Clearly z is a nonvanishing null sequence. If x is a sequence in $\mathcal{A}$, then $x \in \mathcal{A}_n$ for $T-$ almost all n, i.e., $0 < y_n \leq |x_n|$ for $T-$ almost all n. Hence $\frac{z_n}{|x_n|} \leq \frac{y_n t_n}{|x_n|} \leq t_n \leq \frac{1}{n}$ for $T-$ almost all n, whence z converges (stat T) faster than x. ∎

The next result is a generalization of Theorem 2 of [16].

Theorem 2.5. *Suppose $\mathcal{A}$ is a collection of nonvanishing null sequences. There exists a nonvanishing null sequence z which converges (stat T) slower than each x in $\mathcal{A}$ if and only if there exists a sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of subcollections of $\mathcal{A}$, a null sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers, and a strictly increasing sequence $\{N_n\}_{n=1}^{\infty}$ of nonnegative integers such that*

(I) $\sup\{|x_k| : x \in \mathcal{A}_n, N_{n-1} < k \leq N_n\} \leq \varepsilon^2$ *for every n;*
(II) *for each $x \in \mathcal{A}$, $\delta_T(\mathfrak{n}_x) = 1$, where $\mathfrak{n}_x = \cup\{(N_{n-1}, N_n] : x \in \mathcal{A}_n\}$.*

Proof. (i) Necessity. Suppose z is a nonvanishing null sequence that converges (stat T) slower than each x in $\mathcal{A}$. Set $N_n = n$ for $n = 0, 1, 2, ...$; and $\varepsilon_n^2 = |z_n|$ for each $n \geq 1$. Define

$$\begin{aligned} \mathcal{A}_n &= \{x \in \mathcal{A} : |x_n| < |z_n|\} \\ &= \{x \in \mathcal{A} : |x_k| < |z_k|\ ,\ N_{n-1} < k \leq N_n\}. \end{aligned}$$

Then if $\mathcal{A}_n \neq \varnothing$ we have

$$\sup\{|x_k| : x \in \mathcal{A}_n\ ,\ N_{n-1} < k \leq N_n\} = \sup\{|x_k| : x \in \mathcal{A}_n\} \leq |z_n| = \varepsilon_n^2,$$

and if $\mathcal{A}_n = \varnothing$ then the above supremum is $-\infty < \varepsilon^2$. Furthermore, suppose $x \in \mathcal{A}$. Then $\{z_n/x_n\}$ is (stat T) convergent to zero. Therefore $\delta_T\{n \in \mathbb{N} : |x_n| < |z_n|\} = 1$ or $\delta_T\{n \in \mathbb{N} : x \in \mathcal{A}_n\} = 1$, or

$$\delta_T\left(\bigcup\{(N_{n-1}, N_n] : x \in \mathcal{A}_n\}\right) = 1.$$

Hence, $\mathcal{A}$ satisfies (I) and (II).

(ii) Sufficiency. Suppose $\mathcal{A}$, $\{\mathcal{A}_n\}, \{\varepsilon_n\}$, and $\{N_n\}$ satisfy the conditions in the statement of the theorem. Define the sequence z as follows:

$$\begin{aligned} z_1 &= z_2 = ... = z_{N_1} = \varepsilon_1, \\ z_{1+N_1} &= z_{2+N_1} = ... = z_{N_2} = \varepsilon_2, \\ z_{1+N_2} &= z_{2+N_2} = ... = z_{N_3} = \varepsilon_3, \\ &... \end{aligned}$$

Let x be any fixed sequence in $\mathcal{A}$. If $x \in \mathcal{A}_{n_o}$ then $|x_k| \leq \varepsilon_{n_o}^2$ when $N_{n_o-1} < k \leq N_{n_o}$. Hence, $N_{n_o-1} < k \leq N_{n_o}$ implies that

$$\left|\frac{z_k}{x_k}\right| = \frac{\varepsilon_{n_o}}{|x_k|} \geq \frac{\varepsilon_{n_o}}{\varepsilon_{n_o}^2} = \frac{1}{\varepsilon_{n_o}}.$$

It follows that $\lim_{k\in\mathfrak{n}_x} |z_k/x_k| = +\infty$, and the $T-$ density of $\mathfrak{n}_x$ is one by hypothesis. So that z converges (*stat* T) slower than each x in $\mathcal{A}$. ∎

It is natural to compare rates of convergence and (*stat*) rates of convergence. If x converges faster [respectively, slower] than y, than x converges (*stat*) faster [respectively, slower] than y, however, for sequences whose rates of convergence are completely incomparable the inclusion is reversed. We say that x and y *converge at completely incomparable rates* provided that $\lim_n x_n = X,\ \lim_n y_n = Y$,

$$\underline{\lim}_n \left|\frac{x_n - X}{y_n - Y}\right| = 0 \quad \text{and} \quad \overline{\lim}_n \left|\frac{x_n - X}{y_n - Y}\right| = +\infty.$$

If, in the preceding situation, there exist subsets $N_1, N_2 \subseteq \mathbb{N}$, neither having $T-$density zero, such that

$$\lim_{n\in N_1} \left|\frac{x_n - X}{y_n - Y}\right| = 0 \quad \text{and} \quad \lim_{n\in N_2} \left|\frac{x_n - X}{y_n - Y}\right| = +\infty,$$

then we say that x and y *converge (stat T) at completely incomparable rates.*

We now present an analogue of Theorem 3 of [16].

THEOREM 2.6. *Let A be a collection of nonvanishing null sequences. There exists a nonvanishing null sequence z such that for every x in $\mathcal{A}, z$ and x converge (stat T) at completely incomparable rates if there exist two sequences α and β of positive integers such that*

$$1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < ...,$$

a positive null sequence $\{\varepsilon_n\}$, and two sequences $\{\mathcal{A}_n\}$ and $\{\mathfrak{B}_n\}$ of subcollections of $\mathcal{A}$ that satisfy

a) $y_n = \inf\{|x_k| : \ x \in \mathcal{A}_n,\ k \in I_n^\alpha\} > 0$ *for all n;*
b) *for every $x \in \mathcal{A}$, $\mathfrak{n}_x^\alpha = \bigcup [I_n^\alpha :\ x \in \mathcal{A}_n]$ does not have $T-$density zero;*
c) *sup*$\big\{|x_k| : \ x \in \mathfrak{B}_n,\ k \in I_n^\beta\big\} \leq \varepsilon_n^2$ *for all n;*

d) *for every* $x\in\mathcal{A}$, $\mathfrak{n}_x^\beta=\bigcup\left[I_n^\beta:\ x\in\mathfrak{B}_n\right]$ *does not have* $T-$ *density zero, where*

$$\begin{array}{ll} I_1^\alpha=\{1,2,...,\alpha_1\}, & I_1^\beta=\{1+\alpha_1,2+\alpha_1,...,\beta_1\} \\ I_2^\alpha=\{1+\beta_1,2+\beta_1,...,\alpha_2\}, & I_2^\beta=\{1+\alpha_2,2+\alpha_2,...,\beta_2\} \\ I_3^\alpha=\{1+\beta_2,2+\beta_2,...,\alpha_3\}, & \textit{and so on.}\end{array}$$

Proof. Define the sequence z by

$$z_k=\begin{cases}\min\left\{\dfrac{y_n}{n},\dfrac{1}{n}\right\}, & \text{if } k\in I_n^\alpha,\\ \varepsilon_n,\ \text{if } k\in I_n^\beta. & \end{cases}$$

Let x be a fixed element of $\mathcal{A}$. If $x\in\mathcal{A}_n$ and $k\in I_n^\alpha$ then

$$\left|\frac{z_k}{x_k}\right|\le\frac{|z_k|}{|y_n|}\le\frac{1}{n};$$

if $x\in\mathfrak{B}_n$ and $k\in I_n^\beta$, then

$$\left|\frac{z_k}{x_k}\right|\ge\frac{\varepsilon_n}{\varepsilon_n^2}=\frac{1}{\varepsilon_n}.$$

Consequently,

$$\lim_{k\in\mathfrak{n}_x^\alpha}\left|\frac{z_k}{x_k}\right|=0\qquad\text{and}\qquad\lim_{k\in\mathfrak{n}_x^\beta}\left|\frac{z_k}{x_k}\right|=+\infty,$$

and since neither $\mathfrak{n}_x^\alpha$ nor $\mathfrak{n}_x^\beta$ has density zero, it follows that z and x converge (stat) at completely incomparable rates for each x in $\mathcal{A}$. ∎

For countable collections of nonvanishing null sequences there always exits a nonvanishing null sequence z that converges ($statT$) at a rate completely incomparable with every x in $\mathcal{A}$. Namely the following holds.

Corollary 2.7. *If $\mathcal{A}$ is a countable collection of nonvanishing null sequences and T is a mean, then there exists a nonvanishing null sequence z that converges (stat T) at completely incomparable rates with every x in $\mathcal{A}$.*

Proof. Let $\{\varepsilon_n\}$ be a strictly decreasing null sequence and write $\mathcal{A}=\left\{x^{(n)}:\ n\in\mathbb{N}\right\}$, where $x^{(n)}=\{x_{nk}\}_{k=1}^\infty$. Let $\mathcal{A}_n=\mathfrak{B}_n=\left\{x^{(1)},...,x^{(n)}\right\}$, and define $I_1^\alpha,I_1^\beta,I_2^\alpha,I_2^\beta,...$ in such a way that the number of elements in each of these sets is greater than the sum of the number of elements in the proceeding sets. Clearly

$$y_n=\inf\left\{|x_{ik}|:\ i\le n,\ k\in I_n^\alpha\right\}>0$$

since the infimum of a finite set of positive numbers is the smallest element. By the condition on the number of elements in the sets $I_i^{\alpha,\beta}$, we have

$$\bigcup_{n=m}^\infty I_n^\alpha=\mathfrak{n}_x^\alpha=\bigcup\left[I_n^\alpha:\ x\in\mathcal{A}_n\right]$$

does not have $T-$ density zero if $x = x_m$, for each m. Furthermore, since each x is a null sequence, the I_n^{α}'s can be chosen large enough to guarantee that

$$\sup\left\{|x_{ik}| : x^{(i)} \in \mathfrak{B}_n,\ k \in I_n^{\beta}\right\} \le \varepsilon_n^2 \quad \text{for each } n.$$

Finally, it is clear that

$$\mathfrak{n}_x^{\beta} = \bigcup \left[I_n^{\beta} :\ x \in \mathfrak{B}_n\right] = \bigcup_{n=m}^{\infty} I_n^{\beta}$$

does not have $T-$density zero if $x = x_m$, for each m. ∎

Notice that Theorem 4 in [16] shows that the converse of our last theorem is false.

3. Lacunary Statistical Rates of Convergence

By a lacunary sequence we mean an increasing sequence of positive integers $\theta = \{k_r\}$ such that $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Write $I_r := (k_{r-1}, k_r]$, $k_0 = 0$.

The sequence $s = \{s_n\}$ is said to be lacunary statistically convergent to L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left|\{k \in I_r :\ |s_k - L| \ge \varepsilon\}\right| = 0.$$

In this case we write $s_\theta - \lim s = L$ or $s_n \to L(s_\theta)$ ([13, 14]).

A subset K of $\mathbb{N}$ has $\theta-$density if $\delta_\theta(K) := \lim_r |K \cap I_r|\ h_r^{-1}$ exists.

Definition 3.1. *We say that z converges (lacunary stat.) faster than x provided the sequence (z_n/x_n) is lacunary statistically convergent to zero.*

We now present some examples. The first one shows that there exist sequences z and x such that z converges (*stat*) faster than x but z does not converge (*lacunary stat.*) faster than x for some θ. The other example considers the converse of the first example.

Example 3.2. Suppose $\theta = \{k_r\}_{r=0}^{\infty}$ is a lacunary satisfying: $\delta(\bigcup_{r=1}^{\infty} I_{2r}) = 0$. Clearly such a θ exists. Define z and x as follows: $z_n = \frac{1}{n^2}$ for all n,

$$x_n = \begin{cases} \dfrac{1}{n}, & \text{if } n \in \bigcup_{r=0}^{\infty} I_{2r+1}, \\ \dfrac{1}{n^2}, & \text{if } n \in \bigcup_{r=1}^{\infty} I_{2r}. \end{cases}$$

Then $\frac{z_n}{x_n} = \frac{1}{n}$ if $n \in \bigcup_{r=0}^{\infty} I_{2r+1}$ and $\delta(\bigcup_{r=1}^{\infty} I_{2r}) = 0$, so by a result of Fridy [10] ,$\{z_n/x_n\}_{n=1}^{\infty}$ converges (*stat*) to 0, or z converges faster (*stat*) than x. However $\frac{z_n}{x_n} = 1$ if $n \in \bigcup_{r=1}^{\infty} I_{2r}$ so that if $0 < \varepsilon < 1$, for each r, $\frac{1}{h_{2r}} \left| \left\{ n \in I_{2r} : \left| \frac{z_n}{x_n} \right| \geq \varepsilon \right\} \right| = 1$ and hence z does not converge (*lacunary stat.*) faster than x for the given θ.

EXAMPLE 3.3. Let $\{K_n\}_1^{\infty}$ be a strictly increasing sequence of positive integers with the property that the sequence $\left\{ \frac{K_n}{K_1 + \ldots + K_n} \right\}_{n=1}^{\infty}$ is strictly increasing and converges to 1. Let $B_1 = (0, K_1]$, $B_2 = (K_1, K_1 + K_2]$, $B_3 = (K_1 + K_2, K_1 + K_2 + K_3], ...,$ etc.

Define z and x as follows: $z_n = \frac{1}{n^2}$ for all n,

$$x_n = \begin{cases} \dfrac{1}{n^2}, & \text{if } n \in \bigcup\limits_{r=0}^{\infty} B_{2r+1}, \\ \dfrac{1}{n}, & \text{if } n \in \bigcup\limits_{r=1}^{\infty} B_{2r}. \end{cases}$$

Now set

$$\begin{aligned} \theta &= \{k_r\} \\ &= \{0, K_1 + K_2, K_1 + K_2 + K_3 + K_4, K_1 + K_2 + K_3 + K_4 + K_5 + K_6, \ldots\}. \end{aligned}$$

First notice that z does not converge (*stat*) faster than x since $\frac{z_n}{x_n} = 1$ if $n \in \bigcup_{r=0}^{\infty} B_{2r+1}$ and $\delta(\bigcup_{r=0}^{\infty} B_{2r+1}) \neq 0$ since $\frac{K_1 + K_3 + \ldots + K_{2n+1}}{K_1 + K_2 + \ldots + K_{2n+1}} \to 1$ as $n \to \infty$. Finally z does converge (*lacunary stat.*) faster than x for the above θ since

$$\begin{aligned} &\frac{1}{h_r} \left| \{k \in (K_1 + ... + K_{2r-2}, K_1 + ... + K_{2r}] : \frac{z_k}{x_k} = \frac{1}{k}\} \right| = \\ &= \frac{K_{2r}}{K_{2r-1} + K_{2r}} \to 1 \text{ as } r \to \infty. \end{aligned}$$

The following result is an analog of Theorem 2.4.

THEOREM 3.4. *Let $\theta = (k_n)$ be lacunary sequence and let $\mathcal{A}$ be a collection of nonvanishing null sequences. Then there exists a non vanishing null sequence z that lacunary stat. converges faster than each x in $\mathcal{A}$ if and only if there exists a sequence $\{\mathcal{A}\}_{n=1}^{\infty}$ of subcollections of $\mathcal{A}$ such that*

(i) $\lim_n \frac{1}{h_n} |\{k \in I_n : x \in \mathcal{A}_k\}| = 1$ *(i.e., each x in $\mathcal{A}$ is in $\theta-$almost all $\mathcal{A}_k$)*

(ii) *for each n, $y_n := \inf\{|x_n| : x \in \mathcal{A}_n\} > 0$.*

PROOF. Necessity may be proved, by replacing $T-$density by $\theta-$density, in Theorem 2.4. So we just consider sufficiency. Assume that $\mathcal{A}$ is a collection of nonvanishing null sequences and $\{\mathcal{A}_n\}_{n=1}^{\infty}$ is a sequence of subcollections of $\mathcal{A}$ that satisfies (i) and (ii). Now define a sequence $z = \{z_n\}$ by

$$z_n = \begin{cases} \min(\frac{y_n}{h_n}, \frac{1}{h_n}), & \text{if } \mathcal{A}_n \neq \varnothing \\ \frac{1}{h_n}, & \text{if } \mathcal{A}_n = \varnothing \end{cases},$$

By (ii) and the fact that $h_n \to \infty$ as $n \to \infty$ z is a null sequence of positive numbers. If x is a sequence in $\mathcal{A}$, then $\delta_\theta(\{k \in \mathbb{N} : x \in \mathcal{A}_k\}) = 1$. Therefore $0 < y_n \leq |x_n|$ for $\theta-$ almost all n. Hence $\frac{z_n}{|x_n|} \leq \frac{y_n}{|x_n|\, h_n} \leq \frac{1}{h_n}$ for $\theta-$almost all n, whence z lacunary statistically converges faster than x. ☐

The following result is an analog of Theorem 2.5 that can be proved by replacing $T-$density with $\theta-$density.

THEOREM 3.5. *Assume that $\mathcal{A}$ is a collection of nonvanishing null sequences. Then there exists a nonvanishing null sequence z which lacunary statistically converges slower than each x in $\mathcal{A}$ if and only if there exists a sequence $\{\mathcal{A}\}_{n=1}^\infty$ of subcollections of $\mathcal{A}$, a null sequence $\{\varepsilon_n\}_{n=1}^\infty$ of positive numbers, and a strictly increasing sequence $\{N_n\}_{n=1}^\infty$ of nonnegative integers such that*

a) $\sup\{|x_k| : x \in \mathcal{A}_n,\, N_{n-1} < k \leq N_n\} \leq \varepsilon^2$ *for every n and*
b) *for each $x \in \mathcal{A}$, $\delta_\theta(\mathfrak{n}_x) = 1$, where*

$$\mathfrak{n}_x = \cup\{(N_{n-1}, N_n] : x \in \mathcal{A}_n\}.$$

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