GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 73 – 76

A COMPARISON RESULT FOR A LINEAR DIFFERENTIAL EQUATION WITH PIECEWISE CONSTANT DELAY

Juan J. Nieto*
University of Santiago de Compostela, Spain

Abstract. We present a new comparison result for an equation of the form $u'(t) + mu(t) + nu([t]) = \sigma(t)$, $t \in [0, T]$ where $[\,\cdot\,]$ denotes the greatest integer function.

1. Introduction

The study and applications of functional differential equations and in particular of delay differential equations is of great interest. Many times the delay function is assumed to be constant or continuous. However, in some situations the delay is not continuous but piecewise continuous [1, 3, 4, 6, 7, 8, 9].

On the other hand, the theory of differential inequalities plays a prominent role in the study of differential equations.

In this paper we consider a functional differential inequality with piecewise constant delay of the form

$$u'(t) + mu(t) + nu([t]) \leq 0,$$

where $[\cdot]$ is the greatest integer function. We prove a new comparison result obtaining a better estimate than others already known.

2. Preliminaries

Let $T > 0, m > 0$ and $I = [0, T]$. It is well-known that if $u \in C^1(I)$ is such that

$$u'(t) + mu(t) \leq 0, \qquad t \in I,$$

2000 *Mathematics Subject Classification.* 34K06, 34K10.
Key words and phrases. Differential inequalities, functional differential equations, piecewise constant delay, periodic boundary conditions.
*Research partially supported by MCYT/FEDER (Spain), project BFM2001 – 3884.

and

$$u(0) \leq u(T)$$

then $u(t) \leq 0$ for every $t \in I$.

If we consider the functional inequality

$$u'(t) + mu(t) + nu(h(t)) \leq 0, \qquad t \in I, \tag{1}$$

where $n \geq 0$ and $h : I \to I$ with $h(t) \leq t$, $t \in I$ then at least a restriction on n is necessary as the following example shows.

Example 2.1. Consider

$$u : I \to I, \qquad u(t) = t - \frac{T}{2},$$

and

$$h : I \to I, \qquad h(t) = 0, \quad t \in I.$$

Thus, for n sufficiently large we have for any $t \in I$ that

$$u'(t) + mu(t) + nu(h(t)) = 1 + m(t - \frac{T}{2}) - n\frac{T}{2} \leq 1 + (m - n)\frac{T}{2} \leq 0.$$

However, $u(T) = \frac{T}{2} > 0$, and $u(0) = \frac{-T}{2} < u(T)$.

It is possible to give an estimate for n, and we have the following result.

Theorem 2.2. *Let h be continuous. If $u \in C^1(I)$ satisfies the inequality (1), $u(0) \leq u(T)$, and*

$$n\frac{e^{mT} - 1}{m} < 1, \tag{2}$$

then $u(t) \leq 0$, $t \in I$.

This result has been proved independently by Jiang and Wei (Th. 2.1, [2]) and Nieto (Th. 5 and Cor. 2, [5]). In fact in [5] it is considered a more general equation of the form

$$u'(t) + mu(t) + [p(u)](t) = \sigma(t), \qquad t \in I$$

where $p : L^1(I) \to L^1(I)$ is a given functional (not necessarily continuous). Of course, it contains the case $[p(v)](t) = nu(h(t))$.

Note that the case $h(t) = [t]$ is not included in [2] since it is not continuous, but it is a particular case of the results of [5].

The purpose of this note is to consider the important case $h(t) = [t]$ to obtain a better estimate for n and m. Note that the results of [2] are not applicable to this type of delay since the greatest integer function is not continuous.

3. Main Result

Let $k \in \mathbf{N}$, such that $T \in (k, k+1]$. Consider the space E of functions $u : I \to \mathbf{R}$ such that $u \in C(I)$, $u'(t)$ exists and is continuous on $[i-1, i)$, $i = 1, 2, \ldots, k$ and $[k, T)$.

Theorem 3.1. *Suppose that $u \in E$ is such that*

$$u'(t) + mu(t) + nu([t]) \le 0, \qquad t \in I, \tag{3}$$

$$u(0) \le u(T), \tag{4}$$

$$\frac{n(e^m - 1)}{m} < 1. \tag{5}$$

Then $u(t) \le 0$ for every $t \in I$.

Proof. Let $v(t) = e^{mt}u(t)$, $t \in I$. Thus, by (3), for every $t \in I$ we get

$$v'(t) = e^{mt}u'(t) + me^{mt}u(t) \le -ne^{mt}e^{-m[t]}v([t]). \tag{6}$$

Now, for $i = 1, 2, \ldots, k$, consider an interval of the form $[i-1, i)$. For $t \in [i-1, i]$ we can write

$$\begin{aligned} v(t) &= v(i-1) + \int_{i-1}^{t} v'(s)\, ds \le v(i-1) - nv(i-1) \int_{i-1}^{t} e^{m(s-[s])}\, ds \\ &= v(i-1) - nv(i-1)\frac{e^{m(t-i+1)} - 1}{m}. \end{aligned}$$

In particular, for $t = i, i = 1, 2, \ldots, k$ we obtain

$$v(i) \le v(i-1)\Big(1 - \frac{n}{m}(e^m - 1)\Big). \tag{7}$$

Analogously, for $t \in [k, T]$ we have the inequality

$$v(t) \le v(k) - nv(k)\frac{e^{m(t-k)} - 1}{m}.$$

and in particular

$$v(T) \le v(k)\Big(1 - \frac{n}{m}\,(e^{m(T-k)} - 1)\,\Big). \tag{8}$$

We now show that

$$v(i) > 0 \text{ for every } i = 1, 2, \ldots, k \tag{9}$$

is not possible. Indeed, if (9) holds, then using (6) we see that v is decreasing on each subinterval $[i-1, i], i = 1, 2, \ldots, k$ and on $[k, T]$. Thus,

$$v(0) \ge v(1) \ge v(2) \ge \cdots \ge v(k-1) \ge v(k) \ge v(T),$$

and $u(0) = v(0) \ge v(T) = e^{mT}u(T)$ which contradicts (4).

Therefore, there exists $j \in \{0, 1, \ldots, k\}$ such that $v(j) \le 0$. Hence, (5) and (7) imply that

$$v(i) \le 0, i = j+1, \ldots, k,$$

and from (8) we deduce that $v(T) \leq 0$. Using (4) we get that

$$v(0) \leq 0. \tag{10}$$

This implies, in view of (5) and (7), that $v(1) \leq 0$ and $v(i) \leq 0, i = 1, 2, \ldots, k$. Now, by (8) we also have $v(T) \leq 0$. On the other hand, (6) implies that v is decreasing on every subinterval $[i-1, i], i = 1, 2, \ldots, k$ and (8) that it is decreasing on $[k, T]$. Hence $v(t) \leq 0$ for every $t \in I$, and, in consequence, $u(t) \leq 0, t \in I$. □

We point out that estimate (5) is better that estimate (2) since for $T > 1$ we have that $e^{mT} > e^m$. Of course, if $T \leq 1$ then $[t] = 0$ for every $t \in I$ and we have in (3) an ordinary differential inequality.

References

[1] K.L. Cooke and J. Wiener, *A survey of differential equations with piecewise continuous arguments*, in: Delay differential equations and dynamical systems (Claremont, CA, 1990), 1–15. Lecture Notes in Math., 1475, Springer, Berlin, 1991

[2] D. Jiang and J. Wei, *Monotone method for first order and second order periodic boundary value problems and periodic solutions of functional differential equations*, Nonlinear Anal. **50** (2002), 885–898.

[3] S.A.S. Marconato and A. Spezamiglio, *Stability of differential equations with piecewise constant argument via discrete equations*, Dynam. Contin. Discrete Impuls. Systems **7** (2000), 325–333.

[4] H. Matsunaga T. Hara, and S. Sakata, *Global attractivity for a logistic equation with piecewise constant argument*, NoDEA Nonlinear Differential Equations Appl. **8** (2001), 45–52.

[5] J.J. Nieto, *Differential inequalities for functional perturbations of first order ordinary differential equations*, Appl. Math. Lett. **15** (2002), 173–179.

[6] S. Sakata and T. Hara, *Dynamics of a linear differential system with piecewise constant argument*, Dynam. Contin. Discrete Impuls. Systems **7** (2000), 585–594.

[7] G. Seifert, *Almost periodic solutions of certain differential equations with piecewise constant delays and almost periodic time dependence*, J. Differential Equations **164** (2000), 451–458.

[8] J. Wiener, Generalized solutions of functional-differential equations. World Scientific Pub., River Edge, NJ, 1993.

[9] F. Zhang and A. Aimin, *Monotone iterative method for differential equations with piecewise constant arguments*, Portugal. Math. **57** (2000), 345–353.

Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Santiago de Compostela
Santiago de Compostela, 15782
Spain
E-mail: `amnieto@nsc.es`

Received: 21.02.2003