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# OSCILLATION OF HIGHER ORDER DIFFERENCE EQUATIONS VIA COMPARISON

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Abstract. In this paper we shall present some new oscillation criteria for difference equations of the form

$$\Delta^m x(n) + q(n) f(x[n-\tau]) = 0$$

and

$$\Delta^m x(n) = q(t) f(x[n-\tau]) + p(n) F(x[n+\sigma])$$

via comparison with some difference equations of lower order whose oscillatory behavior are known.

## 1. Introduction

In this paper we are concerned with the oscillatory behavior of solutions of higher order difference equations

$$\Delta^m x(n) + q(n) f(x[n-\tau]) = 0 \tag{1.1}$$

and

$$\Delta^m x(n) = q(n) f(x[n-\tau]) + p(n) F(x[n+\sigma]), \tag{1.2}$$

where $m \geq 2, \Delta$ is the forward difference operator defined as follows:

$$\Delta^0 x(n) = x(n), \quad \Delta^m x(n) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} x(n+j), \quad m \geq 1.$$

Further, in what follows it is assumed that

(i) $p, q : \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\} \to \mathbb{R}^+ = (0, \infty)$ for some $n_0 \in \mathbb{N} = \{0, 1, \cdots\}$,
(ii) $\tau$ and $\sigma \geq 0$,

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(iii) $f, F : \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ are continuous functions satisfying $xf(x) > 0$ and $xF(x) > 0$ for $x \neq 0$ and both $f$ and $F$ are nondecreasing.

For $r \in \mathbb{R}$ and $s$ a nonnegative integer, the factorial expression is defined as

$$(r)^{(s)} = \prod_{i=0}^{s-1}(r-i) \quad \text{with} \quad (r)^{(0)} = 1.$$

By a solution of equation (1.1) (or (1.2)), we mean a nontrivial sequence $\{x(n)\}$ satisfying equation (1.1) (or (1.2)) respectively for all $n \in \mathbb{N}(n_0)$, where $n_0$ is some nonnegative integer. A solution $\{x(n)\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and it is nonoscillatory otherwise. An equation is said to be oscillatory if all its solutions are oscillatory.

In recent years, the oscillation of equations (1.1) and (1.2) when $m \geq 1$ has been studied extensively. For recent contributions, we refer the reader to the monographs of Agarwal et. al. [1,2,6] and Györi et. al. [8], also the papers [4,5,7,9–11] and the references cited therein.

The purpose of this paper is to study the oscillatory behavior of all solutions of equations (1.1) and (1.2). The main results are new and independent of the analogous ones known for difference equations (see, for example, [1,2,4–7,10,11] and the references contained therein).

To obtain our results we need the following lemmas in which the first is the discrete analog of the well–known Kiguradze's lemma.

Lemma 1.1. *Let $x(n)$ be defined on $\mathbb{N}(n_0), x(n) > 0$ and $\Delta^m x(n)$ be eventually of one sign on $\mathbb{N}(n_0)$. Then there exist an integer $\ell$ and $n_1 \in \mathbb{N}(n_0), 0 \leq \ell \leq m$ with $m + \ell$ odd for $\Delta^m x(n) \leq 0$ eventually, or $m + \ell$ even for $\Delta^m x(n) \geq 0$ eventually such that*

(1.3)
$$\begin{cases} \ell \leq m-1 \text{ implies } (-1)^{\ell+k}\Delta^k x(n) > 0 \text{ for all } n \in \mathbb{N}(n_1),\ \ell \leq k \leq m-1 \\ \ell \geq 1 \text{ implies } \Delta^k x(n) > 0 \text{ for all } n \in \mathbb{N}(n_1),\ 1 \leq k \leq \ell - 1. \end{cases}$$

Lemma 1.2. *Let $q, \tau$ and $f$ be as in (i), (ii) and (iii) respectively. If the inequality*

$$\left\{\Delta^2 x(n) + q(n)f(x[n-\tau])\right\} \operatorname{sgn} x[n-\tau] \leq 0 \tag{1.4}$$

*has a nonoscillatory solution, then so does the equation*

$$\Delta^2 x(n) + q(n)f(x[n-\tau]) = 0. \tag{1.5}$$

Lemma 1.3. *Let (i) – (iii) hold. If the inequality*

$$\{\Delta x(n) + q(n)f(x[n-\tau])\} \operatorname{sgn} x[n-\tau] \leq 0 \tag{1.6}$$

*has a nonoscillatory solution, then so does the equation*

$$\Delta x(n) + q(n)f(x[n-\tau]) = 0. \tag{1.7}$$

*Also, if the inequality*

$$\{\Delta x(n) - p(n)F(x[n+\sigma])\}\ sgn\ x[n+\sigma] \geq 0 \tag{1.8}$$

*has a nonoscillatory solution, then so does the equation*

$$\Delta x(n) - p(n)F(x[n+\sigma]) = 0.$$

The proof of Lemmas 1.2 and 1.3 may be found in [6,8,10]. Also, these are discrete analog of the results established in [3].

We shall assume that

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0, \tag{1.9}$$

$$-F(-xy) \geq F(xy) \geq F(x)F(y) \quad \text{for} \quad xy > 0 \tag{1.10}$$

and

$$\sum^{\infty}(n-\tau)^{(m-j)} f\left((n-\tau)^{(j-1)}\right) q(n) = \infty, \quad j = 1, 2, \cdots, m. \tag{1.11}$$

For simplicity, we put for all sufficiently large $n$,

$$Q_j(n) = \sum_{r=n}^{\infty} \frac{(r-n+m-j-2)^{(m-j-2)}}{(m-j-2)!} q(r) f\left(\frac{(r-\tau-m+j-1)^{(j-1)}}{j!}\right),$$
$$j = 1, 2, \cdots, m-2,$$

$$Q_{m-1}(n) = q(n) f\left(\frac{(n-\tau-2)^{(m-2)}}{(m-1)!}\right),$$

$$Q_0(n) = q(n) f\left(\sum_{r=n-\tau}^{n-\overline{\tau}} \frac{(r-n+\tau+m-2)^{(m-2)}}{(m-2)!}\right),$$
$$\text{for some} \quad \overline{\tau} > 0 \quad \text{with} \quad \tau > \overline{\tau},$$

$$\overline{Q}_j(n) = \sum_{r=n_1+\tau}^{n} \frac{(r-n_1+m-j-1)^{(m-j-1)}}{(m-j-1)!} q(n)$$
$$f\left(\frac{(r-\tau-m+j-1)^{(j-1)}}{j!}\right),$$
$$j = 1, 2, \cdots, m-1 \quad \text{and some} \quad n_1 \geq n_0,$$

$$Q_j^*(n) = \sum_{r=n}^{\infty} \frac{(r-n+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q(r) f\left(\frac{(r-\tau-m+\ell-1)^{(\ell-1)}}{\ell!}\right),$$
$$j = 1, 2, \cdots, m-1,$$

$$Q_m(n) = q(n) F\left(\frac{(\sigma-\overline{\sigma})^{(m-1)}}{(m-1)!}\right) \quad \text{for some} \quad \overline{\sigma} > 0 \quad \text{with} \quad \sigma > \overline{\sigma}.$$

## 2. Oscillation of Equation (1.1)

In this section we shall present some oscillation results for equation (1.1).

Theorem 2.1. *Let $m$ be even, conditions (i) – (iii), (1.9) and (1.11) hold. If for sufficiently large $n$ all second order difference equations*

$$\Delta^2 z(n) + Q_j(n) f(z[n-\tau]) = 0, \quad j = 1, 3, \cdots, m-1 \tag{2.1; j}$$

*are oscillatory, then equation (1.1) is oscillatory.*

Theorem 2.2. *Let $m$ be odd, conditions (i) – (iii), (1.9) and (1.11) hold. If for all sufficiently large $n$ all second order difference equations* (2.1; $j$), $j = 2, 4, \cdots, m-1$ *are oscillatory, then every solution $\{x(n)\}$ of equation (1.1) is either oscillatory, or $\lim_{n\to\infty} \Delta^i x(n) = 0$ monotonically for $i = 0, 1, \cdots, m-1$. In addition, if there exists a positive integer $\overline{\tau}$ with $\overline{\tau} < \tau$ such that the first order difference equation*

$$\Delta u(n) + Q_0(n) f(u[n-\overline{\tau}]) = 0 \tag{2.2}$$

*is oscillatory, then equation (1.1) is oscillatory.*

Proofs of Theorems 2.1 and 2.2. Assume that equation (1.1) has a nonoscillatory solution $\{x(n)\}$, say, $x(n) > 0$ for $n \ge n_0 \ge 0$. By Lemma 1.1, $x(n)$ satisfies (1.3) for some $\ell \in \{0, 1, 2, \cdots, m-1\}$ with $\ell + m$ odd for $n \ge n_1$, for some $n_1 \ge n_0$.

From the discrete Taylor's formula, it follows that $x$ satisfies the equality

$$\begin{aligned} \Delta^\ell x(n) = {} & \sum_{j=\ell}^{k-1} \frac{(s-n+j-\ell-1)^{(j-\ell)}}{(j-\ell)!} (-1)^{j-\ell} \Delta^j x(s) \\ & + (-1)^{k-\ell} \sum_{r=n}^{s-1} \frac{(r-n+k-\ell-1)^{(k-\ell-1)}}{(k-\ell-1)!} \Delta^k x(r) \end{aligned} \tag{2.3}$$

for $s \ge n \ge n_1, 0 \le \ell \le m-1$ and $0 \le k \le m$. Now, we consider the following three cases:

(I) $\ell \in \{1, 2, \cdots, m-2\}$,
(II) $\ell = m-1$,
(III) $\ell = 0$.

Case (I). Let $\ell \in \{1, 2, \cdots, m-2\}$. From (2.3) with $\ell$ and $k$ replaced by $\ell+1$ and $m$ respectively, we have

$$\begin{aligned} -\Delta^{\ell+1} x(n) = {} & \sum_{j=\ell+1}^{m-1} \frac{(s-n+j-\ell-2)^{(j-\ell-1)}}{(j-\ell-1)!} (-1)^{j-\ell} \Delta^j x(s) \\ & + (-1)^{m-\ell} \sum_{r=n}^{s-1} \frac{(r-n+m-\ell-2)^{(m-\ell-2)}}{(m-\ell-2)!} \Delta^m x(r). \end{aligned} \tag{2.4}$$

Using (1.3) and equation (1.1) in (2.4), we obtain

$$-\Delta^{\ell+1}x(n) \geq \sum_{r=n}^{s-1} \frac{(r-n+m-\ell-2)^{(m-\ell-2)}}{(m-\ell-2)!} q(r) f(x[r-\tau]). \tag{2.5}$$

Letting $s \to \infty$ in (2.5), we get

$$-\Delta^{\ell+1}x(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-\ell-2)^{(m-\ell-2)}}{(m-\ell-2)!} q(r) f(x[r-\tau]). \tag{2.6}$$

Next, from the equality

$$\begin{aligned}\sum_{j=i}^{m-1} &\frac{(-1)^j}{(j-i)!}(n-m+j)^{(j-i)}\Delta^j x(n) \\ &= \sum_{j=i}^{m-1} \frac{(-1)^j}{(j-i)!}(n_1)^{(j-i)}\Delta^j x(n_1+m-j-1) \\ &\quad + \frac{(-1)^{m-1}}{(m-i-1)!}\sum_{s=n_1}^{n-1} s^{(m-i-1)}\Delta^m x(s)\end{aligned}$$

and condition (1.11), one can proceed as in [7] to obtain

$$x(n) \geq \frac{(n-m+\ell-1)^{(\ell-1)}}{\ell!}\Delta^{\ell-1}x(n), \quad n \geq n_1. \tag{2.7}$$

There exists an $n_2 \in \mathbb{N}(n_1+\tau)$ such that

$$x[n-\tau] \geq \frac{(n-\tau-m+\ell-1)^{(\ell-1)}}{\ell!}\Delta^{\ell-1}x[n-\tau], \quad n \geq n_2. \tag{2.8}$$

Using condition (1.9) and inequality (2.7) in inequality (2.6), we have

$$\begin{aligned}-\Delta^{\ell+1}x(n) \geq \sum_{r=n}^{\infty} &\frac{(r-n+m-\ell-2)^{(m-\ell-2)}}{(m-\ell-2)!} q(r) \\ &f\left(\frac{(r-\tau-m+\ell-1)^{(\ell-1)}}{\ell!}\right) f\left(\Delta^{\ell-1}x[n-\tau]\right)\end{aligned} \tag{2.9}$$

for $n \geq n_2$. Set $y(n) = \Delta^{\ell-1}x(n) > 0$ for $n \geq n_2$, and we see that $\{y(n)\}$ satisfies

$$\Delta^2 y(n) + Q_\ell(n) f(y[n-\tau]) \leq 0.$$

Lemma 1.2 now implies that the equation

$$\Delta^2 y(n) + Q_\ell(n) f(y[n-\tau]) = 0$$

has an eventually positive solution. But this contradicts our assumption.

Case (II). Let $\ell = m-1$. It follows from (2.8) that

$$x[n-\tau] \geq \frac{(n-\tau-2)^{(m-2)}}{(m-1)!}\Delta^{m-2}x[n-\tau] \quad \text{for} \quad n \geq n_2. \tag{2.10}$$

Using condition (1.9) and (2.10) in equation (1.1), we have

$$\begin{aligned} -\Delta^m x(n) &= q(n) f(x[n-\tau]) \\ &\geq q(n) f\left(\frac{(n-\tau-2)^{(m-2)}}{(m-1)!}\right) f\left(\Delta^{m-2} x[n-\tau]\right) \\ &= Q_{m-1}(n) f\left(\Delta^{m-2} x[n-\tau]\right) \quad \text{for} \quad n \geq n_2. \end{aligned}$$

The rest of the proof is similar to that of Case (I) and hence omitted.

Case (III). Let $\ell = 0$. This is the case when $m$ is odd. As in [7] one can easily see that condition (1.11) implies that $\lim_{n\to\infty} x(n) = 0$. Therefore, $\lim_{n\to\infty} \Delta^i x(n) = 0$ monotonically for $i = 0, 1, \cdots, m-1$.

It follows from (2.3) with $\ell = 0$ and $k = m-1$ that for $s \geq n \geq n_1$

$$x(n) \geq \sum_{r=n}^{s-1} \frac{(r-n+m-2)^{(m-2)}}{(m-2)!} \Delta^{m-1} x(r).$$

By the hypothesis there exists $\overline{n}_1 \geq n_1$ such that

$$x[n-\tau] \geq \left(\sum_{r=n-\tau}^{n-\overline{\tau}} \frac{(r-n+\tau+m-2)^{(m-2)}}{(m-2)!}\right) \Delta^{m-1} x[n-\overline{\tau}]. \tag{2.11}$$

Using (1.9) and (2.11) in equation (1.1), we obtain

$$\begin{aligned} -\Delta^m x(n) = &\; q(n) f(x[n-\tau]) \\ \geq &\; q(n) f\left(\sum_{r=n-\tau}^{n-\overline{\tau}} \frac{(r-n+\tau+m-2)^{(m-2)}}{(m-2)!}\right) f\left(\Delta^{m-1} x[n-\overline{\tau}]\right), \end{aligned}$$

or

$$\Delta y(n) + Q_0(n) f(y[n-\overline{\tau}]) \leq 0,$$

where $y(n) = \Delta^{m-1} x(n) > 0$ for $n \geq \overline{n}_1$. Lemma 1.3 now implies that the equation

$$\Delta y(n) + Q_0(n) f(y[n-\overline{\tau}]) = 0$$

has an eventually positive solution. But this contradicts our assumption and completes the proofs of Theorems 2.1 and 2.2. ∎

Next, we present the following oscillation criteria for equation (1.1).

THEOREM 2.3. *Let $m$ be even, conditions (i) – (iii), (1.9) and (1.11) hold. If for some $n_1 \geq n_0$*

$$\lim_{n\to\infty} \overline{Q}_j(n) = \infty, \quad j = 1, 3, \cdots, m-1 \tag{2.12; j}$$

*then equation (1.1) is oscillatory.*

THEOREM 2.4. *Let $m$ be odd, conditions (i) – (iii), (1.9) and (1.11) hold. If for some $n_1 \geq n_0$ condition* $(2.12; j)$ *holds with $j = 2, 4, \cdots, m-1$ then every solution $\{x(n)\}$ of equation (1.1) is either oscillatory, or* $\lim_{n\to\infty} \Delta^i x(n) = 0$ *monotonically for $i = 0, 1, \cdots, m-1$. In addition, if there exists a positive integer $\overline{\tau}$ with $\overline{\tau} < \tau$ such that either*

$$(2.13) \qquad \frac{f(x)}{x} \geq 1 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{s=n-\overline{\tau}}^{n-1} Q_0(s) > \left(\frac{\overline{\tau}}{\overline{\tau}+1}\right)^{\overline{\tau}+1},$$

*or*

$$(2.14) \qquad \int_{\pm 0} \frac{du}{f(u)} < \infty \quad \text{and} \quad \sum^{\infty} Q_0(s) = \infty,$$

*then equation (1.1) is oscillatory.*

PROOFS OF THEOREMS 2.3 AND 2.4. Assume that equation (1.1) has a nonoscillatory solution $\{x(n)\}$, say $x(n) > 0$ for $n \geq n_0 \geq 0$. By Lemma 1.1, $x(n)$ satisfies (1.2) for some $\ell \in \{0, 1, 2, \cdots, m-1\}$ with $\ell + m$ odd for $n \geq n_1 \geq n_0$.

Next, we shall consider the following two cases:

(I) $\ell \in \{1, 2, \cdots, m-1\}$,

(II) $\ell = 0$.

The proof of Case (II) is similar to that of Case (III) in Theorems 2.1 and 2.2 except that we apply known result in [8,9] to equation (2.2). Thus, we shall consider Case (I).

Case (I). Let $\ell \in \{1, 2, \cdots, m-1\}$. From equality (2.3) with $k = m$ and $n = n_1$ one can easily find for $s > n_1$,

$$(2.15) \qquad \infty > \Delta^\ell x(n_1) \geq \sum_{r=n_1}^{s-1} \frac{(r - n_1 + m - \ell - 1)^{(m-\ell-1)}}{(m-\ell-1)!} q(r) f(x[r-\tau]).$$

As in the proof of Theorems 2.1 and 2.2, we obtain (2.7) for $n \geq n_2$. Since $\Delta^{\ell-1}x(n)$ is increasing for $n \geq n_1$, there exist $n_3 \geq n_1$ and a constant $c > 0$ such that

$$(2.16) \qquad \Delta^{\ell-1} x[r-\tau] \geq c \quad \text{for} \quad r \geq n_3.$$

Using (1.9), (2.7) and (2.16) in (2.15), we have

$$\begin{aligned} \infty > \Delta^\ell x(n_1) \quad &\geq \sum_{r=n_1+\tau}^{s-1} \frac{(r - n_1 + m - \ell - 1)^{(m-\ell-1)}}{(m-\ell-1)!} q(r) f(c) \\ &\qquad f\left(\tfrac{(r-\tau-m+\ell-1)^{(\ell-1)}}{\ell!}\right) \\ &= \overline{Q}_\ell(s) f(c) \to \infty \quad \text{as} \quad s \to \infty, \end{aligned}$$

which contradicts condition (2.12). This completes the proof. ∎

Now, we present the following comparison theorems.

THEOREM 2.5. *Let $m$ be even, conditions (i) – (iii), (1.9) and (1.11) hold. If for all sufficiently large $n$, all the first order delay difference equations*

$$\Delta z(n) + Q_j^*(n) f(z[n-\tau]) = 0, \quad j = 1, 3, \cdots, m-1 \tag{2.17; j}$$

*are oscillatory, then equation (1.1) is oscillatory.*

THEOREM 2.6. *Let $m$ be odd, conditions (i) – (iii), (1.9) and (1.11) hold. If there exists a positive integer $\overline{\tau}$ with $\overline{\tau} < \tau$ such that for all large $n$ all the equations* $(2.17; j), j = 2, 4, \cdots, m-1$ *and (2.2) are oscillatory, then equation (1.1) is oscillatory.*

PROOF. Assume that equation (1.1) has a nonoscillatory solution $\{x(n)\}$, say, $x(n) > 0$ for $n \geq n_0 \geq 0$. As in the proof of Theorems 2.1 and 2.2, we consider two cases:

(I) $\ell \in \{1, 2, \cdots, m-1\}$,
(II) $\ell = 0$.

The proof of Case (II) is similar to that of Case (III) of Theorems 2.1 and 2.2 and hence will be omitted. Thus, we consider Case (I).

Case (I). Let $\ell \in \{1, 2, \cdots, m-1\}$. It follows from (2.3) that

$$\Delta^\ell x(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q(r) f(x[r-\tau]). \tag{2.18}$$

As in the proof of Theorems 2.1 and (2.2), we obtain (2.8). Using (1.9), (2.8) in (2.18), we have

$$\begin{aligned} \Delta^\ell x(n) \geq \ & \sum_{r=n}^{\infty} \frac{(r-n+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q(r) \\ & f\left(\frac{(r-\tau-m+\ell-1)^{(\ell-1)}}{\ell!}\right) f\left(\Delta^{\ell-1} x[n-\tau]\right), \end{aligned}$$

or

$$\Delta y(n) + Q_\ell^*(n) f(y[n-\tau]) \leq 0 \quad \text{for} \quad n \geq n_1 \geq n_0.$$

Lemma 1.3 now implies that the equation

$$\Delta y(n) + Q_\ell^*(n) f(y[n-\tau]) = 0$$

has an eventually positive solution. But this contradicts our assumption and completes the proof. ∎

The following results are immediate.

COROLLARY 2.7. *Let $m$ be even, conditions (i) – (iii), (1.9) and (1.11) hold. Equation (1.1) is oscillatory if either one of the following conditions holds:*

($e_1$) *$f(x)/x \geq 1$ for $x \neq 0$ and*

$$\liminf_{n\to\infty} \sum_{s=n-\tau}^{n-1} Q_j^*(s) > \left(\frac{\tau}{\tau+1}\right)^{\tau+1}, \quad j = 1, 3, \cdots, m-1 \tag{2.19; j}$$

*or*

($e_2$) *$\int_{\pm 0} du/f(u) < \infty$ and*

$$\sum^{\infty} Q_j^*(s) = \infty, \quad j = 1, 3, \cdots, m-1. \tag{2.20; j}$$

Corollary 2.8. *Let $m$ be odd, conditions (i) – (iii), (1.9) and (1.11) hold. Equation (1.1) is oscillatory if either one of the following conditions is satisfied:*

($o_1$) *condition* $(2.19; j), j = 2, 4, \cdots, m-1$ *and condition (2.13), or*
($o_2$) *condition* $(2.20; j), j = 2, 4, \cdots, m-1$ *and condition (2.14).*

## 3. Oscillation of Equation (1.2)

In this section we shall study the oscillatory behavior of equation (1.2). If equation (1.2) has a nonoscillatory solution $\{x(n)\}$, say, $x(n) > 0$ for $n \geq n_0 \geq 0$, then by Lemma 1.1 there exist an integer $n_1 \geq n_0$ and $\ell \in \{0, 1, \cdots, m\}$ with $\ell + m$ even such that (1.3) holds for $n \geq n_1$. Next, we shall consider the inequalities

$$\Delta^m x(n) \geq q(n) f(x[n-\tau]) \quad \text{when} \quad \ell \in \{0, 1, \cdots, m-1\} \tag{3.1}$$

and

$$\Delta^m x(n) \geq p(n) F(x[n+\sigma]) \quad \text{when} \quad \ell = n \tag{3.2}$$

and obtain the following oscillation results for equation (1.2).

Theorem 3.1. *Let $m$ be even, conditions (i) – (iii), (1.9) – (1.11) hold. If for all sufficiently large $n$, all the second order equations* $(2.1; j), j = 2, 4, \cdots, m-2$ *are oscillatory and there exist positive integers $\overline{\tau}$ and $\overline{\sigma}$ with $\overline{\tau} < \tau$ and $\overline{\sigma} < \sigma$ such that the first order delay equation (2.2) and the first order advanced equation*

$$\Delta y(n) + Q_m(n) F(y[n+\overline{\sigma}]) = 0 \tag{3.3}$$

*are oscillatory, then equation (1.2) is oscillatory.*

Theorem 3.2. *Let $m$ be odd, conditions (i) – (iii), (1.9) – (1.11) hold. If for all sufficiently large $n$, all the second order equations* $(2.1; j), j = 1, 3, \cdots, m-2$ *are oscillatory and there exists a positive integer $\overline{\sigma}$ with $\overline{\sigma} < \sigma$ such that equation (3.3) is oscillatory, then equation (1.2) is oscillatory.*

PROOFS OF THEOREMS 3.1 AND 3.2. Assume that equation (1.2) has a nonoscillatory solution $\{x(n)\}$, say, $x(n) > 0$ for $n \geq n_0 \geq 0$. By Lemma 1.1, $x(n)$ satisfies (1.3) for some $\ell \in \{0, 1, \cdots, m\}$ with $\ell + m$ even for $n \geq n_1$ for some $n_1 \geq n_0$. Now, we distinguish the following four cases:

(I) $\ell \in \{1, 2, \cdots, m-2\}$,
(II) $\ell = m-1$,
(III) $\ell = 0$,
(IV) $\ell = m$.

The proofs of the Cases (I), (II) and (III) are exactly the same as in Section 2 and hence omitted. It remains to consider the Case (IV).

Case (IV). Let $\ell = m$. From (2.3), one can easily see that

$$x(n) = x(s) + \sum_{j=1}^{m-1} \frac{(n-s)^{(j)}}{j!} \Delta^j x(s) + \sum_{r=s}^{n-m} \frac{(n-r-1)^{(m-1)}}{(m-1)!} \Delta^m x(r)$$

and hence

$$x(n) \geq \frac{(n-s)^{(m-1)}}{(m-1)!} \Delta^{m-1} x(s) \quad \text{for} \quad n \geq s \geq n_1.$$

There exists an $n_2 \geq n_1$ such that

$$x[n+\sigma] \geq \frac{(\sigma - \overline{\sigma})^{(m-1)}}{(m-1)!} \Delta^{m-1} x[n + \overline{\sigma}] \quad \text{for} \quad n \geq n_2. \tag{3.4}$$

Using (1.10) and (3.4) in inequality (3.2), we have

$$\Delta^m x(n) \geq q(n) F(x[n+\sigma]) \geq q(n) F\left(\frac{(\sigma-\overline{\sigma})^{(m-1)}}{(m-1)!}\right) F\left(\Delta^{m-1} x[n+\overline{\sigma}]\right),$$

or

$$\Delta v(n) \geq Q_m(n) F(v[n+\overline{\sigma}]) \quad \text{for} \quad n \geq n_2,$$

where $v(n) = \Delta^{m-1} x(n), n \geq n_2$. Lemma 1.3 now implies that the equation

$$\Delta v(n) - Q_m(n) F(v[n+\overline{\sigma}]) = 0$$

has an eventually positive solution. But this contradicts our assumption and completes the proof. ∎

Next, we have the following immediate results.

THEOREM 3.3. *Let $m$ be even, conditions (i) – (iii), and (1.9) – (1.11) hold. If for some $n_1 \geq n_0, \overline{\tau} > 0$ with $\overline{\tau} < \tau$ and some $\overline{\sigma} > 0$ with $\overline{\sigma} < \sigma$, condition* (2.12; $j$), $j = 2, 4, \cdots, m-2$, *condition (2.13) (or (2.14)) and either*

$$\frac{F(x)}{x} \geq 1 \quad \textit{for} \quad x \neq 0 \quad \textit{and} \quad \liminf_{n\to\infty} \sum_{j=n}^{n+\overline{\sigma}-1} Q_m(j) > \left(\frac{\overline{\sigma}-1}{\overline{\sigma}}\right)^{\overline{\sigma}}, \tag{3.5}$$

*or*

$$\int^{\pm\infty} \frac{du}{F(u)} < \infty, \tag{3.6}$$

*then equation (1.2) is oscillatory.*

THEOREM 3.4. *Let $m$ be odd, conditions (i) – (iii), and (1.9) – (1.11) hold. If for some $n_1 \geq n_0$ and some $\overline{\sigma} > 0$ with $\overline{\sigma} < \sigma$, condition $(2.12; j), j = 1, 3, \cdots, m-2$ and condition (3.5) (or (3.6)) hold, then equation (1.2) is oscillatory.*

Here, we note that condition (1.11) implies that

$$\sum^{\infty} q(j) = \infty, \tag{3.7}$$

which is required to ensure oscillation of the advanced superlinear equation (3.3) and hence omitted.

When $F \equiv 0$, i.e., equation (1.2) is reduced to the equation

$$\Delta^m x(n) = q(n) f(x[n-\tau]), \tag{3.8}$$

one can easily obtain the following oscillatory and asymptotic behavior results.

THEOREM 3.5. *Let $m$ be even, conditions (i) – (iii), (1.9) and (1.11) hold. If for some $\overline{\tau} > 0$ with $\overline{\tau} < \tau$ and all large $n$, all equations $(2.17; j), j = 2, 4, \cdots, m-2$ and (2.2) are oscillatory, then every solution $\{x(n)\}$ of equation (3.8) is either oscillatory or $|\Delta^i x(n)| \to \infty$ monotonically as $n \to \infty, i = 0, 1, \cdots, m-1$.*

THEOREM 3.6. *Let $m$ be odd, conditions (i) – (iii), (1.9) and (1.11) hold. If for all large $n$, all the equations $(2.17; j), j = 1, 3, \cdots, m-2$ are oscillatory, then every solution $\{x(n)\}$ of equation (3.8) is either oscillatory, or $|\Delta^i x(n)| \to \infty$ monotonically as $n \to \infty, i = 0, 1, \cdots, m-1$.*

As an illustrative example, we consider the mixed type of difference equation

$$\Delta^m x(n) = q|x[n-2m]|^{\alpha} \text{sgn}\, x[n-2m] + p|x[n+2m]|^{\beta} \text{sgn}\, x[n+2m], \tag{3.9}$$

where $m \geq 2, p, q, \alpha$ and $\beta$ are positive constants with $0 < \alpha \leq 1$ and $\beta \geq 1$. Here, we take $\tau = 2m, \sigma = 2m$ and hence choose $\overline{\tau} = m, \overline{\sigma} = m$. Now, for appropriate choices of the constants involved, one can easily see that equation (3.9) is oscillatory by Theorems 3.3 and 3.4.

We note that none of the known results which have appeared in the literature can be applied to describe the oscillatory behavior of equation (3.9).

Finally, we see that the results of this paper can be extended to more general equations of the form

$$\Delta^m x(n) + q(n) f(x[g(n)]) = 0, \tag{3.10}$$

$$\Delta^m x(n) = q(n)f(x[g(n)]) + p(n)F(x[h(n)]) \tag{3.11}$$

where $p(n), q(n), f$ and $F$ are as in equations (1.1) and (1.2), $g, h \in G = \{g, h : \mathbb{N}(n_0) \to \mathbb{N}$ for some $n_0 \in \mathbb{N} : \lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} h(n) = \infty\}$, $\{g(n)\}$ and $\{h(n)\}$ are nondecreasing sequences, $g(n) \le n$ and $h(n) \ge n$.

Also, we can extend our results to equations of the form

$$\Delta \left(\Delta^{m-1} x(n)\right)^\alpha + q(n)f(x[g(n)]) = 0 \tag{3.12}$$

and

$$\Delta \left(\Delta^{m-1} x(n)\right)^\alpha = q(n)f(x[g(n)]) + p(n)F(x[h(n)]), \tag{3.13}$$

where $\alpha$ is the ratio of positive odd integers, $p(n), q(n), g(n), h(n), f$ and $F$ are as in equations (3.10) and (3.11).

The statements and formulation of results are left to the reader.

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