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ON LANDAU'S THEOREMS

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Abstract. In this paper we give some applications and special cases of a generalization of the Landau's theorem for Frechet-differentiable functions.

1. Introduction

E. Landau has proved the following theorems [11]:

Theorem A. *Let $I \subseteq \mathbf{R}$ be an interval of length not less than 2 and let $f : I \to \mathbf{R}$ be a twice differentiable function satisfying $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ $(x \in I)$. Then*

$$|f'(x)| \leq 2 \quad (x \in I).$$

Furthermore, 2 is the best possible constant in the above inequality.

Theorem B. *Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function satisfying $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ $(x \in I)$. Then*

$$|f'(x)| \leq \sqrt{2} \quad (x \in \mathbf{R}).$$

Furthermore, $\sqrt{2}$ is the best possible constant in the above inequality.

There exists many generalizations of these results. In Section 2 we give some remarks about the generalization of Theorem A given in [9]. Some applications and special cases are given in Section 3.

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2. Landau's theorems for Frechet-differentiable functions

Let X and Y be Banach spaces. Given $a, b \in X$ $(a \neq b)$ define $g : X \to \mathbf{R}$ by

$$g(x) = \|x - a\|^2 + \|b - x\|^2 \quad (x \in X).$$

Let D be a convex subset of X such that $g(x) \leq \|b - a\|^2$ for every $x \in D$ and suppose that $a, b \in \overline{D}$. Furthermore, let $f : X \to Y$ be twice Frechet–differentiable on $\overline{D}$. With these assumptions the following generalizations of Theorems A and B have been proven in [9]:

Theorem C. *If $\|F(x)\| \leq M$ $(x \in \overline{D})$ and $\|F''_{(x)}(h, h)\| \leq N\|h\|^2$ $(h \in X,\ x \in D)$, then*

$$\|F'_{(x)}(b - a)\| \leq 2M + \frac{N}{2}g(x) \leq 2M + \frac{N}{2}\|b - a\|^2 \quad (x \in \overline{D}).$$

Theorem D. *If $\|F''_{(x)}(h, h)\| \leq N\|h\|^2$ $(h \in X,\ x \in D)$, then*

$$\|F'_{(x)}(b - a) - F(b) + F(a)\| \leq \frac{N}{2}g(x) \quad (x \in \overline{D}).$$

We prove now the following generalization of these results.

Theorem 2.1. *Suppose that*

$$\|F''_{(x)}(h, h)\| \leq H(h) \quad (h \in X,\ x \in D), \tag{2.1}$$

where H is a function from X to $\mathbf{R}^+$. Then for all $x \in \overline{D}$

$$\|F_{(x)}(b - a) - F(b) + F(a)\| \leq \frac{1}{2}\left(H(a - x) + H(b - x)\right). \tag{2.2}$$

Under the further assumption

$$\|F(x)\| \leq M \quad (x \in \overline{D}), \tag{2.3}$$

then for all $x \in \overline{D}$

$$\|F'_{(x)}(b - a)\| \leq 2M + \frac{1}{2}\left(H(a - x) + H(b - x)\right). \tag{2.4}$$

Proof. If $x \in \overline{D}$ and $h \in X$ are such that $x + th \in D$ for every t, $0 < t < 1$, then the Taylor's formula holds true:

$$F(x + h) = F(x) + F'_{(x)}(h) + w(x, h)$$

where $w(x, h) = \frac{1}{2}F_{(x+th)}(h, h)$ for some t, $0 < t < 1$. Combining the two formulas for $h = a - x$ and $h = b - x$ we obtain

$$F'_{(x)}(b - a) - F(b) + F(a) = w(x, a - x) - w(b - x). \tag{2.5}$$

Now, (2.5) together with (2.1) implies (2.2). Similarly, (2.5) together with (2.1) and (2.3) implies (2.4). ∎

REMARK 2.2. If the function H in Theorem 2.1 is even $(H(-h) = H(h))$ then for $a = x - h$ and $b = x + h$ we obtain from (2.4):

$$\|F'_{(x)}(2h)\| \le 2M + H(h). \tag{2.6}$$

REMARK 2.3. The inequalities (2.4) and (2.6) hold true if instead of (2.3) we have

$$\|F(b) - F(a)\| \le 2M. \tag{2.7}$$

REMARK 2.4. For $(H(h) = N\|h\|^2$ we obtain Theorems C and D.

3. Some applications

COROLLARY 3.1. *Let $f : [a, b+h] \to \mathbf{R}$ be a differentiable function $(a < b,\ h > 0)$ such that*

$$|\delta_h f'(x)| \le N \quad (x \in (a,b)), \tag{3.8}$$

where $\delta_h g(x) = \frac{1}{h}(g(x+h) - g(x))$. Then

$$|(b-a)\delta_h f(x) - f(b) + f(a)| \le \frac{N}{2}\left[(x-a)^2 + (x-b)^2\right]. \tag{3.9}$$

If we also have

$$m \le f(x) \le M \quad (a \le x \le b+h), \tag{3.10}$$

then

$$(b-a)\,|\delta_h f(x)| \le M - m + \frac{N}{2}\left[(x-a)^2 + (x-b)^2\right]. \tag{3.11}$$

PROOF. This follows from Theorem 2.1 and Remark 2.3 for $X = Y = \mathbf{R}$, $\|x\| = |x|$, $F(x) = \frac{1}{h}\int\limits_x^{x+h} f(t)dt$ $(a \le x \le b)$, $D = (a,b)$, $H(h) = Nh^2$. ∎

COROLLARY 3.2. *Let the conditions of Corollary 3.1 be fulfilled. Then*

$$|\delta_n f(x)| \le \begin{cases} \frac{M-m}{b-a} + \frac{b-a}{2}N, & \text{if} \quad b-a \le \sqrt{\frac{2(M-m)}{N}} \\ \sqrt{2(M-m)N}, & \text{if} \quad b-a \ge \sqrt{\frac{2(M-m)}{N}}. \end{cases} \tag{3.12}$$

PROOF. From (3.11) we get

$$|\delta_h f(x)| \le \frac{M-m}{b-a} + \frac{b-a}{2}N$$

and if $b - a \ge \sqrt{2(M-m)/N}$ we obtain

$$|\delta_h f(x)| \le \sqrt{2(M-m)N}$$

since the function $g(y) = \frac{M-m}{y} + \frac{N}{2}y$ has the minimum $\sqrt{2(M-m)N}$ for $y = \sqrt{2(M-m)/N}$. ∎

Corollary 3.3. *Let $f:\mathbf{R}\to\mathbf{R}$ be a differentiable function such that*

$$m\le f(x)\le M,\quad |\delta_h f'(x)|\le N\quad (x\in\mathbf{R},\ h>0).$$

Then

$$|\delta_h f(x)|\le\sqrt{(M-m)N}\quad (x\in\mathbf{R},\ h>0). \tag{3.13}$$

Proof. Using (2.6) (i. e. (3.11) for $a=x-y$, $b=x+y$), we get

$$|\delta_h f(x)|\le\frac{M-m}{2y}+\frac{yN}{2}. \tag{3.14}$$

The function $g(y)=\frac{M-m}{2y}+\frac{yN}{2}$ has the minimum $\sqrt{(M-m)N}$ for $y=\sqrt{(M-m)/N}$, hence for $y\ge\sqrt{(M-m)/N}$ we get (3.13) from (3.14). □

Corollary 3.4. *Let $f:\mathbf{R}^n\to\mathbf{R}$ be twice differentiable on $\overline{D}$, where $D=\{x\in\mathbf{R}^n;\ a_i<x_i<b_i\}$. Suppose that*

$$\left|\frac{\partial^2 f}{\partial x_i\partial x_j}\right|\le N_{ij}\quad \text{on}\ \ D. \tag{3.15}$$

Then

$$\begin{aligned}\left|\sum_{i=1}^n(b_i-a_i)\frac{\partial f}{\partial x_i}-f(a)+f(b)\right| &\le\frac12\sum_{i,j}N_{ij}\left[(x_i-a_i)(x_j-a_j)+(b_i-x_i)(b_j-x_j)\right]\\ &\le\frac12\sum_{i,j}N_{ij}(b_i-a_i)(b_j-a_j).\end{aligned} \tag{3.16}$$

If, furthermore,

$$m\le f(x)\le M\quad (x\in\overline{D}), \tag{3.17}$$

then

$$\begin{aligned}\left|\sum_{i=1}^n(b_i-a_i)\frac{\partial f}{\partial x_i}\right| &\le M-m\\ &\quad+\frac12\sum_{i,j}N_{ij}\left[(x_i-a_i)(x_j-a_j)+(b_i-x_i)(b_j-x_j)\right]\\ &\le M-m+\frac12\sum_{i,j}N_{ij}(b_i-a_i)(b_j-a_j).\end{aligned} \tag{3.18}$$

PROOF. We use Theorem 2.1 and Remark 2.3 with $X = \mathbf{R}^n$, $Y = \mathbf{R}$, $F = f$, $\|x\| = \sum_{i=1}^{n} |x_i|$ $(x \in \mathbf{R}^n)$, $\|y\| = |y|$ $(y \in \mathbf{R})$. In this case (3.15) implies

$$\left\| F''_{(x)}(h,h) \right\| = \left| \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j \right| \leq \sum_{i,j} N_{ij} |h_i|\ |h_j|.$$

So, Theorem 2.1 implies the first inequalities in (3.16) and (3.18) (note that $\|F(b) - F(a)\| \leq M - m$). The second inequalities follow from the obvious inequality: $ab + cd \leq (a+c)(b+d)$ $(a,b,c,d \geq 0)$. □

COROLLARY 3.5. *Let the conditions of Corollary 3.4 be fulfilled and let* $h = \min\{b_i - a_i;\ 1 \leq i \leq n\}$. *Then*

$$\left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \right| \leq \begin{cases} \frac{M-m}{h} + \frac{h}{2} N, & \text{if } \ h \leq \sqrt{2(M-m)/N} \\ \sqrt{2(M-m)N}, & \text{if } \ h \geq \sqrt{2(M-m)/N} \end{cases} \tag{3.19}$$

where $N = \sum_{i,j} N_{ij}$.

PROOF. We can suppose $b_i - a_i = h$ for every i. Then we get from (3.18)

$$\left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \right| \leq \frac{M-m}{h} + \frac{h}{2} N. \tag{3.20}$$

Now, as in the proof of Corollary 3.2, (3.20) implies (3.19). □

COROLLARY 3.6. *Let* $f : \mathbf{R}^n \to \mathbf{R}$ *be a differentiable function such that*

$$m \leq f(x) \leq M \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| \leq N_{ij} \quad \text{on} \quad \mathbf{R}^n.$$

Then

$$\left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \right| \leq \sqrt{\frac{M-m}{N}}, \tag{3.21}$$

where $N = \sum_{i,j} N_{ij}$.

PROOF. For $b_i = x_i + h_i$, $a_i = x_i - h_i$ $(h_i > 0)$ (3.18) gives

$$\left| \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i} \right| \leq \frac{M-m}{2} + \frac{1}{2} \sum_{i,j} N_{ij} h_i h_j, \tag{3.22}$$

and for $h_1 = \cdots = h_n = h$ we obtain

$$\left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \right| \leq \frac{M-m}{2h} + \frac{hN}{2}. \tag{3.23}$$

Taking the minimum over $h > 0$ of the right–hand side of (3.23) we obtain (3.21). ∎

A simple consequence of (3.19) is the following generalization of a result from [4].

COROLLARY 3.7. *Let $D = \{x \in \mathbf{R}^n;\ 0 < x_i < 1\}$ and let $f : \overline{D} \to \mathbf{R}$ be a twice differentiable function. Suppose that $|f(x)| \le 1$ $(x \in \overline{D})$ and that (3.15) is fulfilled. Then*

$$\left|\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\right| \le \begin{cases} \frac{N+4}{2}, & \text{if } 0 < N \le 4 \\ 2\sqrt{N}, & \text{if } N > 4, \end{cases} \tag{3.24}$$

where $N = \sum_{i,j} N_{ij}$.

COROLLARY 3.8. *Under the assumptions of Corollary 3.6 with*

$$\left|\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \le A \quad (x \in D) \tag{3.25}$$

instead of (3.15), the following inequality holds true:

$$\left|\sum_{i}^{n} \frac{\partial f}{\partial x_i}\right| \le \sqrt{(M-m)A}. \tag{3.26}$$

PROOF. In the case $h_1 = \cdots = h_n = h$ we have

$$\|F''_{(x)}(h,h)\| = h^2 \left|\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \le h^2 A.$$

Thus, instead of (3.22) we obtain

$$\left|\sum_{i}^{n} \frac{\partial f}{\partial x_i}\right| \le \frac{M-m}{2h} + \frac{hA}{2} \tag{3.27}$$

wherefrom (3.26) follows. ∎

REMARK 3.9. Corollary 3.8 is a generalization of a result from [17] where the case $n = 2$ is given.

By using (3.26) and (3.27) we easily obtain the following generalization of a result from [15]:

COROLLARY 3.10. *Let $f : [0,1]^n \to \mathbf{R}$ be a twice differentiable function such that $|f(x)| \le 1$ $(x \in [0,1]^n)$ and*

$$\left|\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \le A \quad (x \in (0,1)^n).$$

Then

$$\left|\sum_i^n \frac{\partial f}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right| \le \begin{cases} 2+\frac{A}{4}, & \text{if } 0<A\le 8 \\ \sqrt{2A}, & \text{if } A\ge 8. \end{cases} \tag{3.28}$$

If f is positive then

$$\left|\sum_i^n \frac{\partial f}{\partial x_i}\left(\frac{1}{2},\ldots,\frac{1}{2}\right)\right| \le \begin{cases} 1+\frac{A}{4}, & \text{if } 0<A\le 4 \\ \sqrt{A}, & \text{if } A\ge 4. \end{cases} \tag{3.29}$$

REMARK 3.11. Analogous improvements of Landau's theorems were given by V. M. Olovyanisnikov (see e. g. [16] where some similar results are given).

ADDITIONAL REMARK. Let us note that results from this paper are given in monograph [13, pp. 45-50]. Some further related results are given in [2, 3, 5, 6, 7, 10, 14, 8].

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