

**BOUNDS FOR THE SIZE OF SETS WITH THE PROPERTY**  
 $D(n)$ 

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**ABSTRACT.** Let  $n$  be a nonzero integer and  $a_1 < a_2 < \dots < a_m$  positive integers such that  $a_i a_j + n$  is a perfect square for all  $1 \leq i < j \leq m$ . It is known that  $m \leq 5$  for  $n = 1$ . In this paper we prove that  $m \leq 31$  for  $|n| \leq 400$  and  $m < 15.476 \log |n|$  for  $|n| > 400$ .

## 1. INTRODUCTION

Let  $n$  be a nonzero integer. A set of  $m$  positive integers  $\{a_1, a_2, \dots, a_m\}$  is called a  $D(n)$ - $m$ -tuple (or a *Diophantine  $m$ -tuple with the property  $D(n)$* ) if  $a_i a_j + n$  is a perfect square for all  $1 \leq i < j \leq m$ .

Diophantus himself found the  $D(256)$ -quadruple  $\{1, 33, 68, 105\}$ , while the first  $D(1)$ -quadruple, the set  $\{1, 3, 8, 120\}$ , was found by Fermat (see [4, 5]). In 1969, Baker and Davenport [1] proved that this Fermat's set cannot be extended to a  $D(1)$ -quintuple, and in 1998, Dujella and Pethő [10] proved that even the Diophantine pair  $\{1, 3\}$  cannot be extended to a  $D(1)$ -quintuple. A famous conjecture is that there does not exist a  $D(1)$ -quintuple. We proved recently that there does not exist a  $D(1)$ -sextuple and that there are only finitely many, effectively computable,  $D(1)$ -quintuples (see [7, 9]).

The question is what can be said about the size of sets with the property  $D(n)$  for  $n \neq 1$ . Let us mention that Gibbs [12] found several examples of Diophantine sextuples, e.g.  $\{99, 315, 9920, 32768, 44460, 19534284\}$  is a  $D(2985984)$ -sextuple.

Define

$$M_n = \sup\{|S| : S \text{ has the property } D(n)\}.$$

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Considering congruences modulo 4, it is easy to prove that  $M_n = 3$  if  $n \equiv 2 \pmod{4}$  (see [3, 13, 15]). On the other hand, if  $n \not\equiv 2 \pmod{4}$  and  $n \notin \{-4, -3, -1, 3, 5, 8, 12, 20\}$ , then  $M_n \geq 4$  (see [6]).

In [8], we proved that  $M_n \leq 32$  for  $|n| \leq 400$  and

$$M_n < 267.81 \log |n| (\log \log |n|)^2 \quad \text{for } |n| > 400.$$

The purpose of the present paper is to improve this bound for  $M_n$ , specially in the case  $|n| > 400$ . We will remove the factor  $(\log \log |n|)^2$ , and also the constants will be considerably smaller.

The above mentioned bounds for  $M_n$  were obtained in [8] by considering separately three types (large, small and very small) of elements in a  $D(n)$ - $m$ -tuple. More precisely, let

$$\begin{aligned} A_n &= \sup\{|S \cap [n^3, +\infty)| : S \text{ has the property } D(n)\}, \\ B_n &= \sup\{|S \cap \langle n^2, |n|^3 \rangle| : S \text{ has the property } D(n)\}, \\ C_n &= \sup\{|S \cap [1, n^2]| : S \text{ has the property } D(n)\}. \end{aligned}$$

In [8], it was proved that  $A_n \leq 21$  and  $B_n < 0.65 \log |n| + 2.24$  for all nonzero integers  $n$ , while  $C_n < 265.55 \log |n| (\log \log |n|)^2 + 9.01 \log \log |n|$  for  $|n| > 400$  and  $C_n \leq 5$  for  $|n| \leq 400$ . The combination of these estimates gave the bound for  $M_n$ .

In the estimate for  $A_n$ , a theorem of Bennett [2] on simultaneous approximations of algebraic numbers was used in combination with a gap principle, while a variant of the gap principle gave the estimate for  $B_n$ . The bound for  $C_n$  (number of "very small" elements) was obtained using the Gallagher's large sieve method [11] and an estimate for sums of characters.

In the present paper, we will significantly improve the bound for  $C_n$  using a result of Vinogradov on double sums of Legendre's symbols. Let us mention that Vinogradov's result, in a slightly weaker form, was used recently, in similar context, by Gyarmati [14] and Sárközy & Stewart [17]. We will prove the following estimates for  $C_n$ .

**PROPOSITION 1.1.** *If  $|n| > 400$ , then  $C_n < 11.006 \log |n|$ . If  $|n| \geq 10^{100}$ , then  $C_n < 8.37 \log |n|$ .*

More detailed analysis of the gap principle used in [8] will lead us to the slightly improved bounds for  $B_n$ .

**PROPOSITION 1.2.** *For all nonzero integers  $n$  it holds  $B_n < 0.6114 \log |n| + 2.158$ . If  $|n| > 400$ , then  $B_n < 0.6071 \log |n| + 2.152$ .*

By combining Propositions 1.1 and 1.2 with the above mentioned estimate for  $A_n$ , we obtain immediately the following estimates for  $M_n$ .

**THEOREM 1.3.** *If  $|n| \leq 400$ , then  $M_n \leq 31$ . If  $|n| > 400$ , then  $M_n < 15.476 \log |n|$ . If  $|n| \geq 10^{100}$ , then  $M_n < 9.078 \log |n|$ .*

## 2. THREE LEMMAS

LEMMA 2.1 (Vinogradov). *Let  $p$  be an odd prime and  $\gcd(n, p) = 1$ . If  $A, B \subseteq \{0, 1, \dots, p-1\}$  and*

$$T = \sum_{x \in A} \sum_{y \in B} \left( \frac{xy + n}{p} \right),$$

then  $|T| < \sqrt{p|A| \cdot |B|}$ .

PROOF. See [18, Problem V.8.c].  $\square$

LEMMA 2.2 (Gallagher). *If all but  $g(p)$  residue classes mod  $p$  are removed for each prime  $p$  in a finite set  $\mathcal{S}$ , then the number of integers which remain in any interval of length  $N$  is at most*

$$(2.1) \quad \left( \sum_{p \in \mathcal{S}} \log p - \log N \right) / \left( \sum_{p \in \mathcal{S}} \frac{\log p}{g(p)} - \log N \right)$$

provided the denominator is positive.

PROOF. See [11].  $\square$

LEMMA 2.3. *If  $\{a, b, c\}$  is a Diophantine triple with the property  $D(n)$  and  $ab + n = r^2$ ,  $ac + n = s^2$ ,  $bc + n = t^2$ , then there exist integers  $e, x, y, z$  such that*

$$ae + n^2 = x^2, \quad be + n^2 = y^2, \quad ce + n^2 = z^2$$

and

$$c = a + b + \frac{e}{n} + \frac{2}{n^2}(abe + rxy).$$

PROOF. See [8, Lemma 3].  $\square$

## 3. PROOF OF PROPOSITION 1.1

Let  $N \geq n^2$  be a positive integer. Since  $|n| > 400$ , we have  $N > 1.6 \cdot 10^5$ . Let  $D = \{a_1, a_2, \dots, a_m\} \subseteq \{1, 2, \dots, N\}$  be a Diophantine  $m$ -tuple with the property  $D(n)$ . We would like to find an upper bound for  $m$  in term of  $N$ . We will use the Gallagher's sieve (Lemma 2.2). Let

$$\mathcal{S} = \{p : p \text{ is prime, } \gcd(n, p) = 1 \text{ and } p \leq Q\},$$

where  $Q$  is sufficiently large. For a prime  $p \in \mathcal{S}$ , let  $C$  denotes the set of integers  $b$  such that  $b \in \{0, 1, 2, \dots, p-1\}$  and there is at least one  $a \in D$  such that  $b \equiv a \pmod{p}$ . Then  $\left( \frac{xy+n}{p} \right) \in \{0, 1\}$  for all distinct  $x, y \in C$ . Here  $\left( \frac{\cdot}{p} \right)$  denotes the Legendre symbol. If  $0 \in C$ , then  $\left( \frac{n}{p} \right) = 1$ . For a given

$x \in C \setminus \{0\}$ , we have  $\left(\frac{xy_0+n}{p}\right) = 0$  for at most one  $y_0 \in C$ . If  $y \neq x, y_0$ , then  $\left(\frac{xy+n}{p}\right) = 1$ . Therefore,

$$\begin{aligned} T &= \sum_{x,y \in C} \left(\frac{xy+n}{p}\right) = \sum_{x \in C} \left(\sum_{y \in C} \left(\frac{xy+n}{p}\right)\right) \\ &\geq \sum_{x \in C} (|C| - 3) \geq |C|(|C| - 3). \end{aligned}$$

On the other hand, Lemma 2.1 implies

$$T < |C| \cdot \sqrt{p}.$$

Thus,  $|C| < \sqrt{p} + 3$  and we may apply Lemma 2.2 with

$$g(p) = \min\{\lfloor \sqrt{p} \rfloor + 3, p\}.$$

Let us denote the numerator and denominator from (2.1) by  $E$  and  $F$ , respectively. By [16, Theorem 9], we have

$$E = \sum_{p \in \mathcal{S}} \log p - \log N < \theta(Q) < 1.01624 Q.$$

The function  $f(x) = \frac{\log x}{\min\{\sqrt{x+3}, x\}}$  is strictly decreasing for  $x > 25$ . Also, if  $Q \geq 118$ , then  $f(p) \geq f(Q)$  for all  $p \leq Q$ .

For  $p \in \mathcal{S}$  it holds  $\gcd(n, p) = 1$ . This condition comes from the assumptions of Lemma 2.1. However, we will show later that  $n$  can be divisible only by a small proportion of the primes  $\leq Q$ . Assume that  $n$  is divisible by at most 5% of primes  $\leq Q$ . Then, for  $Q \geq 118$ , we have

$$\begin{aligned} (3.1) \quad F &\geq \sum_{p \in \mathcal{S}} f(p) - \log N \geq \frac{\log Q}{\sqrt{Q} + 3} \cdot |\mathcal{S}| - \log N \\ &\geq \frac{\log Q}{\sqrt{Q} + 3} \cdot \frac{19}{20} \pi(Q) - \log N > \frac{0.95 Q}{\sqrt{Q} + 3} - \log N. \end{aligned}$$

Since  $F$  has to be positive in the applications of Lemma 2.2, we will choose  $Q$  of the form

$$Q = c_1 \cdot \log^2 N.$$

We have to check whether our assumption on the proportion of primes which divide  $n$  is correct. Suppose that  $n$  is divisible by at least 5% of the primes  $\leq Q$ . Then  $|n| \geq p_1 p_2 \cdots p_{\lceil \pi(Q)/20 \rceil}$ , where  $p_i$  denotes the  $i$ -th prime. By [16, 3.5 and 3.12], we have  $p_{\lceil \pi(Q)/20 \rceil} > R$ , where

$$R = \frac{1}{20} \frac{Q}{\log Q} \log \left( \frac{1}{20} \frac{Q}{\log Q} \right).$$

Assume that  $c_1 \geq 6$ . Then  $Q > 860$  and  $R > 11.77$ . From [16, 3.16], it follows that

$$(3.2) \quad \log |n| > \sum_{p \leq R} \log p > R \left(1 - \frac{1.136}{\log R}\right).$$

Furthermore,  $\frac{1}{20} \frac{Q}{\log Q} > Q^{0.273}$  and  $R > 0.0136 Q$ . Hence, (3.2) implies  $\log R > 7.793$  and therefore

$$\log N \geq 2 \log |n| > 0.01466 Q \geq 0.08796 \log^2 N,$$

contradicting the assumption that  $N > 1.6 \cdot 10^5$ .

Therefore, we have that  $n$  is divisible by at most 5% of the primes  $\leq Q$ , and hence we have justified the estimate (3.1).

Under the assumption that  $c_1 \geq 6$ , the inequality (3.1) implies

$$F > 0.861 \sqrt{Q} - \log N = (0.861 \sqrt{c_1} - 1) \log N$$

and

$$\frac{E}{F} < \frac{1.017 c_1}{0.861 \sqrt{c_1} - 1} \cdot \log N.$$

For  $c_1 = 6$  we obtain

$$(3.3) \quad \frac{E}{F} < 5.503 \log N.$$

Assume now that  $N \geq 10^{200}$  and  $c_1 \geq 4$ . Then  $Q > 848303$  and we can prove in the same manner as above that  $n$  is divisible by at most 1% of the primes  $\leq Q$ . This fact implies

$$\frac{E}{F} < \frac{1.017 c_1}{0.986 \sqrt{c_1} - 1} \cdot \log N.$$

For  $c_1 = 4.11$  we obtain

$$(3.4) \quad \frac{E}{F} < 4.185 \log N.$$

Setting  $N = n^2$  in (3.3) and (3.4), we obtain the statements of Proposition 1.1.  $\square$

#### 4. PROOF OF PROPOSITION 1.2

We may assume that  $|n| > 1$ . Let  $\{a, b, c, d\}$  be a  $D(n)$ -quadruple such that  $n^2 < a < b < c < d$ . We apply Lemma 2.3 on the triple  $\{b, c, d\}$ . Since  $b > n^2$  and  $be + n^2 \geq 0$ , we have that  $e \geq 0$ . If  $e = 0$ , then  $d = b + c + 2\sqrt{bc + n} < 2c + 2\sqrt{c(c-1) + n} < 4c$ , contradicting the fact that  $d > 4.89c$  (see [8, Lemma 5]).

Hence  $e \geq 1$  and

$$(4.1) \quad d > b + c + \frac{2bc}{n^2} + \frac{2t\sqrt{bc}}{n^2}.$$

Lemma 2.3 also implies

$$(4.2) \quad c \geq a + b + 2r.$$

From  $r^2 \geq ab - \sqrt[4]{ab}$  and  $ab \geq 30$  it follows that  $r > 0.96a$ , and (4.2) implies  $c > 3.92a$ . Similarly,  $bc \geq 42$  implies  $t > 0.969\sqrt{bc}$  and, by (4.1),  $d > b + c + 3.938\frac{bc}{n^2} > 4.938c + b$ .

Assume now that  $\{a_1, a_2, \dots, a_m\}$  is a  $D(n)$ - $m$ -tuple and  $n^2 < a_1 < a_2 < \dots < a_m < |n|^3$ . We have

$$a_3 > 3.92a_1, \quad a_i > 4.938a_{i-1} + a_{i-2}, \quad \text{for } i = 4, 5, \dots, m.$$

Therefore,  $a_m > \alpha_m a_1$ , where the sequence  $(\alpha_k)$  is defined by

$$(4.3) \quad \alpha_k = 4.938\alpha_{k-1} + \alpha_{k-2}, \quad \alpha_2 = 1, \quad \alpha_3 = 3.92.$$

Solving the recurrence (4.3), we obtain  $\alpha_k \approx \beta\gamma^{k-3}$ , with  $\beta \approx 3.964355$ ,  $\gamma \approx 5.132825$ . More precisely,

$$|\alpha_k - \beta\gamma^{k-3}| < \frac{1}{\beta\gamma^{k-3}}.$$

From  $|n|^3 - 1 \geq a_m > \alpha_m a_1 \geq \alpha_m(n^2 + 1)$ , it follows  $\alpha_m \leq |n| - \frac{1}{|n|}$  and  $\beta\gamma^{m-3} < |n|$ . Hence,

$$(4.4) \quad m < \frac{1}{\log \gamma} \log |n| + 3 - \frac{\log \beta}{\log \gamma}.$$

For the above values of  $\beta$  and  $\gamma$  we obtain

$$m < 0.6114 \log |n| + 2.158.$$

Assume now that  $|n| > 400$ . Then  $bc > ab > 400^4$ , which implies  $c > 3.999999a$  and  $d > 4.999999c + b$ . Therefore, in this case the relation (4.4) holds with  $\beta \approx 4.042648$ ,  $\gamma \approx 5.192581$ , and we obtain

$$m < 0.6071 \log |n| + 2.152.$$

□

REMARK 4.1. The constants in Theorem 1.3 can be improved, for large  $|n|$ , by using formula (2.26) from [16] in the estimate for the sum  $\sum_{p \in \mathcal{S}} f(p)$ . In that way, it can be proved that for every  $\varepsilon > 0$ ,  $F > (2 - \varepsilon)\sqrt{Q} - \log N$  holds for sufficiently large  $Q$ .

Also, in the proof of Proposition 1.2, for sufficiently large  $|n|$  we have  $c > (4 - \varepsilon)a$  and  $d > (5 - \varepsilon)c + b$ , which leads to  $B_n < \left( \frac{1}{\log(\frac{5+\sqrt{29}}{2})} + \varepsilon \right) \log |n|$ .

These results imply that for every  $\varepsilon > 0$  there exists  $n(\varepsilon)$  such that for  $|n| > n(\varepsilon)$  it holds

$$M_n < \left( 2 + \frac{1}{\log(\frac{5+\sqrt{29}}{2})} + \varepsilon \right) \log |n|.$$

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