Continued fractions and Newton's approximations

TAKAO KOMATSU*

Abstract. *We generalise the relationship between continued fractions and Newton's approximations.*

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1. Introduction

$\theta = [\ a_0;\ a_1,\ a_2,\ \ldots\ ]$ denotes the continued fraction expansion of θ, where

$$\begin{aligned} \theta &= a_0 + \theta_0, & a_0 &= \lfloor \theta \rfloor, \\ 1/\theta_{n-1} &= a_n + \theta_n, & a_n &= \lfloor 1/\theta_{n-1} \rfloor \qquad (n = 1,\ 2,\ \ldots). \end{aligned}$$

The k-th convergent $p_k/q_k = [\ a_0;\ a_1,\ \ldots,\ a_k\ ]$ of θ is then given by the recurrence relations

$$\begin{aligned} p_k &= a_k p_{k-1} + p_{k-2} \quad (k = 0,\ 1,\ \ldots), & p_{-2} &= 0, & p_{-1} &= 1, \\ q_k &= a_k q_{k-1} + q_{k-2} \quad (k = 0,\ 1,\ \ldots), & q_{-2} &= 1, & q_{-1} &= 0. \end{aligned}$$

As well-known, continued fraction expansion of a quadratic irrational has its periodic form. Especially, the continued fraction expansion of the type $\sqrt{D}$, where D is a positive integer not being any square, is represented as

$$\sqrt{D} = [\ a_0;\ \overline{a_1,\ a_2,\ \ldots,\ a_s}\ ],$$

where

$$a_i = a_{s-i} \quad (i = 1, 2, \ldots, s-1) \qquad \text{and} \qquad a_s = 2a_0\,.$$

See Theorem 3 in [2], pp. 317.

Elezović [1] establishes the following relationship between such a continued fraction expansion and Newton's approximation.

Proposition 1. ([1], Theorem 1)

$$\frac{1}{2}\left(\frac{p_{nr-1}}{q_{nr-1}} + \frac{q_{nr-1}}{p_{nr-1}}D\right) = \frac{p_{2nr-1}}{q_{2nr-1}} \quad (n = 1, 2, \ldots),$$

where

$$r = \begin{cases} s, & \textit{if } s \textit{ is odd} \\ s/2, & \textit{if } s \textit{ is even.} \end{cases}$$

*Faculty of Education, Mie University, 1515 Kamihama, Tsu, Mie 514-8507, Japan, e-mail: komatsu@edu.mie-u.ac.jp

Moreover, the following remark is added there without any proof.

Proposition 2. ([1], Remark 1) *For $D = a^2 + 4$ with an odd $a \geq 3$*

$$\frac{1}{2}\left(\frac{p_k}{q_k} + \frac{q_k}{p_k}D\right) = \begin{cases} \dfrac{p_{2k+1}}{q_{2k+1}}, & \text{if } k = 5n-1 \\ \dfrac{p_{2k-1}}{q_{2k-1}}, & \text{if } k = 5n-2 \\ \dfrac{p_{2k+3}}{q_{2k+3}}, & \text{if } k = 5n \\ \dfrac{(a-2)p_{2k+2} + p_{2k+1}}{(a-2)q_{2k+2} + q_{2k+1}}, & \text{if } k = 5n+1 \\ \dfrac{p_{2k+1} - (a-2)p_{2k}}{q_{2k+1} - (a-2)q_{2k}}, & \text{if } k = 5n-3. \end{cases}$$

In this paper we shall give the complete proof of such a result in an extended form.

2. Main theorem

Theorem 1. *Let $\sqrt{D} = [\ a_0;\ \overline{a_1,\ a_2,\ \ldots,\ a_{s-1},\ 2a_0}\ ]$. If for $i = 0, 1, \ldots, \lfloor s/2 \rfloor$*

$$\frac{1}{2}\left(\frac{p_{ns+i-1}}{q_{ns+i-1}} + \frac{q_{ns+i-1}}{p_{ns+i-1}}D\right) = \frac{\alpha_i p_{2ns+2i} + p_{2ns+2i-1}}{\alpha_i q_{2ns+2i} + q_{2ns+2i-1}} \quad (n = 0, 1, 2, \ldots)$$

and

$$\frac{1}{2}\left(\frac{p_{ns-i-1}}{q_{ns-i-1}} + \frac{q_{ns-i-1}}{p_{ns-i-1}}D\right) = \frac{p_{2ns-2i-1} - \beta_i p_{2ns-2i-2}}{q_{2ns-2i-1} - \beta_i q_{2ns-2i-2}} \quad (n = 1, 2, \ldots),$$

then

$$\alpha_i = \beta_i \qquad (i = 0, 1, \ldots, \lfloor s/2 \rfloor).$$

Remark 1. *In fact, from*

$$\frac{1}{2}\left(\frac{p_{i-1}}{q_{i-1}} + \frac{q_{i-1}}{p_{i-1}}D\right) = \frac{\alpha_i p_{2i} + p_{2i-1}}{\alpha_i q_{2i} + q_{2i-1}}$$

we have

$$\alpha_i = -\frac{(p_{i-1}^2 + q_{i-1}^2 D)q_{2i-1} - 2p_{i-1}q_{i-1}p_{2i-1}}{(p_{i-1}^2 + q_{i-1}^2 D)q_{2i} - 2p_{i-1}q_{i-1}p_{2i}}.$$

From

$$\frac{1}{2}\left(\frac{p_{s-i-1}}{q_{s-i-1}} + \frac{q_{s-i-1}}{p_{s-i-1}}D\right) = \frac{p_{2s-2i-1} - \beta_i p_{2s-2i-2}}{q_{2s-2i-1} - \beta_i q_{2s-2i-2}}$$

we have

$$\beta_i = -\frac{(p_{s-i-1}^2 + q_{s-i-1}^2 D)q_{2(s-i)-1} - 2p_{s-i-1}q_{s-i-1}p_{2(s-i)-1}}{(p_{s-i-1}^2 + q_{s-i-1}^2 D)q_{2(s-i-1)} - 2p_{s-i-1}q_{s-i-1}p_{2(s-i-1)}}.$$

Immediately, *Proposition 1* is a corollary when $\alpha_0 = 0$ (and $\alpha_{s/2} = 0$ if s is even). The case $i = 1$ implies the following corollary.

Corollary 1. *When* $\sqrt{D} = \sqrt{a_0^2 + t} = [\ a_0;\ \overline{a_1,\ a_2,\ \dots,\ a_{s-1},\ 2a_0}\ ]$ $(t > 0)$*,*

$$\frac{1}{2}\left(\frac{p_{ns}}{q_{ns}} + \frac{q_{ns}}{p_{ns}}D\right) = \frac{\alpha p_{2ns+2} + p_{2ns+1}}{\alpha q_{2ns+2} + q_{2ns+1}} \quad (n = 1, 2, \dots)$$

and

$$\frac{1}{2}\left(\frac{p_{ns-2}}{q_{ns-2}} + \frac{q_{ns-2}}{p_{ns-2}}D\right) = \frac{p_{2ns-3} - \alpha p_{2ns-4}}{q_{2ns-3} - \alpha q_{2ns-4}} \quad (n = 1, 2, \dots)\,,$$

where

$$\alpha = \frac{2a_0 - a_1 t}{(a_1 a_2 + 1)t - 2a_0 a_2}\,.$$

3. Some lemmas

Lemma 1.

$$\frac{1}{\theta_n} = \frac{\sqrt{D} + k_n}{l_n} \quad (n = 0, 1, 2, \dots)\,,$$

where for $n \geq 0$

$$\begin{cases} k_n = (-1)^{n-1}(q_n q_{n-1} D - p_n p_{n-1})\,, \\ l_n = (-1)^n (q_n^2 D - p_n^2)\,. \end{cases}$$

Moreover,

$$\begin{cases} D q_n = k_n p_n + l_n p_{n-1}\,, \\ p_n = k_n q_n + l_n q_{n-1} \end{cases}$$

and

$$\frac{1}{2}\left(\frac{p_n}{q_n} + \frac{q_n}{p_n}D\right) = \frac{\dfrac{p_n + k_n q_n}{l_n q_n} p_n + p_{n-1}}{\dfrac{p_n + k_n q_n}{l_n q_n} q_n + q_{n-1}}\,.$$

Proof. Firstly, note that k_n and l_n $(n \geq 1)$ satisfy the recurrence relations

$$k_n = a_n l_{n-1} - k_{n-1}, \quad l_n = l_{n-2} - a_n^2 l_{n-1} + 2a_n k_{n-1}, \quad D - k_n^2 = l_n l_{n-1}$$

with $k_{-1} = 0$, $l_{-1} = 1$, $k_0 = a_0$ and $l_0 = D - a_0^2$ (*c.f.* [2], pp.313–315). Hence,

$$\begin{aligned} &k_0 = -(1 \cdot 0 \cdot D - a_0 \cdot 1) = a_0, \\ &l_0 = 1^2 D - a_0^2 = t, \\ &l_1 = -(a_1^2 D - (a_0 a_1 + 1)^2) = 1 - a_1^2 t + 2a_1 a_0 \end{aligned}$$

are valid. Now, k_{n+1} is induced from k_n and l_n, and l_{n+2} is from l_n, l_{n+1} and k_{n+1}, because

$$\begin{aligned} (-1)^n (q_{n+1} q_n D - p_{n+1} p_n) &= (-1)^n (a_{n+1}(q_n^2 D - p_n^2) + (q_n q_{n-1} D - p_n p_{n-1})) \\ &= (-1)^n (a_{n+1}(-1)^n l_n + (-1)^{n-1} k_n) \\ &= a_{n+1} l_n - k_n = k_{n+1} \end{aligned}$$

and

$$\begin{aligned}(-1)^{n+2}(q_{n+2}^2D-p_{n+2}^2)&=(-1)^{n+2}(a_{n+2}^2(q_{n+1}^2D-p_{n+1}^2)+(q_n^2D-p_n^2)\\&\quad+2a_{n+2}(q_{n+1}q_nD-p_{n+1}p_n))\\&=(-1)^{n+2}(a_{n+2}^2(-1)^{n+1}l_{n+1}+(-1)^nl_n+2a_{n+2}(-1)^nk_{n+1})\\&=l_n-a_{n+2}^2l_{n+1}+2a_{n+2}k_{n+1}=l_{n+2}\,.\end{aligned}$$

Therefore, the first part of *Lemma* is proved.

By using this we have

$$\begin{aligned}k_np_n+l_np_{n-1}=&\quad(-1)^{n-1}(p_nq_nq_{n-1}D-p_n^2p_{n-1})+(-1)^n(p_{n-1}q_n^2D-p_n^2p_{n-1})\\=&\quad(-1)^{n-1}q_n(p_nq_{n-1}-p_{n-1}q_n)D=Dq_n\end{aligned}$$

and

$$\begin{aligned}k_nq_n+l_nq_{n-1}=&\quad(-1)^{n-1}(q_n^2q_{n-1}D-p_np_{n-1}q_n)+(-1)^n(q_n^2q_{n-1}D-p_n^2q_{n-1})\\=&\quad(-1)^{n-1}p_n(p_nq_{n-1}-p_{n-1}q_n)=p_n\,.\end{aligned}$$

Hence, finally we obtain

$$\frac{1}{2}\left(\frac{p_n}{q_n}+\frac{q_n}{p_n}D\right)=\frac{\dfrac{p_n}{q_n}p_n+Dq_n}{\dfrac{p_n}{q_n}q_n+p_n}=\frac{\left(\dfrac{p_n}{q_n}+k_n\right)p_n+p_{n-1}}{\left(\dfrac{p_n}{q_n}+k_n\right)q_n+q_{n-1}}=\frac{\dfrac{p_n+k_nq_n}{l_nq_n}p_n+p_{n-1}}{\dfrac{p_n+k_nq_n}{l_nq_n}q_n+q_{n-1}}.$$

□

Lemma 2. *For* $i=0,1,2,\dots,\lfloor s/2\rfloor$

$$\begin{array}{lll}k_{ns+i}=k_i, & l_{ns+i}=l_i, & (n=0,1,2,\dots)\\ k_{ns-i-1}=k_i, & l_{ns-i-1}=l_{i-1} & (n=1,2,\dots)\,.\end{array}$$

Proof. The first two identities are clear because

$$\frac{1}{\theta_{ns+i}}=\frac{1}{\theta_i}\quad(n=0,1,2,\dots)\,.$$

By $a_{ns-i}=a_i$ one has

$$\theta_{ns-i-1}=\frac{\sqrt{D}-k_i}{l_i}\qquad\text{or}\qquad\frac{1}{\theta_{ns-i-1}}=\frac{\sqrt{D}+k_i}{l_{i-1}}\quad(n=1,2,\dots)\,,$$

yielding the second two results. □

4. Proof of the theorem

Proof of Theorem. By *Lemmas 1* and *2*, the left-hand side of the first identity is

$$\frac{1}{2}\left(\frac{p_{ns+i-1}}{q_{ns+i-1}}+\frac{q_{ns+i-1}}{p_{ns+i-1}}D\right)=\frac{\frac{p_{ns+i-1}+k_{ns+i-1}q_{ns+i-1}}{l_{ns+i-1}q_{ns+i-1}}p_{ns+i-1}+p_{ns+i-2}}{\frac{p_{ns+i-1}+k_{ns+i-1}q_{ns+i-1}}{l_{ns+i-1}q_{ns+i-1}}q_{ns+i-1}+q_{ns+i-2}}$$
$$=\frac{\frac{p_{ns+i-1}+k_{i-1}q_{ns+i-1}}{l_{i-1}q_{ns+i-1}}p_{ns+i-1}+p_{ns+i-2}}{\frac{p_{ns+i-1}+k_{i-1}q_{ns+i-1}}{l_{i-1}q_{ns+i-1}}q_{ns+i-1}+q_{ns+i-2}}\,.$$

On the other hand, the right-hand side is

$$\frac{\alpha_i p_{2ns+2i}+p_{2ns+2i-1}}{\alpha_i q_{2ns+2i}+q_{2ns+2i-1}}=[\,a_0;\ a_1,\ \dots,\ a_{2ns+2i},\ \alpha_i\,]$$
$$=\frac{[\,a_{ns+i};\ a_{ns+i+1},\ \dots,\ a_{2ns+2i},\ \alpha_i\,]p_{ns+i-1}+p_{ns+i-2}}{[\,a_{ns+i};\ a_{ns+i+1},\ \dots,\ a_{2ns+2i},\ \alpha_i\,]q_{ns+i-1}+q_{ns+i-2}}$$
$$=\frac{[\,a_i;\ a_{i+1},\ \dots,\ a_{ns+2i},\ \alpha_i\,]p_{ns+i-1}+p_{ns+i-2}}{[\,a_i;\ a_{i+1},\ \dots,\ a_{ns+2i},\ \alpha_i\,]q_{ns+i-1}+q_{ns+i-2}}\,.$$

Note that

$$\begin{pmatrix}a_i&1\\1&0\end{pmatrix}\begin{pmatrix}a_{i+1}&1\\1&0\end{pmatrix}\cdots\begin{pmatrix}a_{ns+2i}&1\\1&0\end{pmatrix}\begin{pmatrix}\alpha_i&1\\1&0\end{pmatrix}$$
$$=\begin{pmatrix}p_{i-1}&p_{i-2}\\q_{i-1}&q_{i-2}\end{pmatrix}^{-1}\begin{pmatrix}p_{ns+2i}&p_{ns+2i-1}\\q_{ns+2i}&q_{ns+2i-1}\end{pmatrix}\begin{pmatrix}\alpha_i&1\\1&0\end{pmatrix}$$
$$=(-1)^i\times\begin{pmatrix}q_{i-2}p_{ns+2i}-p_{i-2}q_{ns+2i}&q_{i-2}p_{ns+2i-1}-p_{i-2}q_{ns+2i-1}\\-q_{i-1}p_{ns+2i}+p_{i-1}q_{ns+2i}&-q_{i-1}p_{ns+2i-1}+p_{i-1}q_{ns+2i-1}\end{pmatrix}\begin{pmatrix}\alpha_i&1\\1&0\end{pmatrix}\,.$$

Hence, the denominator of $[\,a_i;\ a_{i+1},\ \dots,\ a_{ns+2i},\ \alpha_i\,]$ is equal to

$$(-1)^i\big((-q_{i-1}p_{ns+2i}+p_{i-1}q_{ns+2i})\alpha_i-q_{i-1}p_{ns+2i-1}+p_{i-1}q_{ns+2i-1}\big)$$
$$=(-1)^i\big((-q_{i-1}(k_{ns+2i}q_{ns+2i}+l_{ns+2i}q_{ns+2i-1})+p_{i-1}q_{ns+2i})\alpha_i$$
$$-q_{i-1}(k_{ns+2i-1}q_{ns+2i-1}+l_{ns+2i-1}q_{ns+2i-2})+p_{i-1}q_{ns+2i-1}\big)$$
$$=(-1)^i\big((-q_{i-1}k_{2i}+p_{i-1})\alpha_i q_{ns+2i}$$
$$+(-q_{i-1}l_{2i}\alpha_i-q_{i-1}k_{2i-1}+p_{i-1})q_{ns+2i-1}-q_{i-1}l_{2i-1}q_{ns+2i-2}\big)$$
$$=(q_{i-1}k_{2i}-p_{i-1})\alpha_i(A_1q_{ns+i}+A_2q_{ns+i-1})$$
$$+(q_{i-1}l_{2i}\alpha_i+q_{i-1}k_{2i-1}-p_{i-1})(A_3q_{ns+i}+A_4q_{ns+i-1})$$
$$+q_{i-1}l_{2i-1}(A_5q_{ns+i}+A_6q_{ns+i-1})\,,$$

where

$$A_1=A_1^{(i)}=q_{i-1}p_{2i}-p_{i-1}q_{2i},\qquad A_2=A_2^{(i)}=q_ip_{2i}-p_iq_{2i},$$
$$A_3=A_3^{(i)}=q_{i-1}p_{2i-1}-p_{i-1}q_{2i-1},\qquad A_4=A_4^{(i)}=q_ip_{2i-1}-p_iq_{2i-1},$$
$$A_5=A_5^{(i)}=q_{i-1}p_{2i-2}-p_{i-1}q_{2i-2},\qquad A_6=A_6^{(i)}=q_ip_{2i-2}-p_iq_{2i-2}\,.$$

Thus, the coefficient of q_{ns+i} is

$$((q_{i-1}k_{2i}-p_{i-1})A_1+q_{i-1}l_{2i}A_3)\alpha_i+(q_{i-1}k_{2i-1}-p_{i-1})A_3+q_{i-1}l_{2i-1}A_5=0$$

if and only if

$$\alpha_i = -\frac{A_3(q_{i-1}k_{2i-1} - p_{i-1}) + A_5 q_{i-1} l_{2i-1}}{A_1(q_{i-1}k_{2i} - p_{i-1}) + A_3 q_{i-1} l_{2i}}\,.$$

Then, the coefficient of q_{ns+i-1} $\times(A_1(q_{i-1}k_{2i} - p_{i-1}) + A_3 q_{i-1} l_{2i})$ is

$$\begin{aligned}
& -(q_{i-1}k_{2i} - p_{i-1})(q_{i-1}k_{2i-1} - p_{i-1})A_2A_3 - (q_{i-1}k_{2i} - p_{i-1})q_{i-1}l_{2i-1}A_2A_5 \\
& -q_{i-1}l_{2i}(q_{i-1}k_{2i-1} - p_{i-1})A_4A_3 - q_{i-1}^2 l_{2i}l_{2i-1}A_4A_5 \\
& +(q_{i-1}k_{2i-1} - p_{i-1})(q_{i-1}k_{2i} - p_{i-1})A_4A_1 + (q_{i-1}k_{2i-1} - p_{i-1})q_{i-1}l_{2i})A_4A_3 \\
& +q_{i-1}l_{2i-1}(q_{i-1}k_{2i} - p_{i-1})A_6A_1 + q_{i-1}^2 l_{2i-1}l_{2i})A_6A_3 \\
= \; & (q_{i-1}k_{2i-1} - p_{i-1})(q_{i-1}k_{2i} - p_{i-1})(-1)^{i-1} + q_{i-1}l_{2i-1}(q_{i-1}k_{2i} - p_{i-1})(-1)^i a_{2i} \\
& +q_{i-1}^2 l_{2i-1}l_{2i}(-1)^i \\
= \; & (-1)^i(q_{i-1}^2 D - p_{i-1}^2) = -l_{i-1}
\end{aligned}$$

because

$$-A_2A_3 + A_1A_4 = (-1)^{i-1}, \quad -A_2A_5 + A_1A_6 = (-1)^i a_{2i}, \quad -A_4A_5 + A_3A_6 = (-1)^i\,.$$

Similarly, by using $p_i q_{i-1} - p_{i-1}q_i = (-1)^{i-1}$ and $p_{2i}q_{2i-1} - p_{2i-1}q_{2i} = -1$, the numerator of $[\ a_i;\ a_{i+1},\ \ldots,\ a_{ns+2i},\ \alpha_i\ ]$ is equal to

$$\begin{aligned}
& -(q_{i-2}k_{2i} - p_{i-2})\alpha_i(A_1 q_{ns+i} + A_2 q_{ns+i-1}) \\
& -(q_{i-2}l_{2i}\alpha_i + q_{i-2}k_{2i-1} - p_{i-2})(A_3 q_{ns+i} + A_4 q_{ns+i-1}) \\
& -q_{i-2}l_{2i-1}(A_5 q_{ns+i} + A_6 q_{ns+i-1}) \\
= \; & -\tfrac{l_{i-1}q_{ns+i} + (k_{i-1} - k_i)q_{ns+i-1}}{A_1(q_{i-1}k_{2i} - p_{i-1}) + A_3 q_{i-1}l_{2i}} = -\tfrac{p_{ns+i-1} + k_{i-1}q_{ns+i-1}}{A_1(q_{i-1}k_{2i} - p_{i-1}) + A_3 q_{i-1}l_{2i}}\,.
\end{aligned}$$

Therefore, we obtain

$$[\ a_i;\ a_{i+1},\ \ldots,\ a_{ns+2i},\ \alpha_i\ ] = \frac{p_{ns+i-1} + k_{i-1}q_{ns+i-1}}{l_{i-1}q_{ns+i-1}}\,.$$

Next, by *Lemmas 1* and *2*, the left-hand side of the second identity is

$$\frac{\left(\frac{p_{ns-i-1}}{q_{ns-i-1}} + k_{ns-i-1}\right)p_{ns-i-1} + l_{ns-i-1}p_{ns-i-2}}{\left(\frac{p_{ns-i-1}}{q_{ns-i-1}} + k_{ns-i-1}\right)q_{ns-i-1} + l_{ns-i-1}q_{ns-i-2}}$$

$$= \frac{\left(\frac{p_{ns-i-1}}{q_{ns-i-1}} + k_i\right)p_{ns-i-1} + l_{i-1}p_{ns-i-2}}{\left(\frac{p_{ns-i-1}}{q_{ns-i-1}} + k_i\right)q_{ns-i-1} + l_{i-1}q_{ns-i-2}}\,.$$

On the other hand, the right-hand side of the second identity is

$$\begin{aligned}
& \tfrac{(a_{2ns-2i-1} - \beta_i)p_{2ns-2i-2} + p_{2ns-2i-3}}{(a_{2ns-2i-1} - \beta_i)q_{2ns-2i-2} + q_{2ns-2i-3}} = [\ a_0;\ a_1,\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1} - \beta_i\ ] \\
= \; & \tfrac{[\ a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1} - \beta_i\ ]p_{ns-i-1} + p_{ns-i-2}}{[\ a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1} - \beta_i\ ]q_{ns-i-1} + q_{ns-i-2}}\,,
\end{aligned}$$

where

$$
\begin{aligned}
& [\, a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1}-\beta_i \,] \\
= & [\, a_{s-i};\ a_{s-i+1},\ \ldots,\ a_{(n+1)s-2i-2},\ a_{(n+1)s-2i-1}-\beta_i \,] \\
= & \tfrac{q_i(p_{ns-2i-1}-\beta_i p_{ns-2i-2})+p_i(q_{ns-2i-1}-\beta_i q_{ns-2i-2})}{q_{i-1}(p_{ns-2i-1}-\beta_i p_{ns-2i-2})+p_{i-1}(q_{ns-2i-1}-\beta_i q_{ns-2i-2})}
\end{aligned}
$$

because

$$
\begin{aligned}
& \begin{pmatrix} a_{s-i} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{s-i+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{(n+1)s-2i-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{(n+1)s-2i-1}-\beta_i & 1 \\ 1 & 0 \end{pmatrix} \\
= & \begin{pmatrix} a_{ns+i} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{ns+i-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{2i+2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{2i+1}-\beta_i & 1 \\ 1 & 0 \end{pmatrix} \\
= & \left(\begin{pmatrix} p_{2i+1} & p_{2i} \\ q_{2i+1} & q_{2i} \end{pmatrix}^{-1} \begin{pmatrix} p_{ns+i} & p_{ns+i-1} \\ q_{ns+i} & q_{ns+i-1} \end{pmatrix} \right)^t \begin{pmatrix} a_{2i+1}-\beta_i & 1 \\ 1 & 0 \end{pmatrix} \\
= & \begin{pmatrix} q_i(p_{ns-2i-1}-\beta_i p_{ns-2i-2})+p_i(q_{ns-2i-1}-\beta_i q_{ns-2i-2}) & * \\ q_{i-1}(p_{ns-2i-1}-\beta_i p_{ns-2i-2})+p_{i-1}(q_{ns-2i-1}-\beta_i q_{ns-2i-2}) & ** \end{pmatrix} .
\end{aligned}
$$

Here, X^t denotes the transpose of the matrix X. $*$ or $**$ denotes the omission of that element. Hence, the denominator of

$$[\, a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1}-\beta_i \,]$$

equals

$$
\begin{aligned}
& q_{i-1}(k_{ns-2i-1}q_{ns-2i-1}+l_{ns-2i-1}q_{ns-2i-2} \\
& -\beta_i(k_{ns-2i-2}q_{ns-2i-2}+l_{ns-2i-2}q_{ns-2i-3}))+p_{i-1}(q_{ns-2i-1}-\beta_i q_{ns-2i-2}) \\
= & (q_{i-1}k_{2i}+p_{i-1}-\beta_i q_{i-1}l_{2i})q_{ns-2i-1}+(q_{i-1}l_{2i-1}+\beta_i q_{i-1}k_{2i}-\beta_i p_{i-1})q_{ns-2i-2} \\
= & (q_{i-1}k_{2i}+p_{i-1}-\beta_i q_{i-1}l_{2i})(A_4 q_{ns-i-1}+A_3 q_{ns-i-2}) \\
& -(q_{i-1}l_{2i-1}+\beta_i q_{i-1}k_{2i}-\beta_i p_{i-1})(A_2 q_{ns-i-1}+A_1 q_{ns-i-2}) .
\end{aligned}
$$

Therefore, the coefficient of q_{ns-i-2} is nullified if and only if

$$
\begin{aligned}
\beta_i & = -\tfrac{A_1 q_{i-1}l_{2i-1}-A_3(q_{i-1}k_{2i}+p_{i-1})}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}} \\
& = -\tfrac{A_3(q_{i-1}k_{2i-1}-p_{i-1})+A_5 q_{i-1}l_{2i-1}}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}} = \alpha_i .
\end{aligned}
$$

Then, the coefficient of q_{ns-i-1} in the denominator of

$$[\, a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1}-\beta_i \,]$$

is

$$
\begin{aligned}
& (q_{i-1}k_{2i}+p_{i-1})A_4 - q_{i-1}l_{2i-1}A_2 - ((q_{i-1}k_{2i}-p_{i-1})A_2+q_{i-1}l_{2i}A_4)\beta_i \\
= & \tfrac{(-1)^{i-1}(q_{i-1}^2 k_{2i}^2 - p_{i-1}^2 + q_{i-1}^2 l_{2i-1}l_{2i})}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}} = \tfrac{l_{i-1}}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}} .
\end{aligned}
$$

Similarly, the numerator of $[\ a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1}-\beta_i\ ]$ equals

$$\begin{aligned}&(q_{i-1}k_{2i}+p_{i-1}-\beta_i q_{i-1}l_{2i})(A_4 q_{ns-i-1}+A_3 q_{ns-i-2})\\&-(q_{i-1}l_{2i-1}+\beta_i q_{i-1}k_{2i}-\beta_i p_{i-1})(A_2 q_{ns-i-1}+A_1 q_{ns-i-2})\\=\ &\tfrac{2k_i q_{ns-i-1}+l_{i-1}q_{ns-i-2}}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}}=\tfrac{p_{ns-i-1}+k_i q_{ns-i-1}}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}}\,.\end{aligned}$$

Therefore, we finally obtain

$$[\ a_{ns-i};\ a_{ns-i+1},\ \ldots,\ a_{2ns-2i-2},\ a_{2ns-2i-1}-\beta_i\ ]=\frac{p_{ns-i-1}+k_i q_{ns-i-1}}{l_{i-1}q_{ns-i-1}}\,.$$

Of course, using the definitions of k_n, l_n, A_1, A_3 and A_5 we conclude that

$$\begin{aligned}\alpha_i=\beta_i\ &=-\tfrac{A_3(q_{i-1}k_{2i-1}-p_{i-1})+A_5 q_{i-1}l_{2i-1}}{A_1(q_{i-1}k_{2i}-p_{i-1})+A_3 q_{i-1}l_{2i}}\\&=-\tfrac{(p_{i-1}^2+q_{i-1}^2 D)q_{2i-1}-2p_{i-1}q_{i-1}p_{2i-1}}{(p_{i-1}^2+q_{i-1}^2 D)q_{2i}-2p_{i-1}q_{i-1}p_{2i}}\,.\end{aligned}$$

□

5. Some interesting patterns

Example 1. (Proof of *Proposition 2*) *Since*

$$\sqrt{D}=\sqrt{a^2+4}=[\ a;\ \overline{\frac{a-1}{2},\ 1,\ 1,\ \frac{a-1}{2},\ 2a}\ ]$$

with an odd $a\ge 3$, by Corollary 1 *we have*

$$\alpha_1=\frac{2a-(a-1)/2\cdot 4}{((a-1)/2+1)\cdot 4-2a}=1\,.$$

It is easier to calculate α_2 by using α_i in the proof of Theorem 1. *Since*

$$\frac{1}{\theta_0}=\frac{\sqrt{D}+a}{4}\quad\text{and}\quad\frac{1}{\theta_1}=\frac{\sqrt{D}+(a-2)}{a}\,,$$

by Lemma 2

$$k_4=k_0=a,\quad l_4=l_{-1}=1,\quad k_3=k_1=a-2,\quad l_3=l_0=4\,.$$

Notice that

$$q_1=a_1=\frac{a-1}{2},\quad p_1=a_0a_1+1=\frac{a^2-a+2}{2}$$

and

$$\begin{aligned}&A_5^{(2)}=q_1p_2-p_1q_2=-1,\quad A_3^{(2)}=q_1p_3-p_1q_3=-a_3=-1,\\&A_1^{(2)}=q_1p_4-p_1q_4=-(a_3a_4+1)=-\tfrac{a+1}{2}\,.\end{aligned}$$

Hence,

$$\alpha_2=-\frac{(-1)(\frac{a-1}{2}(a-2)-\frac{a^2-a+2}{2})+(-1)\frac{a-1}{2}\cdot 4}{-\frac{a+1}{2}(\frac{a-1}{2}\cdot a-\frac{a^2-a+2}{2})+(-1)\frac{a-1}{2}\cdot 1}=-\frac{-a+2}{1}=a-2\,.$$

Therefore, one has Proposition 2.

Example 2. *For $D = a^2 + 4$ with an even a*

$$\frac{1}{2}\left(\frac{p_k}{q_k} + \frac{q_k}{p_k}D\right) = \frac{p_{2k+1}}{q_{2k+1}}.$$

Proof. *Since $\sqrt{a^2+4} = [\ a;\ \overline{a/2,\ 2a}\ ]$, the result follows by* Proposition 1.

Example 3. *For $D = a^2 - 4$ with an even $a \geq 6$*

$$\frac{1}{2}\left(\frac{p_k}{q_k} + \frac{q_k}{p_k}D\right) = \begin{cases} \dfrac{p_{2k+1}}{q_{2k+1}}, & \text{if } k \text{ is odd;} \\ \dfrac{3p_{2k+2} + (a/2+1)p_{2k+1}}{3q_{2k+2} + (a/2+1)q_{2k+1}}, & \text{if } k = 4n; \\ \dfrac{(a/2+1)p_{2k+1} - 3p_{2k}}{(a/2+1)q_{2k+1} - 3q_{2k}}, & \text{if } k = 4n-2. \end{cases}$$

Remark 2. *When $a = 4$, from $\sqrt{a^2-4} = \sqrt{12} = [\ 3;\ \overline{2,\ 6}\ ]$, by* Proposition 1 *we have*

$$\frac{1}{2}\left(\frac{p_k}{q_k} + \frac{q_k}{p_k}D\right) = \frac{p_{2k+1}}{q_{2k+1}}.$$

Proof. *Since*

$$\sqrt{a^2-4} = \sqrt{(a-1)^2 + (2a-5)} = [\ a-1;\ \overline{1,\ \frac{a-4}{2},\ 1,\ 2(a-1)}\ ]$$

with an even $a \geq 6$, one gets

$$\alpha_1 = \frac{2(a-1) - (2a-5)}{((a-4)/2+1)(2a-5) - 2(a-1)(a-4)/2} = \frac{6}{a+2}.$$

Example 4. ([1], Remark 1, pp.33) *For $D = a^2 - 4$ with an odd $a \geq 5$*

$$\frac{1}{2}\left(\frac{p_k}{q_k} + \frac{q_k}{p_k}D\right) = \begin{cases} \dfrac{p_{2k+1}}{q_{2k+1}}, & \text{if } k = 6n-1, 6n-3, 6n-4, 6n-5; \\ \dfrac{3p_{2k+2} + \frac{a-1}{2}p_{2k+1}}{3q_{2k+2} + \frac{a-1}{2}q_{2k+1}}, & \text{if } k = 6n; \\ \dfrac{\frac{a-1}{2}p_{2k+1} - 3p_{2k}}{\frac{a-1}{2}q_{2k+1} - 3q_{2k}}, & \text{if } k = 6n-2. \end{cases}$$

Proof. *Since*

$$\sqrt{a^2-4} = \sqrt{(a-1)^2 + (2a-5)} = [\ a-1;\ \overline{1,\ \frac{a-3}{2},\ 2,\ \frac{a-3}{2},\ 1,\ 2(a-1)}\ ]$$

with an odd $a \geq 5$, one gets

$$\alpha_1 = \frac{2(a-1) - (2a-5)}{((a-3)/2+1)(2a-5) - 2(a-1)(a-3)/2} = \frac{6}{a-1}.$$

Since

$$\frac{1}{\theta_0} = \frac{\sqrt{D} + (a-1)}{2a-5}, \quad \frac{1}{\theta_1} = \frac{\sqrt{D} + (a-4)}{4} \quad \text{and} \quad \frac{1}{\theta_2} = \frac{\sqrt{D} + (a-2)}{a-2},$$

by Lemma 2

$$k_4 = k_1 = a - 4, \quad l_4 = l_0 = 2a - 5, \quad k_3 = k_2 = a - 2, \quad l_3 = l_1 = 4\,.$$

Notice that

$$q_1 = a_1 = \frac{a-1}{2}, \quad p_1 = a_0 a_1 + 1 = \frac{a^2 - a + 2}{2}$$

and

$$A_5^{(2)} = -1, \quad A_3^{(2)} = -a_3 = -2, \quad \textit{and} \quad A_1^{(2)} = -(a_3 a_4 + 1) = -(a-2)\,.$$

Hence,

$$\alpha_2 = -\frac{(-2)((a-2)-a) + (-1)\cdot 4}{-(a-2)((a-4)-a) + (-2)(2a-5)} = 0\,.$$

References

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