(m, n)-rings as algebras with only one operation

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Abstract. *In this paper a class of (m, n)-rings with a left and right zero is described as a variety of algebras of type $< 3m + n - 5, 0 >$.*

Key words: $n-$*groupoid,* $n-$*semigroup,* $n-$*quasigroup,* $n-$*group,* $\{i, j\}-$*neutral operation, inversing operation,* $(m, n)-$*ring*

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1. Preliminaries

A notion of an $n-$group was introduced by W. Dörnte in [7] as a generalization of the notion of a group. See, also [3], [1], [10].

Definition 1. *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. We say that (Q, A) is a Dörnte $n-$group [briefly: $n-$group] iff it is an $n-$semigroup and an $n-$quasigroup as well.*

Proposition 1. (see [15]) *Let $n \geq 2$ and let (Q, A) be an $n-$groupoid. Then the following statements are equivalent:*

(i) (Q, A) *is an $n-$group;*

(ii) there are mappings $^{-1}$ *and* $\mathbf{e}$ *of the sets Q^{n-1} and Q^{n-2}, respectively, into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]*

(a) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$,

(b) $A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$ *and*

(c) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$; *and*

(iii) there are mappings $^{-1}$ *and* $\mathbf{e}$ *of the sets Q^{n-1} and Q^{n-2}, respectively, into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ [of the type $< n, n-1, n-2 >$]*

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($\overline{a}$) $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$,

($\overline{b}$) $A(x, a_1^{n-2}, \mathbf{e}(a_1^{n-2})) = x$ *and*

($\overline{c}$) $A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$.

Remark 1. $\mathbf{e}$ *is a* $\{1, n\}-$*neutral operation of an* $n-$*grupoid* (Q, A) *iff algebra* $(Q, \{A, \mathbf{e}\})$ *of type* $< n, n-2 >$ *satisfies the laws* (b) *and* $(\overline{b})$ *from* Proposition 1 *(cf. [12]). The notion of* $\{i, j\}-$*neutral operation* $(i, j \in \{1, \dots, n\}, i < j)$ *of an* $n-$*groupoid is defined in a similar way (cf. [12]). Every* $n-$*groupoid has at most one* $\{i, j\}-$*neutral operation (cf. [12]). In every* $n-$*group,* $n \geq 2$*, there is a* $\{1, n\}-$*neutral operation (cf. [12]). There are* $n-$*groups without* $\{i, j\}-$*neutral operations with* $\{i, j\} \neq \{1, n\}$ *(cf. [14]). In [14],* $n-$*groups with* $\{i, j\}-$*neutral operations, for* $\{i, j\} \neq \{1, n\}$ *are described. Operation* $^{-1}$ *from* Proposition 1 *is a generalization of the inversing operation in a group. In fact, if* (Q, A) *is an* $n-$*group,* $n \geq 2$*, then for every* $a \in Q$ *and for every sequence* a_1^{n-2} *over* Q

$$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}),$$

where E *is a* $\{1, 2n-1\}-$*neutral operation of the* $(2n-1)-$*group* $(Q, \overset{2}{A})$; $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$ *(cf. [13]). (For* $n = 2$*,* $a^{-1} = \mathsf{E}(a)$*;* a^{-1} *is the inverse element of the element* a *with respect to the neutral element* $\mathbf{e}(\emptyset)$ *of the group* (Q, A)*.)*

Proposition 2. (see [14]) *Let* $n \geq 3$*, let* (Q, A) *be an* n*-group and* $\mathbf{e}$ *its* $\{1, n\}$*-neutral operation. Then the following statements are equivalent:*

(i) (Q, A) *is a commutative* n*-group,*

(ii) $\mathbf{e}$ *is an* $\{i, j\}$*-neutral operation of the* n*-group* (Q, A) *for every* $\{i, j\} \subseteq \{1, \dots, n\}$*,* $i < j$*.*

Proposition 3. (see [16]) *Let* (Q, A) *be an* m*-group,* $^{-1}$ *its inversing operation,* $\mathbf{e}$ *its* $\{1, m\}$*-neutral operation and let* $m \geq 2$*. Also let*

(o) $\quad {}^{-1}\!A(x, a_1^{m-2}, y) = z \overset{def}{\Leftrightarrow} A(z, a_1^{m-2}, y) = x$

for $x, y, z \in Q$ *and for every sequence* a_1^{m-2} *over* Q*. Then, for all* $x, y, z \in Q$ *and for every sequence* a_1^{m-2} *over* Q *the following equalities hold*

(1) $\quad {}^{-1}\!A(x, a_1^{m-2}, y) = A(x, a_1^{m-2}, (a_1^{m-2}, y)^{-1})$,

(2) $\quad \mathbf{e}(a_1^{m-2}) = {}^{-1}\!A(z, a_1^{m-2}, z)$,

(3) $\quad (a_1^{m-2}, x)^{-1} = {}^{-1}\!A({}^{-1}\!A(z, a_1^{m-2}, z), a_1^{m-2}, x)$ *and*

(4) $\quad A(x, a_1^{m-2}, y) = {}^{-1}\!A(x, a_1^{m-2}, {}^{-1}\!A({}^{-1}\!A(z, a_1^{m-2}, z), a_1^{m-2}, y))$.

Proposition 4. (see [16]) *Let* $n \geq 2$ *and let* (Q, B) *be an* $n-$*groupoid. Let also the following laws*

$B(B(x, z, b_1^{n-2}), B(y, a_1^{n-2}, z), a_1^{n-2}) = B(x, y, b_1^{n-2})$ *and*

$B(a, c_1^{n-2}, B(B(B(z, c_1^{n-2}, z), c_1^{n-2}, b), c_1^{n-2}, B(B(z, c_1^{n-2}, z), c_1^{n-2}, a))) = b$

hold in the $n-$*groupoid* (Q, B)*. Then, there is an* $n-$*group* (Q, A) *such that the following equality holds* ${}^{-1}\!A = B$*.*

Definition 2. (see [2],[5],[8] and [9]) *Let (Q,A) be a commutative $m-$group and let $m \geq 2$. Let also (Q,M) be an $n-$groupoid ($n-$semigroup in [2], [4]) and let $n \geq 2$. We say that (Q,A,M) is an $(m,n)-$*ring *iff for every $i \in \{1,\ldots,n\}$ and for every $a_1^{n-1}, b_1^m \in Q$ the following equality holds*

$$M(a_1^{i-1}, A(b_1^m), a_i^{n-1}) = A(\overline{M(a_1^{i-1}, b_j, a_i^{n-1})\big|}^{m}_{j=1}).$$

Proposition 5. *Let (Q,A,M) be an $(m,n)-$ring. Then, there is* at most one *element $o \in Q$ such that for every $a^{n-1} \in Q$ the following equalities hold*

$$M(o, a_1^{n-1}) = o \quad \textit{and} \quad M(a_1^{n-1}, o) = o.$$

See [4].

Proposition 6. *Let (Q,A,M) be an $(m,n)-$ring and* **O** *the $\{1,m\}-$neutral operation of the $m-$group (Q,A). Also, let o be the element of the set Q such that*

$$(\hat{o}) \qquad M(o, a_1^{n-1}) = M(a_1^{n-1}, o) = o$$

for all $a_1^{n-1} \in Q$. Then the following equality holds

$$\mathbf{O}(\overset{m-2}{o}) = o.$$

See [4] [$\mathbf{O}(\overset{m-2}{o}) = \overline{o}$ for $m{>}2$; $\overline{a}$ is skewed to a [7]].

Sketch of the proof.

a) $M(A(\overset{m}{o}), a_1^{n-1}) = A(\overline{M(o, a_1^{n-1}|}^{m}) = A(\overset{m}{o})$,
$M(a_1^{n-1}, A(\overset{m}{o})) = A(\overline{M(a_1^{n-1}, o|}^{m}) = A(\overset{m}{o})$ [: *Definition 2.*, (ô)].
b) By a) and *Proposition 5.*, we conclude that the following equality holds $A(\overset{m}{o}) = o$.
c) $A(\mathbf{O}(\overset{m-2}{o}), \overset{m-2}{o}, o) = o$, $A(o, \overset{m-2}{o}, o) = o$ [: *Proposition 1.*, *Remark 1.*,b)].
d) By c), *Definition 1.*, *Proposition 1.* and *Remark 1.*, we conclude that the following equality holds $\mathbf{O}(\overset{m-2}{o}) = o$. (For $m = 2 : \overset{m-2}{a} = \emptyset$.)

Remark 2.

(a) *Note that element o determined in* Proposition 5. *(*Proposition 6.*) is called a* left *and* right zero*, respectively, or a* two-side zero.

(b) *An element $z \in Q$ is called* zero *of (Q,A,M) iff for each $i \in \{1,\ldots,n\}$ and for every $a_1^{n-1} \in Q$ the following equality holds $M(a_1^{i-1}, z, a_i^{n-1}) = z$.*

(c) *Let (Q,A,M) be an (m,n)-ring, o its two-side zero and $m{>}2$. Then, by* Proposition 1.*,* Proposition 6. *and* Definition 2.*, we conclude that for each $x \in Q$ and for each $i \in \{1,\ldots,m\}$ the following equality holds $A(\overset{i-1}{o}, x, \overset{m-i}{o}) = x$. Whence, we conclude that there is a group $(Q,+)$ such that for every $x_1^m \in Q$ the following equality holds $A(x_1^m) = x_1 + ... + x_m$ (cf. [6]).*

2. Results

Theorem 1. *Let (Q,T,o) be an algebra of the type $< 3m+n-5, 0 >$, and let $m, n \geq 2$. Also let*

(a) $\boldsymbol{\alpha}(x, a_1^{m-2}, y) \overset{def}{=} T\left(x, T(T(\overset{2m-1}{o}, a_1^{m-2}, \overset{n-2}{o}), y, \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}), \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}\right),$

(b) $\boldsymbol{\beta}(x, a_1^{m-2}, y) \overset{def}{=} T\left(x, y, \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}\right)$ *and*

(c) $\boldsymbol{\gamma}(x, b_1^{n-2}, y) \overset{def}{=} T\left(T(x, o, y, \overset{3m-6}{o}, b_1^{n-2}), x, \overset{3m+n-7}{o}\right)$

for every $x, y, a_1^{m-2}, b_1^{n-2} \in Q$.

Furthermore, let the following laws

(i) $\boldsymbol{\beta}(\boldsymbol{\beta}(x, z, b_1^{m-2}), \boldsymbol{\beta}(y, a_1^{m-2}, z), a_1^{m-2}) = \boldsymbol{\beta}(x, y, b_1^{m-2}),$

(ii) $\boldsymbol{\beta}\left(a, c_1^{m-2}, \boldsymbol{\beta}\left(\boldsymbol{\beta}\left(\boldsymbol{\beta}(z, c_1^{m-2}, z), c_1^{m-2}, b\right), c_1^{m-2}, \boldsymbol{\beta}(\boldsymbol{\beta}(z, c_1^{m-2}, z), c_1^{m-2}, a)\right)\right) = b,$

(iii) $\boldsymbol{\alpha}(x_{\varphi(1)}, \ldots, x_{\varphi(m)}) = \boldsymbol{\alpha}(x_1^m)$ *for all permutations* φ *on* $\{1, \ldots, m\}$,

(iv) $\boldsymbol{\gamma}(x_1^{i-1}, \boldsymbol{\beta}(y_1^m), x_i^{n-1}) = \boldsymbol{\beta}(\overline{\boldsymbol{\gamma}(x_1^{i-1}, y_j, x_i^{n-1})\big|_{j=1}}^{m})$ *for all* $i \in \{1, \ldots, n\}$,

(v) $T(x, \overset{3m-4}{o}, b_1^{n-2}) = x,$

(vi) $T(o, o, y, \overset{3m-6}{o}, b_1^{n-2}) = o$ *and*

(vii) $T(x, x, \overset{3m+n-7}{o}) = o$

hold in the algebra (Q,T,o) of the type $< 3m+n-5, 0 >$. Then $(Q, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ is an (m,n)-ring and o is its two-side zero.

Proof.

1) By $(b), (i), (ii)$ and *Proposition 4.*, we conclude that there is an $m-$group (Q,A) such that the following equality holds ${}^{-1}A = \boldsymbol{\beta}$.

2) $A = \boldsymbol{\alpha}$. The sketch of the proof:

$$\begin{aligned}
\boldsymbol{\alpha}(x, a_1^{m-2}, y) &= T(x, T(T(\overset{2m-1}{o}, a_1^{m-2}, \overset{n-2}{o}), y, \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}), \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}) \\
&= T(x, T(\boldsymbol{\beta}(o, a_1^{m-2}, o), y, \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}), \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}) \\
&= T(x, \boldsymbol{\beta}(\boldsymbol{\beta}(o, a_1^{m-2}, o), a_1^{m-2}, y), \overset{2m-3}{o}, a_1^{m-2}, \overset{n-2}{o}) \\
&= \boldsymbol{\beta}(x, a_1^{m-2}, \boldsymbol{\beta}(\boldsymbol{\beta}(o, a_1^{m-2}, o), a_1^{m-2}, y)) \\
&= {}^{-1}A(x, a_1^{m-2}, {}^{-1}A({}^{-1}A(o, a_1^{m-2}, o), a_1^{m-2}, y)) = A(x, a_1^{m-2}, y)
\end{aligned}$$

[: (a), (b), 1), *Proposition 3.*].

3) By 1), 2), (iii), (iv) and *Definition 2*, we conclude that $(Q, \boldsymbol{\alpha}, \boldsymbol{\gamma})$ is an $(m,n)-$ring.

4) For all $a_1^{n-1} \in Q$ the following equalities hold

$$\boldsymbol{\gamma}(o, a_1^{n-1}) = o \quad \text{and } \boldsymbol{\gamma}(a_1^{n-1}, o) = o.$$

The sketch of the proof:

$$\begin{aligned}
\boldsymbol{\gamma}(o, b_1^{n-2}, y) =&\ T(T(o, o, y, \overset{3m-6}{o}, b_1^{n-2}), o, \overset{3m+n-7}{o}) \\
=&\ T(o, o, \overset{3m+n-7}{o}) = o; \\
\boldsymbol{\gamma}(x, b_1^{n-2}, o) =&\ T(T(x, o, o, \overset{3m-6}{o}, b_1^{n-2}), x, \overset{3m+n-7}{o}) \\
=&\ T(x, x, \overset{3m+n-7}{o}) = o;
\end{aligned}$$

[: (c),(v),(vi),(vii)]. □

Theorem 2. *Let (Q, A, M) be an $(m,n)-$ring,* $\mathbf{O}$ *be the $\{1, m\}-$neutral operation of the $m-$group (Q, A), and $^{-}$ the inversing operation of the $m-$group (Q, A). Also, let o be the element of the set Q such that*

(a) $M(o, a_1^{n-1}) = M(a_1^{n-1}, o) = o$ *for all* $a_1^{n-1} \in Q$ *(i.e., let o be a two-side zero).*

(o) $T(x, y, z, a_1^{m-2}, b_1^{m-2}, c_1^{m-2}, d_1^{n-2}) \overset{def}{=}$
$$A(^{-1}A(M(x, d_1^{n-2}, z), a_1^{m-2}, M(y, d_1^{n-2}, z)), b_1^{m-2}, {}^{-1}A(x, c_1^{m-2}, y)).$$[1]
for each $x, y, z, a_1^{m-2}, b_1^{m-2}, c_1^{m-2}, d_1^{n-2} \in Q$

Then, the following identities hold:

(1) $\mathbf{O}(b_1^{m-2}) = T(x, x, y, \overset{2}{\overline{a_1^{m-2}|}}, b_1^{m-2}, c_1^{n-2});$

(2) $^{-}(b_1^{m-2}, x) = T(T(\overset{2}{u}, v, \overset{2}{\overline{a_1^{m-2}|}}, b_1^{m-2}, c_1^{n-2}), x, o, \overset{2}{\overline{d_1^{m-2}|}}, b_1^{m-2}, e_1^{n-2});$

(3) $^{-1}A(x, b_1^{m-2}, y) = T(x, y, o, \overset{2}{\overline{a_1^{m-2}|}}, b_1^{m-2}, c_1^{n-2});$

(4) $A(x, b_{1,}^{m-2}, y) =$
$$T(x, T(T(\overset{2}{u}, v, \overset{2}{\overline{a_1^{m-2}|}}, b_{1,}^{m-2} c_1^{n-2}), y, o, \overset{2}{\overline{d_1^{m-2}|}}, b_{1,}^{m-2} e_1^{n-2}), o, \overset{2}{\overline{p_1^{m-2}|}}, b_{1,}^{m-2} q_1^{n-2});$$

(5) $M(x, b_{1,}^{n-2} y) = T(T(x, o, y, \overset{m-2}{o}, a_{1,}^{m-2} \overset{m-2}{o}, b_1^{n-2}), x, o, \overset{2}{\overline{c_1^{m-2}|}}, a_{1,}^{m-2} d_1^{n-2});$

(6) $T(x, \overset{2}{o}, \overset{2}{\overline{a_1^{m-2}|}}, \overset{m-2}{o}, b_1^{n-2}) = x;$

(7) $T(\overset{2}{o}, x, \overset{2}{\overline{a_1^{m-2}|}}, \overset{m-2}{o}, b_1^{n-2}) = o;$ *and*

(8) $T(\overset{2}{x}, y, \overset{2}{\overline{a_1^{m-2}|}}, \overset{m-2}{o}, b_1^{n-2}) = o.$

[1] cf. [11]/(7) or [10]/10/(11)

Sketch of the proof.

1) $T(x,x,y,\overset{2}{\overline{a_1^{m-2}|}},b_1^{m-2},c_1^{n-2})$
$= A({}^{-1}\!A(M(x,c_1^{n-2},y),a_1^{m-2},M(x,c_1^{n-2},y)),a_1^{m-2},{}^{-1}\!A(x,a_1^{m-2},x))$
$= A(\mathbf{O}(a_1^{m-2}),a_1^{m-2},{}^{-1}A(x,a_1^{m-2},x)) = {}^{-1}\!A(x,a_1^{m-2},x)) = \mathbf{O}(a_1^{m-2})$
[: (o), *Proposition 3.*, *Proposition 1.*, *Remark 1.*].

2) $T(T(\overset{2}{u},v,\overset{2}{\overline{a_1^{m-2}|}},b_{1,}^{m-2}c_1^{n-2}),x,o,\overset{2}{\overline{d_1^{m-2}|}},b_{1,}^{m-2}e_1^{n-2})$

$= T(\mathbf{O}(b_1^{m-2}),x,o,\overset{2}{\overline{d_1^{m-2}|}},b_{1,}^{m-2}e_1^{n-2})$
$= A({}^{-1}\!A(M(\mathbf{O}(b_1^{m-2}),e_{1,}^{n-2}o),d_{1,}^{m-2}M(x,e_{1,}^{n-2}o)),d_{1,}^{m-2}A(\mathbf{O}(b_1^{m-2}),b_{1,}^{m-2}x))$
$= A({}^{-1}\!A(o,d_{1,}^{m-2},o)d_{1,}^{m-2},A(\mathbf{O}(b_1^{m-2}),b_{1,}^{m-2}-(b_{1,}^{m-2}x)))$
$= A(\mathbf{O}(d_1^{m-2}),d_{1,}^{m-2}-(b_{1,}^{m-2}x)) = -(b_{1,}^{m-2}x)$
[: (o), 1), (a), *Proposition 3.*, *Proposition 1.*, *Remark 1.*].

3) $T(x,y,o,\overset{2}{\overline{a_1^{m-2}|}},b_1^{m-2},c_1^{n-2})$
$= A({}^{-1}\!A(M(x,c_1^{n-2},o),a_1^{m-2},M(y,c_1^{n-2},o)),a_1^{m-2},{}^{-1}\!A(x,b_1^{m-2},y))$
$= A({}^{-1}\!A(o,a_1^{m-2},o),a_1^{m-2},{}^{-1}\!A(x,b_1^{m-2},y))$
$= A(\mathbf{O}(a_1^{m-2}),a_1^{m-2},{}^{-1}A(x,b_1^{m-2},y))$
$={}^{-1}\!A(x,b_1^{m-2},y))$
[: (o), (a), *Proposition 3.*, *Proposition 1.*, *Remark 1.*].

4) $T(x,T(T(\overset{2}{u},v,\overset{2}{\overline{a_1^{m-2}|}},b_{1,}^{m-2}c_1^{n-2}),y,o,\overset{2}{\overline{d_1^{m-2}|}},b_{1,}^{m-2}e_1^{n-2}),o,\overset{2}{\overline{p_1^{m-2}|}},b_{1,}^{m-2}q_1^{n-2})$

$= T(x,-(b_1^{m-2},y),o,\overset{2}{\overline{p_1^{m-2}|}},b_1^{m-2},q_1^{n-2})$
$= A({}^{-1}\!A(M(x,q_{1,}^{n-2}o),p_{1,}^{m-2}M(-(b_{1,}^{m-2}y),q_{1,}^{n-2}o)),p_1^{m-2},\ {}^{-1}\!A(x,b_{1,}^{m-2}-(b_{1,}^{m-2}y)))$
$= A({}^{-1}\!A(o,p_1^{m-2},o),p_1^{m-2},{}^{-1}A(x,b_1^{m-2},-(b_1^{m-2},y)))$
$= A(\mathbf{O}(p_1^{m-2}),p_1^{m-2},{}^{-1}A(x,b_1^{m-2},-(b_1^{m-2},y)))$
$= {}^{-1}\!A(x,b_1^{m-2},-(b_1^{m-2},y)) = A(x,b_1^{m-2},-(b_1^{m-2},-(b_1^{m-2},y)))$
$= A(x,b_1^{m-2},y)$
[: (o), 2), (a), *3.*, *Definition 1.*, *Remark 1.*].

5_1) $M(x,b_1^{n-2},y) = {}^{-1}A(M(x,b_1^{n-2},y),\overset{m-2}{o},\mathbf{O}(\overset{m-2}{o}))$
$= {}^{-1}\!A(M(x,b_1^{n-2},y),\overset{m-2}{o},o) = {}^{-1}A(M(x,b_1^{n-2},y),\overset{m-2}{o},M(o,b_1^{n-2},y))$
$= A({}^{-1}\!A(M(x,b_1^{n-2},y),\overset{m-2}{o},M(o,b_1^{n-2},y)),a_1^{m-2},\mathbf{O}(a_1^{m-2}))$
$= A({}^{-1}\!A(M(x,b_1^{n-2},y),\overset{m-2}{o},M(o,b_1^{n-2},y)),a_1^{m-2},A(x,a_1^{m-2},-(a_1^{m-2},x)))$

$= A(A(^{-1}A(M(x,b_1^{n-2},y),\overset{m-2}{o},M(o,b_1^{n-2},y)),a_1^{m-2},x),a_1^{m-2},-(a_1^{m-2},x))$

$= A(A(^{-1}A(M(x,b_{1,}^{n-2}y),\overset{m-2}{o},M(o,b_{1,}^{n-2}y)),a_{1,}^{m-2-1}A(x,\overset{m-2}{o},o)),a_{1,}^{m-2}-(a_{1,}^{m-2}x))$

$= {}^{-1}A(A(^{-1}A(M(x,b_1^{n-2},y),\overset{m-2}{o},M(o,b_1^{n-2},y)),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},o)),a_1^{m-2},x)$

$= {}^{-1}A(T(x,o,y,\overset{m-2}{o},a_1^{m-2},\overset{m-2}{o},b_1^{n-2}),a_1^{m-2},x)$

[: *Proposition 1.*, *Remark 1.*, *Proposition 6.*, (a), (o)].

$5_2)$ $X \overset{def}{=} T(x,o,y,\overset{m-2}{o},a_1^{m-2},\overset{m-2}{o},b_1^{n-2})$.

$5_3)$ $M(x,b_1^{n-2},y) = {}^{-1}A(X,a_1^{m-2},x) = A(\mathbf{O}(c_1^{m-2}),c_1^{m-2},{}^{-1}A(X,a_1^{m-2},x))$

$= A(^{-1}A(o,c_1^{m-2},o),c_1^{m-2},{}^{-1}A(X,a_1^{m-2},x))$

$= A(^{-1}A(M(X,d_1^{n-2},o),c_1^{m-2},M(x,d_1^{n-2},o)),c_1^{m-2},{}^{-1}A(X,a_1^{m-2},x))$

$= T(X,x,o,\overset{2}{\overline{c_1^{m-2}|}},a_1^{m-2},d_1^{n-2})$

$= T(T(x,o,y,\overset{m-2}{o},a_1^{m-2},\overset{m-2}{o},b_1^{n-2}),x,o,\overset{2}{\overline{c_1^{m-2}|}},a_1^{m-2},d_1^{n-2})$

[: 5_1), 5_2), *Proposition 1.*, *Remark 1.*, *Proposition 3.*, (a), (o)].

6) $T(x,\overset{2}{o},\overset{2}{\overline{a_1^{m-2}|}},\overset{m-2}{o},b_1^{n-2})$

$= A(^{-1}A(M(x,b_1^{n-2},o),a_1^{m-2},M(o,b_1^{n-2},o)),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},o))$

$= A(^{-1}A(o,a_1^{m-2},o),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},o))$

$= A(\mathbf{O}(a_1^{m-2}),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},o)) = {}^{-1}A(x,\overset{m-2}{o},o)$

$= {}^{-1}A(x,\overset{m-2}{o},\mathbf{O}(\overset{m-2}{o})) = x; {}^{-1}A(x,\overset{m-2}{o},\mathbf{O}(\overset{m-2}{o})) = z \Leftrightarrow$

$A(z,\overset{m-2}{o},\mathbf{O}(\overset{m-2}{o})) = x \Leftrightarrow z = x$

[: (o), (a), *Proposition 3.*, *Proposition 1.*, *Remark 1.*, *Proposition 6.*].

7) $T(\overset{2}{o},x,\overset{2}{\overline{a_1^{m-2}|}},\overset{m-2}{o},b_1^{n-2})$

$= A(^{-1}A(M(o,b_1^{n-2},x),a_1^{m-2},M(o,b_1^{n-2},x)),a_1^{m-2},{}^{-1}A(o,\overset{m-2}{o},o))$

$= A(\mathbf{O}(a_1^{m-2}),a_1^{m-2},{}^{-1}A(o,\overset{m-2}{o},o)) = {}^{-1}A(o,\overset{m-2}{o},o) = \mathbf{O}(\overset{m-2}{o}) = o$

[: (o), *Proposition 3.*, *Proposition 1.*, *Remark 1.*, *Proposition 6.*].

8) $T(\overset{2}{x},y,\overset{2}{\overline{a_1^{m-2}|}},\overset{m-2}{o},b_1^{n-2})$

$= A(^{-1}A(M(x,b_1^{n-2},y),a_1^{m-2},M(x,b_1^{n-2},y)),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},x))$

$= A(\mathbf{O}(a_1^{m-2}),a_1^{m-2},{}^{-1}A(x,\overset{m-2}{o},x)) = {}^{-1}A(x,\overset{m-2}{o},x) = \mathbf{O}(\overset{m-2}{o}) = o$

[: (o), *Proposition 3.*, *Proposition 1.*, *Remark 1.*, *Proposition 6.*]. □

Remark 3. *The operation* **O** *has been described in* Theorem 2. *by using only the operation* T. *Bearing in mind* Proposition 6., *(1) from* Theorem 2., *as well as the convention that* $a_1^\circ = \emptyset$, *we find that for each* $x, y, c_1^{n-2} \in Q$, *(1) reduces to the following equality*

$$(m) \qquad o = T(x, x, y, c_1^{n-2}).$$

Hence, bearing in mind Theorem 2., *we find out that in* $(2, n)-$*rings the operations* $A, ^{-1}A, -$ *and* M *can also be described by using just one* $(n+1)-$ *ary operation. In addition, in the case* $m = 2$, *the constant* $o \in Q$ *can be eliminated from equalities (6)–(8) in* Theorem 2. *by using* (m).

In [11] rings [(2,2)–rings] have been described as 3–groupoids with one law.

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