# Single facility minisum location on curves

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**Abstract**. *The minisum location problem is well-known and had extensively been studied in the case of the unknown location being situated somewhere in the plane. Also the accompanying Weiszfeld iteration [3] is nowadays well understood, even for noneuclidean distances [1, 2]. We introduce as a side condition that the unknown optimal location point may only lie on some given curve in the plane. For a piece at a straight line and for a circle numerical methods are developed and numerical examples got by them are given.*

**Key words:** *location, single facility minisum location, side conditions*

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## 1. The problem for general curves

Let points be given

$$(x_i, y_i) \quad (i = 1, \ldots, n \geq 3) \tag{1}$$

on the plane, not all of them on a straight line. Looking for an optimal location point $(x, y)$ means to determine $(x, y)$ such that the sum of Euclidean distances $d_i(x, y)$ from point $i$ to the unknown location is minimized. Thus, with

$$d_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2} \quad (i = 1, \ldots, n) \tag{2}$$

the optimization problem

$$F(x, y) = \sum_{i=1}^{n} d_i(x, y) \longrightarrow \min \tag{3}$$

has to be solved. The optimal location $(x, y)$ lies in the convex hull of the given points; up to very special cases it is unique because $F$ is strictly convex [3]. Thus, the conditions

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$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0 \tag{4}$$

are necessary and sufficient for the minimum; they can be written as

$$x = \frac{\displaystyle\sum_{i=1}^{n} \frac{x_i}{d_i(x,y)}}{\displaystyle\sum_{i=1}^{n} \frac{1}{d_i(x,y)}}, \qquad y = \frac{\displaystyle\sum_{i=1}^{n} \frac{y_i}{d_i(x,y)}}{\displaystyle\sum_{i=1}^{n} \frac{1}{d_i(x,y)}}. \tag{5}$$

It is well-known [1, 2] that the Weiszfeld iteration procedure based on (5), i.e.

$$x^{(k+1)} = \frac{\displaystyle\sum_{i=1}^{n} \frac{x_i}{d_i(x^{(k)}, y^{(k)})}}{\displaystyle\sum_{i=1}^{n} \frac{1}{d_i(x^{(k)}, x^{(k)})}}, \qquad y^{(k+1)} = \frac{\displaystyle\sum_{i=1}^{n} \frac{y_i}{d_i(x^{(k)}, y^{(k)})}}{\displaystyle\sum_{i=1}^{n} \frac{1}{d_i(x^{(k)}, y^{(k)})}} \tag{6}$$

$$k = 0, 1, 2, \ldots$$

with e.g.

$$\big(x^{(0)}, y^{(0)}\big) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right)$$

linearly converges to $(x, y)$ up to very special cases.

We now consider the problem of determining $(x, y)$ with (3) not within the convex hull of the given points but only on some curve given in its parametric form by

$$x = f(t),\ y = g(t),\ t \in [u, v],\ -\infty < u, v < \infty\,. \tag{7}$$

First, we will consider this general case and later we will go into details if (7) is a piece of a straight line or some circle.

The side conditions (7) can be inserted into (3) to give the objective function

$$F(t) = \sum_{i=1}^{n} d_i\big(f(t), g(t)\big) \longrightarrow \min \tag{8}$$

with just one variable $t$. The necessary condition $\frac{\partial F}{dt} = 0$ for a minimum (or maximum) gives

$$\sum_{i=1}^{n} \frac{f'(t)\big(f(t) - x_i\big) + g'(t)\big(g(t) - y_i\big)}{d_i\big(f(t), g(t)\big)} = 0\,. \tag{9}$$

Some $t$ with (9) and with $\frac{d^2F}{dt^2} > 0$ will give a local minimum of $F$ and $\big(x = f(t), y = g(t)\big)$ will be the optimal location on the given curve. Now (9) cannot generally be solved. Thus, we will discuss two special cases of (7), namely the straight line and the circle.

## 2. The case of a straight line

A finite piece of a straight line is given by

$$f(t) = t,\ g(t) = ax + b,\ -\infty < u \le t \le v < \infty \tag{10}$$

where $a$ and $b$ are certain given values. The necessary condition (9) is

$$\sum_{i=1}^{n} \frac{(t - x_i) + a(at + b - y_i)}{d_i\big(f(t), g(t)\big)} = 0 \tag{11}$$

where

$$d_i\big(f(t), g(t)\big) = \sqrt{(t - x_i)^2 + (at + b - y_i)^2}\,. \tag{12}$$

As normally

$$\frac{d^2F}{dt^2} = \sum_{i=1}^{n} \frac{(ax_i + b - y_i)^2}{d_i^3\big(f(t), g(t)\big)} > 0\,;$$

if not all of the given points are on the given straight line, we know that $F$ is strictly convex and that a solution of (11) will give the global minimum. If this one is in $[u, v]$, the iteration

$$t^{(k+1)} = \frac{\displaystyle\sum_{i=1}^{n} \frac{-a(b - y_i) + x_i}{d_i\big(f(t^{(k)}), g(t^{(k)})\big)}}{\displaystyle(1 + a^2)\sum_{i=1}^{n} \frac{1}{d_i\big(f(t^{(k)}), g(t^{(k)})\big)}} \tag{13}$$

corresponding to (6) empirically converges to the solution. The proof is open.

Let us consider some example with $n = 6$. The given points are

$$\begin{array}{c|cccccc} i & 1 & 2 & 3 & 4 & 5 & 6 \\ x_i & 0 & 2 & 5 & 4 & 4 & 10 \\ y_i & 0 & 4 & 5 & 1 & -3 & 3 \end{array} \tag{14}$$

We set $u = -2$, $v = 5$ and consider the four cases

| | | |
|---|---|---|
| (A) | $a = 1/2$, | $b = 0$ |
| (B) | $a = 1/2$, | $b = 1$ |
| (C) | $a = -1/2$, | $b = -1$ |
| (D) | $a = -1/2$, | $b = 3$ |

With $t^{(0)} = 1$ we found by (13) after $K$ iterations the following results up to four correct decimals

| | $K$ | $x = t$ | $y = at + b$ | $F(t)$ |
|---|---|---|---|---|
| (A) | 10 | 3.6978 | 1.8489 | 22.4501 |
| (B) | 12 | 3.3908 | 2.6954 | 23.1450 |
| (C) | 19 | 2.2645 | -2.1322 | 31.6916 |
| (D) | 25 | 4.0000 | 1.0000 | 22.1764 |

The given data and the results are visualized in *Figure 1.*

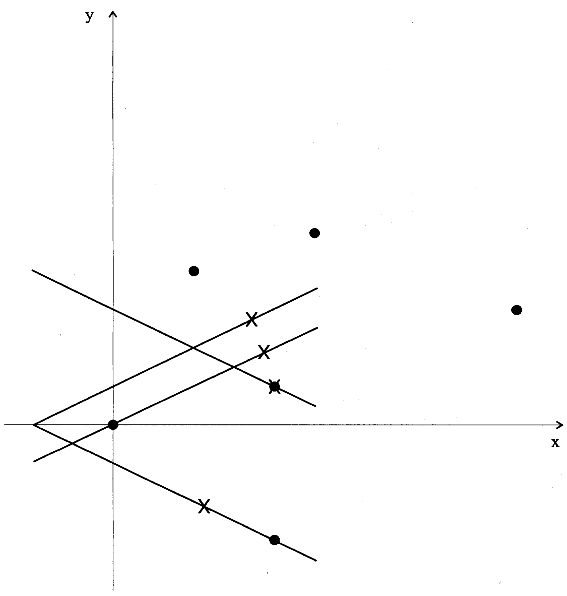


Figure 1.

## 3. The case of a circle

A circle with center $(a, b)$ and radius $r$ is given by

$$x = f(t) = a + r\cos t, \quad y = g(t) = b + r\sin t, \quad 0 \le t \le 2\pi \tag{15}$$

The necessary condition (9) gives

$$\sum_{i=1}^{n} \frac{-r\sin t(a + r\cos t - x_i) + r\cos t(b + r\sin t - y_i)}{d_i\big(f(t), g(t)\big)} = 0 \tag{16}$$

with

$$d_i\big(f(t), g(t)\big) = \sqrt{(a + r\cos t - x_i)^2 + (b + r\sin t - y_i)^2}\ .$$

Some canonical fixpoint iteration for the solution of (16) is

$$t^{(k+1)} = \operatorname{atan} \left( \frac{\displaystyle\sum_{i=1}^{n} \frac{b + r\sin(t^{(k)}) - y_i}{d_i(f(t^{(k)}), g(t^{(k)}))}}{\displaystyle\sum_{i=1}^{n} \frac{a + r\cos(t^{(k)}) - x_i}{d_i(f(t^{(k)}), g(t^{(k)}))}} \right) \tag{17}$$

$$k = 0, 1, 2, \dots\ .$$

But as with $t^{(k+1)}$ also $t^{(k+1)} + \pi$ fits (17), it is advisable to take at each iteration that value of those two which gives a smaller value of $F$.

For example, we consider again the given points (14). We always use $(a, b) = (5, 3)$ and four different radii $r = 1, 3, 5, 8$. The results of (17) with $t^{(0)} = 1$ (verified to give the global minimum) are

| $r$ | $K$ | $t$ | $a + r\cos t$ | $b + r\sin t$ | $F(t)$ |
|---|---|---|---|---|---|
| 1 | 9 | 3.9401 | 4.3022 | 2.2837 | 22.9006 |
| 3 | 14 | 4.0692 | 3.2007 | .5994 | 23.4086 |
| 5 | 24 | 3.8365 | 1.1596 | -.2017 | 28.4016 |
| 8 | 24 | 4.0418 | .0281 | -3.2674 | 42.0335 |

The results are visualized in *Figure 2*.

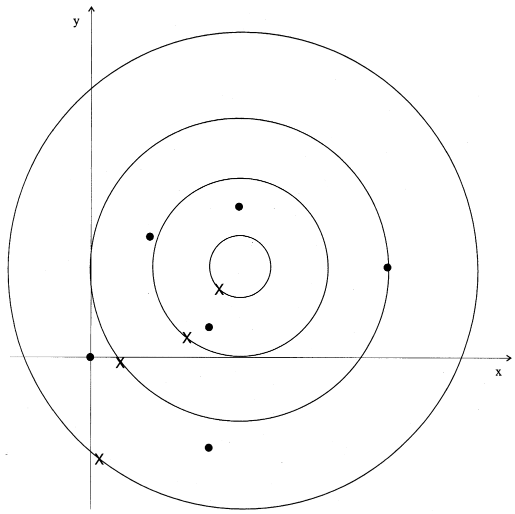


Figure 2.

Note that in this case (in contrast to the case of a straight line) the equation (16) empirically characterizes just one minimum and one maximum. Thus, the decision to take either $t^{(k+1)}$ or $t^{(k+1)} + \pi$ at each iteration is essential. A proof of the uniqueness of a minimum and a proof for the convergence of the iterations are open. It might be of interest that we were not able to design a convergent iteration scheme for the case of a given ellipse $x(t) = a + p\cos t$, $y(t) = b + q\sin t$ $(p \neq q)$; also we verified that in this case several minima and maxima may occur.

## References

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