On algebraic equations concerning semi-tangential polygons

MIRKO RADIĆ*

Abstract. *Some properties of equations* (5) *and* (6) *are proved (*Theorem *1–2) and it was established that the positive roots of these equations are radii of a sequence of tangential semi-polygons which have the same lengths of tangents.*

Key words: *algebraic equation, semi-tangential polygon*

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1. Preliminaries

First, on notations which will be used.

Symbol $S_j(x_1,\dots,x_n)$. Let $x_1,\dots,x_n$ be real numbers, and let j be an integer such that $1 \le j \le n$. Then $S_j(x_1,\dots,x_n)$ is the sum of all $\binom{n}{j}$ products of the form $x_{i_1}\cdot\dots\cdot x_{i_j}$, where $i_1,\dots,i_j$ are different elements of the set $\{1,\dots,n\}$, that is

$$S_j(x_1,\dots,x_n) = \sum_{1\le i_1<\dots<i_j\le n} x_{i_1}\cdot\dots\cdot x_{i_j}. \tag{1}$$

Of course, $S_1(x_1,\dots,x_n) = x_1+\dots+x_n$.

Semi-polygon. Let $A_1,\dots,A_n$ be any given different points in a plane. Then the union

$$A_1A_2 \cup A_2A_3 \cup \dots \cup A_{n-1}A_n \cup S \tag{2}$$

of line segments $A_1A_2,\dots,A_{n-1}A_n$ and the set S which is either ab empty set or a segment A_nA_1, will be called a semi-polygon and denoted by $A_1\cdots A_n$ or briefly by $\underline{A}$.

So, each polygon may be termed a semi-polygon, but not conversely, if S is an empty set.

If $A_1\cdots A_n$ is a semi-polygon which is not a polygon, then its vertices A_1 and A_n will be called end-vertices.

*Department of Mathematics, University of Rijeka, Omladinska 14, HR-51 000 Rijeka, Croatia

Tangential semi-polygon. A semi-polygon $A_1 \cdots A_n$ will be called a tangential semi-polygon if there is a circle $\mathcal{C}$ such that each side of $\underline{A}$ lies on a tangent line of $\mathcal{C}$ and, in case $A_1 \cdots A_n$ is not a polygon, the end-vertices A_1 and A_n lie on $\mathcal{C}$.

Now something about the angles which play an important role in the following.

Let $A_1 \cdots A_n$ be a tangential semi-polygon and let C be the centre of its inscribed circle. In case $\underline{A}$ is a polygon, then

$$\beta_i = \angle CA_iA_{i+1}, \ \ i = 1, \dots, n \tag{3}$$

and in case $\underline{A}$ is not a polygon, then

$$\beta_i = \angle CA_{i+1}A_{i+2}, \ \ i = 1, \dots, n-2. \tag{4}$$

Of course, in any case, for each β_i there holds $\beta_i < \frac{\pi}{2}$, since no two of the consecutive vertices are the same.

2. On some algebraic equations

In what follows, for brevity, S_j^n will be written instead of $S_j(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are real numbers different from zero, and T_j^n will be written nstead of $S_j(\operatorname{tg}\beta_1, \dots, \operatorname{tg}\beta_n)$. So,

$$S_j^n = S_j(t_1, \dots, t_n), \ \ T_j^n = S_j(\operatorname{tg}\beta_1, \dots, \operatorname{tg}\beta_n), \ \ j = 1, \dots, n.$$

Also, the symbol $\hat{n}$ will be used which is defined as follows: If n is a natural number, then

$$\hat{n} = \begin{cases} n \text{ if } n \text{ is odd} \\ n-1 \text{ if } n \text{ is even.} \end{cases}$$

The number s in the expression $(-1)^s$ will always be given by

$$s = (1 + 3 + 5 + \cdots + \hat{n}) + 1.$$

Theorem 1. *Let the following two equations be given*

$$\frac{x^n - S_2^n x^{n-2} + S_4^n x^{n-4} - \cdots + (-1)^s S_{n-1}^n x}{S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \cdots + (-1)^s S_n^n} = \lambda, \ \ n \text{ is odd} \tag{5}$$

$$\frac{S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \cdots + (-1)^s S_{n-1}^n x}{-x^n + S_2^n x^{n-2} - S_4^n x^{n-4} + \cdots + (-1)^s S_n^n} = \lambda, \ \ n \text{ is even,} \tag{6}$$

where λ is any given positive number. Then the number of positive roots of the first equation is $\frac{n+1}{2}$, and of the second $\frac{n}{2}$. For each positive root x_i of those equations there holds

$$\min\{t_1, \dots, t_n\} \operatorname{tg}\frac{\varphi}{n} \le x_i \le \max\{t_1, \dots, t_n\} \operatorname{tg}\frac{\varphi + (n-1)\pi}{n}, \tag{7}$$

where $\varphi = \operatorname{arctg}\lambda$.

Proof. We shall use the following two trigonometric equalities

$$\operatorname{tg}(\beta_1 + \cdots + \beta_n) = \frac{T_1^n - T_3^n + T_5^n - \cdots + (-1)^s T_n^n}{1 - T_2^n + T_4^n - \cdots + (-1)^s T_{n-1}^n}, \quad n \text{ is odd}, \tag{8}$$

$$\operatorname{tg}(\beta_1 + \cdots + \beta_n) = \frac{T_1^n - T_3^n + T_5^n - \cdots + (-1)^s T_{n-1}^n}{1 - T_2^n + T_4^n - \cdots + (-1)^s T_n^n}, \quad n \text{ is even}, \tag{9}$$

which can be easily proved by induction on n.

First we prove the following lemma.

Lemma 1. *For each integer* $k \in \left\{0, 1, \ldots, \frac{\hat{n}-1}{2}\right\}$ *there are angles* $\beta_1^{(k)}, \ldots, \beta_n^{(k)}$ *such that*

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = \varphi + k\pi, \tag{10}$$

$$t_1 \operatorname{tg} \beta_1^{(k)} = \cdots = t_n \operatorname{tg} \beta_n^{(k)}. \tag{11}$$

Proof. We need to prove that there are angles $\beta_1^{(k)}, \ldots, \beta_n^{(k)}$ satisfying (10) and they have the property that there exists a positive number x_k such that

$$t_1 \operatorname{tg} \beta_1^{(k)} = \cdots = t_n \operatorname{tg} \beta_n^{(k)} = x_k \tag{12}$$

or

$$\operatorname{tg} \beta_i^{(k)} = \frac{x_k}{t_i}, \quad i = 1, \ldots, n.$$

Thus we have the condition

$$\sum_{i=1}^{n} \operatorname{arctg} \frac{x_k}{t_i} = \varphi + k\pi$$

which obviously can be fulfiled since the function $\operatorname{arctg} x$ is continuous for every real number x. So, our lemma is proved. □

Now, if in (8) and (9) we replace $\beta_1 + \cdots + \beta_n$ by $\varphi + k\pi$ and $\operatorname{tg} \beta_i$ by $\frac{x}{t_i}$, $i = 1, \ldots, n$, we shall get the equations which can be written as (5) and (6). Each x_k given by (12) is a positive root of the corresponding equation.

In proving that inequalities (7) hold well, we shall use the following obvious fact: If $u_1, \ldots, u_n$ are positive numbers, then

$$\min\{u_1, \ldots, u_n\} \leq \frac{u_1 + \cdots + u_n}{n} \leq \max\{u_1, \ldots, u_n\}.$$

So from (10) it follows that

$$\min\{\beta_1^{(k)}, \ldots, \beta_n^{(k)}\} \leq \frac{\varphi + k\pi}{n} \leq \max\{\beta_1^{(k)}, \ldots, \beta_n^{(k)}\} \tag{13}$$

and from (11) we see that $t_i < t_j$ implies $\beta_i^{(k)} > \beta_j^{(k)}$. Thus the following holds:

$$\text{if } \ t_i = \min\{t_1, \ldots, t_n\}, \ \text{ then } \ \beta_i^{(k)} = \max\{\beta_1^{(k)}, \ldots, \beta_n^{(k)}\}$$

$$\text{if } \ t_j = \max\{t_1, \ldots, t_n\}, \ \text{ then } \ \beta_j^{(k)} = \min\{\beta_1^{(k)}, \ldots, \beta_n^{(k)}\},$$

and in this case

$$t_i \operatorname{tg}\beta_i^{(k)} = t_j \operatorname{tg}\beta_j^{(k)}.$$

Now, using (12) and (13), it is obvious that

$$\min\{t_1,\dots,t_n\}\operatorname{tg}\frac{\varphi+k\pi}{n} \le x_k \le \max\{t_1,\dots,t_n\}\operatorname{tg}\frac{\varphi+k\pi}{n}.$$

Since $\varphi \le \varphi + k\pi \le \frac{\hat{n}-1}{2}\pi$ for each $k = 0,1,\dots,\frac{\hat{n}-1}{2}$, the proof of *Theorem 1* is complete. □

The following corollaries may also be interesting.

Corollary 1. *Equations* (5) *and* (6) *have all real roots. For each negative root* x_k *there holds*

$$\max\{t_1,\dots,t_n\}\operatorname{tg}\left(\frac{\varphi}{n}+\frac{(n-1)\pi}{n}\right) \le x_k \le \min\{t_1,\dots,t_n\}\operatorname{tg}\left(\frac{\varphi}{n}+\frac{(n+1)\pi}{2n}\right).$$

Proof. In the same way as in *Lemma 1* it can be shown that for each $k \in \left\{\frac{\hat{n}+1}{2},\dots,n-1\right\}$ there are angles $\beta_1^{(k)},\dots,\beta_n^{(k)}$ such that

$$\beta_1^{(k)}+\dots+\beta_n^{(k)} = \varphi + k\pi$$

$$t_1\operatorname{tg}\beta_1^{(k)} = \dots = t_n\operatorname{tg}\beta_n^{(k)} = x_k,$$

but now x_k is negative since $\varphi + k\pi > n\frac{\pi}{2}$.

So, we have the following situation: if $k = 0,1,\dots,\frac{\hat{n}-1}{2}$, we get positive roots, if $k = \frac{\hat{n}+1}{2},\dots,n-1$, we get negative roots, if $k = n,\dots,n+\frac{\hat{n}-1}{2}$, we again get all positive roots, and so on.

For example, if $n = 5$, then for $k = 0,1,2$ we get positive roots, for $k = 3,4$ negative, and so on. □

Corollary 2. *Let* λ *in equations* (5) *and* (6) *be negative and let* φ *be the least positive angle such that* $\varphi = \operatorname{arctg}\lambda$. *Then we have angles* $\varphi + k\pi$ *for* $k = 0,1,2,\dots$ *and the situation is like when* $\lambda > 0$.

For example, if $\lambda = -3$, $n = 5$, then for $k = 0,1$ we get positive roots, and for $k = 2,3,4$ negative.

Corollary 3. *Let* λ *in equations* (5) *and* (6) *be zero. Then we have the following two equations*

$$x^n - S_2^n x^{n-2} + S_4^n x^{n-4} - \dots + (-1)^s S_{n-1}^n x = 0, \quad n \text{ is odd} \tag{14}$$

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \dots + (-1)^s S_{n-1}^n x = 0, \quad n \text{ is even}, \tag{15}$$

and the angles are $k\pi$, $k = 0,1,\dots$ *For* $k = 0$ *we get the root equal to zero. For* $k = 1,\dots,\frac{\hat{n}-1}{2}$ *we get positive roots, and for* $k = \frac{\hat{n}+1}{2},\dots,\hat{n}-1$ *negative. For each positive root there holds*

$$\min\{t_1,\dots,t_n\}\operatorname{tg}\frac{\pi}{n} \le x_k \le \max\{t_1,\dots,t_n\}\operatorname{tg}\frac{(\hat{n}-1)\pi}{n},$$

and for each negative root

$$\max\{t_1,\dots,t_n\}\,\mathrm{tg}\,\frac{(\hat{n}+1)\pi}{2n} \le x_k \le \min\{t_1,\dots,t_n\}\,\mathrm{tg}\,\frac{(\hat{n}-1)\pi}{n}.$$

Corollary 4. *Let λ in equations* (5) *and* (6) *be ∞. Then we have the following two equations*

$$S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \dots + (-1)^s S_n^n = 0, \quad n \text{ is odd}, \tag{16}$$

$$x^n - S_2^n x^{n-2} + S_4^n x^{n-4} - \dots + (-1)^{s+1} S_n^n = 0, \quad n \text{ is even}, \tag{17}$$

and the angles are $(2k-1)\frac{\pi}{2}$, $k = 1, 2, \dots$ The situation is similar to the one in Corollary 3*.*

Before stating with the following theorem let us remark that the angle φ will be as in *Theorem* 1, $\varphi = \mathrm{arctg}\,\lambda$, and the expressions

$$U_1^{(n)}(x),\ V_1^{(n)}(x),\ U_2^{(n)}(x),\ V_2^{(n)}(x)$$

will be as follows

$$\begin{aligned}
U_1^{(n)}(x) &= x^n - S_2^n x^{n-2} + S_4^n x^{n-4} - \dots + (-1)^s S_{n-1}^n x, \quad n \text{ is odd}\\
V_1^{(n)}(x) &= S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \dots + (-1)^s S_n^n, \quad n \text{ is odd}\\
U_2^{(n)}(x) &= S_1^n x^{n-1} - S_3^n x^{n-3} + S_5^n x^{n-5} - \dots + (-1)^s S_{n-1}^n, \quad n \text{ is even}\\
V_2^{(n)}(x) &= -x^n + s_2^n x^{n-2} - S_4^n x^{n-4} + \dots + (-1)^s S_n^n
\end{aligned}$$

Thus equations (5) and (6) can be written as

$$U_1^{(n)}(x) - \lambda V_1^{(n)}(x) = 0, \tag{18}$$

$$U_2^{(n)}(x) - \lambda V_2^{(n)}(x) = 0. \tag{19}$$

Theorem 2. *Let m, n, q be positive integers such that $mq = n$ and let $t_1, \dots, t_n$ be positive numbers such that*

$$t_{i+jm} = t_i, \quad i = 1, \dots, m, \quad j = 1, \dots, q-1. \tag{20}$$

Then depending on which of the following three possibilities occurs

$$\begin{gathered}
m \text{ is odd, } n \text{ is odd, } m|n\\
m \text{ is odd, } n \text{ is even, } m|n\\
m \text{ is even, } n \text{ is even, } m|n
\end{gathered}$$

one of the following three assertions holds

$$\left(U_1^{(m)}(x) - \tau V_1^{(m)}(x)\right) \ \Big| \ \left(U_1^{(n)}(x) - \lambda V_1^{(n)}(x)\right) \tag{21}$$

$$\left(U_1^{(m)}(x) - \tau V_1^{(m)}(x)\right) \ \Big| \ \left(U_2^{(n)}(x) - \lambda V_2^{(n)}(x)\right) \tag{22}$$

$$\left(U_2^{(m)}(x) - \tau V_2^{(m)}(x)\right) \ \Big| \ \left(U_2^{(n)}(x) - \lambda V_2^{(n)}(x)\right) \tag{23}$$

where $\tau = \operatorname{tg}\frac{\varphi}{q}$, *and* $|$ *is a symbol for divides.*

Of course, in the expressions $U_1^{(m)}(x)$, $V_1^{(m)}(x)$, $U_2^{(m)}(x)$, $V_2^{(m)}(x)$ *stand* m *instead of* n. *So, for example*

$$U_1^{(m)}(x) = x^m - S_2^m x^{m-2} + S_4^m x^{m-4} - \cdots + (-1)^s S_{m-1}^m x,$$

where $S_j^m = S_j(t_1, \ldots, t_m)$, $j = 2, 4, \ldots, m-1$.

Proof. From

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = \varphi + k\pi, \quad k = 0, 1, \ldots, n-1$$

$$\begin{aligned} t_1 \operatorname{tg}\beta_1^{(k)} &= \cdots = t_m \operatorname{tg}\beta_m^{(k)} \\ = t_1 \operatorname{tg}\beta_{m+1}^{(k)} &= \cdots = t_m \operatorname{tg}\beta_{2m}^{(k)} \\ \vdots \quad & \ddots \quad \vdots \\ = t_1 \operatorname{tg}\beta_{(q-1)m}^{(k)} &= \cdots = t_m \operatorname{tg}\beta_{qm}^{(k)} = x_k \end{aligned}$$

it follows that

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = q\left(\beta_1^{(k)} + \cdots + \beta_m^{(k)}\right).$$

Accordingly

$$\varphi + k\pi = q\left(\frac{\varphi}{q} + \frac{k}{q}\pi\right), \quad k = 0, q, 2q, \ldots, (m-1)q$$

that is

$$\beta_1^{(k)} + \cdots + \beta_n^{(k)} = \frac{\varphi}{q} + \frac{k}{q}\pi, \quad k = 0, q, 2q, \ldots, (m-1)q.$$

Thus *Theorem* 2 is proved. □

Before stating with some corollaries from it, here is an example.

Let $n = 6$, $t_1 = t_4 = 1$, $t_2 = t_5 = 2$, $t_3 = t_6 = 3$, $\varphi = \frac{\pi}{3}$. Thus $m = 3$, $q = 2$, $\lambda = \sqrt{3}$, $\tau = \operatorname{tg}\frac{\pi}{6} = \frac{\sqrt{3}}{3}$, and

$$U_1^{(3)}(x) - \frac{\sqrt{3}}{3}V_1^{(3)}(x) = x^3 - 2\sqrt{3}x^2 - 11x + 2\sqrt{3},$$

$$U_2^{(6)}(x) - \sqrt{3}V_2^{(6)}(x) = \sqrt{3}x^6 + 12x^5 - 58\sqrt{3}x^4 - 144x^3 + 193\sqrt{3}x^2 + 132x - 36\sqrt{3},$$

$$\left(U_2^{(6)}(x) - \sqrt{3}V_2^{(6)}(x)\right) : \left(U_1^{(3)}(x) - \frac{\sqrt{3}}{3}V_1^{(3)}(x)\right) = \sqrt{3}x^3 + 18x^2 - 11\sqrt{3}x - 18.$$

Corollary 5. *Let* $\lambda = 0$. *If* (20) *is fulfiled, then*

$$\begin{aligned} &U_1^{(m)}(x) \,|\, U_1^{(n)}(x), && \textit{when } m \textit{ is odd, } n \textit{ is odd} \\ &U_1^{(m)}(x) \,|\, U_2^{(n)}(x), && \textit{when } m \textit{ is odd, } n \textit{ is even} \\ &U_2^{(m)}(x) \,|\, U_2^{(n)}(x), && \textit{when } m \textit{ is even, } n \textit{ is even.} \end{aligned}$$

Corollary 6. *Let* $\lambda = \infty$. *If* (20) *is fulfiled, then*

$$\begin{aligned} &V_1^{(m)}(x) \mid V_1^{(n)}(x), && \text{when } m \text{ is odd, } n \text{ is odd} \\ &V_1^{(m)}(x) \mid V_2^{(n)}(x), && \text{when } m \text{ is odd, } n \text{ is even} \\ &V_2^{(m)}(x) \mid V_2^{(n)}(x), && \text{when } m \text{ is even, } n \text{ is even.} \end{aligned}$$

Corollary 7. *Let condition* (20) *in* Theorem 2 *be replaced by*

$$\begin{aligned} S_j(t_1,\dots,t_m) &= S_j(t_{m+1},\dots,t_{2m}) = \dots \\ &= S_j(t_{(q-1)m},\dots,t_{qm}), \quad j = 1,\dots,m. \end{aligned} \tag{24}$$

Then (21)*,* (22) *and* (23) *hold, too. Also* Corollary 5 *and* Corollary 6 *hold, too.*

Proof. It is easy to see that each $S_j(t_1,\dots,t_n)$, $j = 1,\dots,n$ can be expressed as a sum of the products such that each factor is of the form

$$S_i(t_{1+k},\dots,t_{m+k}),$$

where $i \in \{1,\dots,m\}$, $k \in \{0,1,\dots,(q-1)m-1\}$. So, for example, if $n = 12$, $m = 2$, $j = 3$, then

$$\begin{aligned} S_3(t_1,\dots,t_{12}) = {} & S_3(t_1,\dots,t_6) + S_3(t_7,\dots,t_{12}) \\ & + S_1(t_1,\dots,t_6)S_2(t_7,\dots,t_{12}) \\ & + S_1(t_7,\dots,t_{12})S_2(t_1,\dots,t_6). \end{aligned}$$

Thus the essentiall in the expressions $U_1^{(m)}(x),\dots,V_2^{(n)}(x)$ remains unchanged. □

Example. Let $n = 6$, $t_1 = 1$, $t_2 = 3$, $t_3 = \frac{16}{5}$, $t_4 = 2$, $t_5 = 6$, $t_6 = \frac{6}{5}$. Then

$$t_1 + t_2 + t_3 = t_4 + t_5 + t_6, \quad t_1t_2t_3 = t_4t_5t_6.$$

If $\varphi = \pi$, then $\lambda = 0$, $\tau = \operatorname{tg}\frac{\pi}{2} = \infty$, and we have

$$U_2^{(6)}(x) = 14.4x^4 - 242.4x^2 + 297.6,$$

$$V_1^{(3)}(x) = 7.2x^2 - 9.6$$

$$U_2^{(6)}(x) : V_1^{(3)}(x) = 2x^2 - 31.$$

3. Some properties of tangential semi-polygons

An essential characteristic of a tangential semi-polygon expresses the following theorem.

Theorem 3. *Let $t_1, \dots, t_n$ be any given lengths (in fact positive numbers) and let λ be any given real number or either ∞ or $-\infty$. Further, let $\beta_1, \dots, \beta_n$ be angles such that*

$$0 < \beta_i < \frac{\pi}{2}, \quad i = 1, \dots, n$$
$$\operatorname{tg}(\beta_1 + \cdots + \beta_n) = \lambda. \tag{25}$$

If v denotes the number of all tangential semi-polygons whose tangents have the lengths $t_1, \dots, t_n$ and the angles $\beta_1, \dots, \beta_n$ satisfy (25)*, then the following assertions hold:*

1) If $\lambda > 0$ or $\lambda = \infty$, then $v = \frac{n+1}{2}$ if n is odd, and $v = \frac{n}{2}$ if n is even.

2) If $\lambda = 0$, then $v = \frac{\hat{n}-1}{2}$.

Analogously in the case when $\lambda < 0$ or $\lambda = -\infty$.

Proof. Follows from *Theorem* 1 and *Theorem* 2 and theirs corollaries. □

Example. Let $n = 6$, $t_1 = \cdots = t_6 = 1$, $\lambda = \infty$. Then we have the equation

$$x^6 - 15x^4 + 15x^2 - 1 = 0$$

whose positive roots are

$$x_1 = \operatorname{tg}\frac{\pi}{6} = 0,267949192$$
$$x_2 = \operatorname{tg}\frac{\pi}{4} = 1$$
$$x_3 = \operatorname{tg}\frac{5\pi}{12} = 3.732050808$$

and these are the radii of the corresponding tangential semi-polygons. The first polygon "lie" on five semicircles, the second one on three, and the third one on one. (The first is showen in figure below. Its end-vertices are denoted by 1 and 8.)

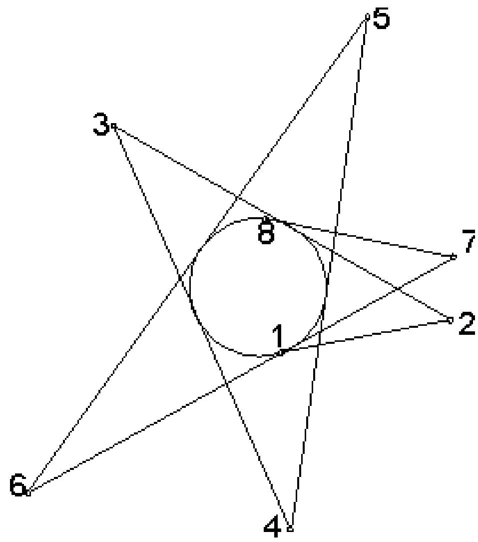

Figure 1.

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