Stability and optimality in parametric convex programming models*

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Abstract. *Equivalent conditions for structural stability are given for convex programming models in terms of three point-to-set mappings. These mappings are then used to characterize locally optimal parameters. For Lagrange models and, in particular, LFS models the characterizations are given relative to general (possibly unstable) perturbations.*

Key words: *stable convex model, unstable convex model, optimal parameter, Lagrange multiplier function, LFS function*

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1. Introduction

"Stability" and "optimality" are the fundamental notions in optimization and applied mathematics. Although not uniquely defined, they are often expressed in terms of each other. For example "structural optima" is defined for "stable" perturbations [21], while "stable" models require that the set of optimal solutions be non-empty and bounded [18, 19, 20]. This paper studies both stability and optimality in convex models using three point-to-set mappings.

Stability is essentially defined as lower semi-continuity of the feasible set mapping. New equivalent conditions for stability are established here using two other point-to-set mappings. The conditions are useful, in particular, when one wants to know whether the feasible set mapping is continuous for a specific type of perturbations of specific coefficients. Let us recall that stability, relative to arbitrary

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perturbations of *all* coefficients in the system is equivalent to the existence of a Slater point, which is also equivalent to the notions of "regularity" [14, 15] and "C-stability", as recently shown in [17]. These results for linear systems were given in [8]. For construction of stable perturbations in perturbed linear systems and applications see [24].

The basic optimality problem in the study of convex parametric models (henceforth abbreviated: models) is to describe, and possibly calculate, the parameters that optimize the optimal value function. These parameters have been characterized for stable perturbations in [18, 19, 20]. Under "input constraint qualifications" the necessary conditions for locally optimal parameter are simplified [20]. Some of the optimality conditions have been recently extended to abstract settings in [1]. This extension has many potential applications from dynamic formulations of management and operations research problems to control theory [4, 7]. Our contribution to the topic of optimality is a characterization of locally optimal parameters in the absence of stability. We show that such characterization is possible for "Lagrange models". Such models include all convex models for which the constraints have "locally flat surfaces" (the so-called LFS constraints, e.g., [11, 12]) and, in particular, all linear models. It is important to observe that characterizations of locally optimal parameters are different for stable and unstable perturbations.

In view of the Liu-Floudas transformation [10], every mathematical program with twice continuously differentiable functions can be transformed into a partly convex program and these can be considered and studied as convex models, e.g., [22, 23, 25]. This means that our results are applicable to the general optimization problems.

2. Stability

Consider the system of perturbed constraints

$$\begin{aligned} f^i(x,\theta) &\le 0, \qquad i \in P = \{1, \dots, q\}, \\ A(\theta)x &= b(\theta), \\ x &\ge 0 \end{aligned} \tag{C,θ}$$

where $f^i : \mathbb{R}^{n+p} \longrightarrow \mathbb{R}$ are continuous functions defined (with finite values) on the entire $\mathbb{R}^{n+p}$, $i \in P$, $A(\theta)$ is an $m \times n$ matrix and $b(\theta)$ is an m-tuple, both with elements that are continuous functions of $\theta \in \mathbb{R}^p$. We assume that $f^i(\cdot,\theta) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i \in P$, are convex functions for every θ. The vector θ is referred to as the parameter and $x \in \mathbb{R}^n$ is the decision variable. As θ varies so does the set of feasible decisions

$$\mathrm{F} : \theta \longrightarrow \mathrm{F}(\theta) = \{x \in \mathbb{R}^n : f^i(x,\theta) \le 0,\ i \in P,\ A(\theta)x = b(\theta),\ x \ge 0\}.$$

The changes are described by the point-to-set mapping $\mathrm{F} : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ defined as $\mathrm{F} : \theta \longrightarrow \mathrm{F}(\theta)$. With every point (vector) θ, the mapping F associates the feasible set $\mathrm{F}(\theta)$. The set of all θ's for which $\mathrm{F}(\theta)$ is not empty is called the feasible set of parameters and it is denoted by $\mathsf{F} = \{\theta : \mathrm{F}(\theta) \ne \emptyset\}$. As $\theta \in \mathsf{F}$ varies so do many other objects that can be described by point-to-set mappings. These include the *index* mappings

$$P^{=} : \theta \longrightarrow P^{=}(\theta) = \{i \in P \ : \ x \in \mathrm{F}(\theta) \Rightarrow f^i(x,\theta) = 0\}$$

and

$$N^{=} : \theta \longrightarrow N^{=}(\theta) = \{i \in N \ : \ x \in \mathrm{F}(\theta) \Rightarrow x_i = 0\}$$

where $N = \{1, \dots, n\}$. By analogy with the terminology used in [20, 24], for every θ, we call these indices, respectively, the minimal index set of inequality constraints and the minimal index set of variables. We will also use $\mathrm{F}^{=} : \theta \longrightarrow \mathrm{F}^{=}(\theta)$ where

$$\mathrm{F}^{=}(\theta) = \{x : f^i(x,\theta) = 0, \ i \in P^{=}(\theta), \ A(\theta)x = b(\theta), \ x_j = 0, \ j \in N^{=}(\theta)\}.$$

This mapping is used in characterizations of optimal variables, e.g., [3, 18, 25]. Stability of a system of constraints and optimality of parameters will be studied around an arbitrary but fixed $\theta^\star \in \mathsf{F}$. For this purpose we also need $\mathrm{F}^{=}_\star : \theta \longrightarrow \mathrm{F}^{=}_\star(\theta)$ where

$$\mathrm{F}^{=}_\star(\theta) = \{x : f^i(x,\theta) \le 0, \ i \in P^{=}(\theta^\star), \ A(\theta)x = b(\theta), \ x_j \ge 0, \ j \in N^{=}(\theta^\star)\}.$$

This mapping is also important in the study of Lagrange multiplier functions, e.g., [18, 25]. Let us recall how lower semi-continuity of a general point-to-set mapping $\Gamma : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ is defined (see, e.g., [2, 5, 9]): The mapping Γ is lower semi-continuous at $\theta^\star \in \mathbb{R}^p$ if for each open set $A \subset \mathbb{R}^n$, satisfying $A \cap \Gamma(\theta^\star) \neq \emptyset$, there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ such that $A \cap \Gamma(\theta) \neq \emptyset$ for each $\theta \in N(\theta^\star)$.

Definition 1. *The perturbed convex system (C,θ) is said to be structurally stable at* $\theta^\star \in \mathsf{F}$ *if the point-to-set mapping* $\mathrm{F} : \theta \longrightarrow \mathrm{F}(\theta)$ *is lower semi-continuous at* $\theta^\star$.

Remark 1. *It is well known that* Γ *is lower semi-continuous if, and only if, it is open. Since the mapping* F *is closed,* F *is continuous if, and only if, it is lower semi-continuous.*

In mathematical programming models describing real-life situations not necessarily all coefficients are perturbed. Typically, one wants to study the effect of some specific perturbations on the system. For example, one may be interested in how changes of certain technological coefficients in linear programs influence the system. In this case $\theta = \theta(t)$ can be specified to be one or more elements such as $\theta = (a_{1,2}, a_{4,3}, b_2)$. It is also meaningful to talk only about perturbations that preserve lower semi-continuity of the feasible set mapping. In this case we may restrict the notion of lower semi-continuity and structural stability to only a part of the neighborhood. Following [19, 20] we say that the mapping is lower semi-continuous at $\theta^\star \in \mathbb{R}^p$, *relative* to a set S that contains $\theta^\star$, if for each open set $A \subset \mathbb{R}^n$, satisfying $A \cap \Gamma(\theta^\star) \neq \emptyset$, there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ such that $A \cap \Gamma(\theta) \neq \emptyset$ for each $\theta \in N(\theta^\star) \cap S$. We also say that the system (C,θ) is structurally stable at $\theta^\star \in \mathsf{F}$, relative to a set S that contains $\theta^\star$, if $\mathrm{F} : \theta \longrightarrow \mathrm{F}(\theta)$ is lower semi-continuous at $\theta^\star$ relative to S. Such S is called a region of stability at $\theta^\star$, e.g., [18, 25]. For the sake of simplicity we assume, throughout the paper, that S is a connected set.

A useful necessary condition for structural stability follows. (Throughout the paper, the inclusion sign "$\subset$" includes "$=$", i.e., the inclusion may not be strict.)

Lemma 1. *If the perturbed convex system (C,θ) is structurally stable at $\theta^\star \in \mathsf{F}$, relative to a subset S of F that contains $\theta^\star$, then there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ such that $P^=(\theta) \subset P^=(\theta^\star)$ and $N^=(\theta) \subset N^=(\theta^\star)$ for every $\theta \in N(\theta^\star) \cap S$.*

Proof. It is an easy adaptation of the proof given in [20] for convex inequality constraints and the one from [24] for linear constraints. □

Let us begin the study of the system (C,θ) by considering the mappings F and $\mathrm{F}^=$. The behavior of these mappings is essentially different. There are examples showing that one may be lower semi-continuous but not the other, e.g., [25]. Still, it is easy to show that the necessary condition for lower semi-continuity of F, given in *Lemma 1*, is also a necessary condition for lower semi-continuity of $\mathrm{F}^=$.

Let us consider perturbations around $\theta^\star$ that locally belong to the set

$$R(\theta^\star) = \{\theta \in \mathsf{F} \ : \mathrm{F}^=(\theta) \subset \mathrm{F}^=_\star(\theta)\}.$$

We have the following new result:

Lemma 2. *Consider the perturbed convex system (C,θ) around some $\theta^\star \in \mathsf{F}$. If* F *or* $\mathrm{F}^=$ *is lower semi-continuous at $\theta^\star$ relative to $R(\theta^\star)$ then* $\mathrm{F}^=$ *coincides with* $\mathrm{F}^=_\star$ *on $N(\theta^\star) \cap R(\theta^\star)$, where $N(\theta^\star)$ is some neighborhood of $\theta^\star$.*

Proof. Using *Lemma 1*, we know that lower semi-continuity of F or $\mathrm{F}^=$ implies the existence of a neighborhood $N(\theta^\star)$ such that $P^=(\theta) \subset P^=(\theta^\star)$ and $N^=(\theta) \subset N^=(\theta^\star)$ for every $\theta \in N(\theta^\star) \cap R(\theta^\star)$. Hence $\mathrm{F}^=_\star(\theta) \subset \mathrm{F}^=(\theta)$ for such θ. The reverse inclusion is true by definition of $R(\theta^\star)$. Hence $\mathrm{F}^=$ and $\mathrm{F}^=_\star$ coincide on $R(\theta^\star)$ close to $\theta^\star$. □

For perturbations in the set $R(\theta^\star)$ we also have the following result:

Theorem 1. *Consider the perturbed convex system (C,θ) around some $\theta^\star \in \mathsf{F}$. If $\mathrm{F}^= : \theta \longrightarrow \mathrm{F}^=(\theta)$ is lower semi-continuous at $\theta^\star$ relative to $R(\theta^\star)$, then so is $\mathrm{F} : \theta \longrightarrow \mathrm{F}(\theta)$.*

Proof. Using *Lemma 2*, we know that $\mathrm{F}^=$ and $\mathrm{F}^=_\star$ coincide on $R(\theta^\star)$ close to $\theta^\star$. So $\mathrm{F}^=_\star$ is also lower semi-continuous and this implies lower semi-continuity of F; e.g., [2, 20]. □

If F is lower semi-continuous, then so is $\mathrm{F}^=$, provided that certain inequality constraints are "faithfully convex" in the variable x. A function $f^i(\cdot,\theta) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i \in P$, is said to be faithfully convex at θ if one can represent it in the form

$$f^i(x,\theta) = \varphi^i(B^i(\theta)x + c^i(\theta),\theta) + [d^i(\theta)]^{\mathrm{T}}x + \alpha^i(\theta)$$

where $\varphi^i(\cdot,\theta)$ is strictly convex, $B^i(\theta)$, $c^i(\theta)$, $d^i(\theta)$ and $\alpha^i(\theta)$ are, respectfully, matrices, vectors and scalars of appropriate dimensions. Faithfully convex functions have been extensively studied in the literature, e.g., [3, 16], but not in the context of stability.

Definition 2. *The perturbed convex system (C,θ) has $P^=$-faithfully convex constraints at θ if the constraints $f^i(\cdot,\theta)$, $i \in P^=(\theta)$, are faithfully convex at θ.*

A "technical" lemma follows. "Null(A)" denotes the null-space of a matrix A.

Lemma 3. *Consider the perturbed convex system (C,θ) with $P^{=}$-faithfully convex constraints at some $\theta \in \mathsf{F}$. If $x \in \mathrm{F}^{=}(\theta)$ and $y \in \mathrm{F}(\theta)$, then*

$$x - y \in \mathrm{Null}\left(\bigcap_{i \in P^{=}(\theta)} \begin{bmatrix} B^i(\theta) \\ (d^i(\theta))^T \end{bmatrix} \right).$$

Proof. Since $\mathrm{F}(\theta) \subset \mathrm{F}^{=}(\theta)$ and $\mathrm{F}^{=}(\theta)$ is a convex set, it follows that $x + \alpha(y - x) \in \mathrm{F}^{=}(\theta)$ for all $\alpha \geq 0$ sufficiently small. Hence $f^i(x+\alpha(y-x),\theta) = 0,\ i \in P^{=}(\theta)$. This means that $y - x$ is in the cone of directions of constancy of $f^i(\cdot,\theta)$ at x (e.g., [3, 25]). But the latter, for faithfully convex functions, is the null-space of the matrix $[[B^i(\theta)]^{\mathrm{T}},\ d^i(\theta)]^{\mathrm{T}}$, e.g., [3]. (An alternative proof of this lemma is given in [17].) ☐

The following result was proved for linear systems in [6]. We prove it here for $P^{=}$-faithfully convex constraints.

Theorem 2. *Consider the perturbed convex system (C,θ) around some $\theta^\star \in \mathsf{F}$ and some subset S of F that contains $\theta^\star$. Assume that the system has $P^{=}$-faithfully convex constraints at every $\theta \in S$. If F is lower semi-continuous at $\theta^\star$ relative to S, then so is $\mathrm{F}^{=}$.*

Proof. Take an $x^\star \in \mathrm{F}^{=}(\theta^\star)$ and a sequence $\theta^k \in S$. There exists a $y \in \mathrm{F}(\theta^\star)$ such that

$$\begin{aligned} f^i(y,\theta^\star) &< 0, \quad i \in P \setminus P^{=}(\theta^\star), \\ A(\theta^\star)y &= b(\theta^\star), \\ y_i &> 0, \quad i \in N \setminus N^{=}(\theta^\star). \end{aligned}$$

Hence there exists $0 < \epsilon < 1$ such that $z = y + \epsilon(x^\star - y)$ satisfies

$$\begin{aligned} f^i(z,\theta^\star) &< 0, \quad i \in P \setminus P^{=}(\theta^\star), \\ A(\theta^\star)z &= b(\theta^\star), \text{ because } A(\theta^\star)x^\star = b(\theta^\star), \\ z_i &> 0, \quad i \in N \setminus N^{=}(\theta^\star). \end{aligned}$$

Also, by convexity, we have

$$\begin{aligned} f^i(z,\theta^\star) &\leq (1-\epsilon)f^i(y,\theta^\star) + \epsilon f^i(x^\star,\theta^\star) \leq 0, \quad i \in P^{=}(\theta^\star), \\ z_i &= (1-\epsilon)y_i + \epsilon x_i^\star \geq 0, \quad i \in N^{=}(\theta^\star). \end{aligned}$$

This means that $z \in \mathrm{F}(\theta^\star)$. Since F is lower semi-continuous, there exist sequences $y^k \in \mathrm{F}(\theta^k)$, $y^k \longrightarrow y$ and $z^k \in \mathrm{F}(\theta^k)$, $z^k \longrightarrow z$. For the same ϵ construct the new sequence $x^k = y^k + \frac{(z^k - y^k)}{\epsilon}$. Clearly $x^k \longrightarrow x^\star$. In order to complete the proof we have to show that $x^k \in \mathrm{F}^{=}(\theta^k)$ for large k's. This is true because, for every

$i \in P^=(\theta^k)$, we have

$$\begin{aligned} f^i(x^k,\theta^k) &= \varphi^i(B^i(\theta^k)[y^k + \frac{(z^k - y^k)}{\epsilon}] + c^i(\theta^k),\theta^k) + [d^i(\theta^k)]^{\mathrm{T}}[y^k + \frac{(z^k - y^k)}{\epsilon}] \\ &\quad + \alpha^i(\theta^k) \\ &= \varphi^i(B^i(\theta^k)y^k + c^i(\theta^k),\theta^k) + [d^i(\theta^k)]^{\mathrm{T}}y^k + \alpha^i(\theta^k), \text{ by Lemma 3,} \\ &= f^i(y^k,\theta^k), \text{ by definition,} \\ &= 0, \text{ because } y^k \in \mathrm{F}(\theta^k) \subset \mathrm{F}^=(\theta^k). \end{aligned}$$

Also

$$A(\theta^k)x^k = b(\theta^k), \text{ since } A(\theta^k)y^k = A(\theta^k)z^k = b(\theta^k).$$

Finally, since F is lower semi-continuous, for every large k, we have $N^=(\theta^k) \subset N^=(\theta^\star)$ and hence $y_i^k = z_i^k = 0,\ i \in N^=(\theta^k)$. □

The faithful convexity is a crucial assumption in *Theorem 2*. Without it the theorem does not hold. This is confirmed by the next example.

Example 1. *Consider the system*

$$f(x,\theta) \le 0,\ 0 \le x \le 1$$

where $f(x,\theta) = \theta^2(|x| - 1)$, *if* $|x| > 1$, *and 0 otherwise. Here* $\mathrm{F}(\theta)$ *is the same for every* θ *and hence* F *is lower semi-continuous at* $\theta^\star = 0$. *But* $\mathrm{F}^=(\theta) = \mathbb{R}$ *for* $\theta = \theta^\star$, *and* $-1 \le x \le 1$ *otherwise. Hence* $\mathrm{F}^=$ *is not lower semi-continuous. Note that the constraint* f *is not faithfully convex at* $\theta \ne \theta^\star$. □

Let us now include the mapping $\mathrm{F}^=_\star$ in our study. Lower semi-continuity of this mapping is required in many statements related to input constraint qualifications, optimality conditions, and Lagrange multiplier functions; e.g., [18, 19, 20]. It is used to characterize structurally stable convex and linear systems; e.g., [18, 20]. (See also *Theorem 6* below.) The mappings $\mathrm{F}^=$ and $\mathrm{F}^=_\star$ coincide for the system introduced in *Example 1* but generally these mappings are different. The assumptions of *Theorem 2* alone do not guarantee that $\mathrm{F}^=_\star$ is lower semi-continuous. However, this is the case if perturbations are restricted to the set $R(\theta^\star)$:

Theorem 3. *Consider the perturbed convex system (C,θ) around some* $\theta^\star \in \mathsf{F}$. *Assume that the system has* $P^=$*-faithfully convex constraints at every* $\theta \in R(\theta^\star)$. *If* F *is lower semi-continuous at* $\theta^\star$ *relative to* $R(\theta^\star)$, *then so is* $\mathrm{F}^=_\star$.

Proof. Again, using *Lemma 2*, we have that $\mathrm{F}^=$ and $\mathrm{F}^=_\star$ coincide on $R(\theta^\star)$ close to $\theta^\star$. The proof now follows from *Theorem 2*. □

Finally, let us now focus on the mappings $\mathrm{F}^=_\star$ and $\mathrm{F}^=$. If the former is lower semi-continuous then so is the latter:

Theorem 4. *Consider the perturbed convex system (C,θ) around some* $\theta^\star \in \mathsf{F}$. *If* $\mathrm{F}^=_\star$ *is lower semi-continuous at* $\theta^\star$ *then so is* $\mathrm{F}^=$.

Proof. Take an open set A such that $A \cap \mathrm{F}^=(\theta^\star) \ne \emptyset$. Since $\mathrm{F}^=(\theta^\star) = \mathrm{F}^=_\star(\theta^\star)$, also $A \cap \mathrm{F}^=_\star(\theta^\star) \ne \emptyset$. But lower semi-continuity of $\mathrm{F}^=_\star$ implies that so is F; e.g., [2, 20]. Hence $P^=(\theta) \subset P^=(\theta^\star)$ and $N^=(\theta) \subset N^=(\theta^\star)$ for every $\theta \in N(\theta^\star)$, by *Lemma 1*. This further implies $\mathrm{F}^=_\star(\theta) \subset \mathrm{F}^=(\theta)$ and hence $A \cap \mathrm{F}^=(\theta) \ne \emptyset$ for every $\theta \in N(\theta^\star)$. □

The lower semi-continuity implications between the three mappings, proved in this section, are depicted in *Figure 1.* They are stated at an arbitrary but fixed $\theta^\star \in \mathsf{F}$. The signs, if any, along an arrow denote the assumptions that are needed in the proof of the implications. Thus "$\mathbf{fc}^=$" stands for the assumption that the system (C,θ) has $P^=$-faithfully convex constraints at every perturbation θ, while "**R**" means that the implication holds relative to the set $R(\theta^\star)$. Since lower semi-continuity is a local property, only perturbations in some neighborhood of $\theta^\star$ are considered.

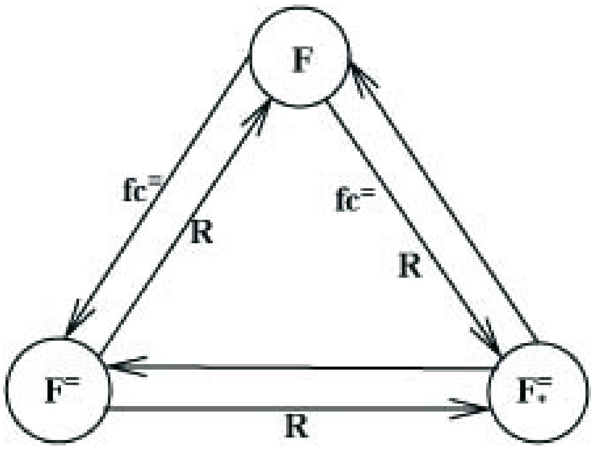

Figure 1. *Lower semi-continuity implications*

3. Optimality

So far, we have studied perturbed systems of constraints. Now we include an objective function in the study. Consider the convex parametric programming model

$$\begin{aligned} &\min_{(x)} \ f(x,\theta) \\ &s.t. \end{aligned} \qquad \text{(P,}\theta\text{)}$$

$$f^i(x,\theta) \le 0, \quad i \in P = \{1,\ldots,q\}.$$

For the sake of simplicity we only consider inequality constraints in this section. The objective function f and all the constraints $f^i : \mathbb{R}^{n+p} \longrightarrow \mathbb{R}$ are assumed to be continuous functions defined (with finite values) on the entire $\mathbb{R}^{n+p}$, $i \in P$. It is assumed that $f(\cdot,\theta), f^i(\cdot,\theta) : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i \in P$, are convex functions for every θ. The notation related to the new formulation is easily adjusted. For instance, given θ, the feasible set is $\mathrm{F}(\theta) = \{x \in \mathbb{R}^n : f^i(x,\theta) \le 0,\ i \in P\}$. Notation for the feasible set of parameters F and for the minimal index set of active constraints $P^=(\theta)$ remains the same, while $\mathrm{F}^=(\theta) = \{x \in \mathbb{R}^n : f^i(x,\theta) = 0,\ i \in P^=(\theta)\}$ and $\mathrm{F}^=_\star(\theta) = \{x \in \mathbb{R}^n : f^i(x,\theta) \le 0,\ i \in P^=(\theta^\star)\}$ around an arbitrary but fixed $\theta^\star \in \mathsf{F}$. A new point-to-set player is the mapping

$$\mathrm{F}^{\mathrm{o}} : \theta \longrightarrow \mathrm{F}^{\mathrm{o}}(\theta) = \{x^{\mathrm{o}}(\theta)\}$$

where $x^{\circ}(\theta)$ denotes an optimal solution of the convex (fixed θ) program (P,θ). (This mapping associates, with every θ, the set of all optimal solutions.) With the introduction of the objective function we also have the optimal value function $f^{\circ}(\theta) = f(x^{\circ}(\theta), \theta)$. The main objective of this section is to characterize parameters θ that locally minimize the function $f^{\circ}(\theta)$:

Definition 3. *Consider the convex model (P,θ). A parameter $\theta^{\star} \in \mathsf{F}$ is a locally optimal parameter if $f^{o}(\theta^{\star}) \leq f^{o}(\theta)$ for every $\theta \in N(\theta^{\star}) \cap \mathsf{F}$, where $N(\theta^{\star})$ is some neighborhood of $\theta^{\star}$.*

Locally optimal parameters can be characterized using the Lagrangian

$$\mathsf{L}(x, u; \theta) = f(x, \theta) + \sum_{i \in P} u_i f^i(x, \theta).$$

Given $\theta \in \mathsf{F}$ we say that $(x^{\star}, u^{\star}),\ x^{\star} \in \mathbb{R}^n,\ u^{\star} \in \mathbb{R}^q,\ u^{\star} \geq 0$, is a saddle point of the Lagrangian if

$$\mathsf{L}(x^{\star}, u; \theta) \leq \mathsf{L}(x^{\star}, u^{\star}; \theta) \leq \mathsf{L}(x, u^{\star}; \theta)$$

for every $x \in \mathbb{R}^n$ and every $u = (u_i),\ u_i \geq 0,\ i \in P$. It is well known that the set of all multipliers $u^{\star} = u^{\circ}(\theta)$ is the same at every optimal solution $x^{\star} = x^{\circ}(\theta)$. Hence we can study the point-to-set mapping

$$\mathcal{L} : \theta \longrightarrow \mathcal{L}(\theta) = \{u^{\circ}(\theta)\}.$$

Definition 4. *Consider the convex model (P,θ) and some subset S of F. If at every $\theta \in S$ there exists an optimal solution $x^{\circ}(\theta)$ and if $\mathcal{L}(\theta) \neq \emptyset$, then the model is said to be a Lagrange model on S.*

There are convex models that are not Lagrange at any $\theta \in \mathsf{F}$. The next example illustrates a model that is a Lagrange model at only one parameter.

Example 2. *Consider*

$$\min_{(x_1,x_2,x_3)} \ x_1 - \theta x_3 \quad \text{s.t.} \qquad \text{(NL,}\theta\text{)}$$

$$\begin{aligned} (x_1 - 1)^2 + (x_2 - 1)^2 - 1 &\leq 0, \\ (x_1 - 1)^2 + (x_2 + 1)^2 - 1 &\leq 0, \\ -(1 + \theta^2)x_1 + x_3 &\leq 0, \\ -x_3 &\leq 0. \end{aligned}$$

One can show that this is a Lagrange model only at the root of $\theta^3 + \theta = 1$ (call it $\theta^{\star}$). The optimal value function here is $f^{o}(\theta) = 1 - \theta^3 - \theta$, for $\theta \geq 0$, and 1 otherwise. The feasible set mapping is lower semi-continuous around $\theta^{\star}$. Indeed, one can find that $\mathrm{F}(\theta) = \{(1, 0, x_3)^T \in \mathbb{R}^3 : 0 \leq x_3 \leq 1 + \theta^2\}$. Later we will check, using an optimality condition, that $\theta^{\star}$ is an optimal parameter relative to the set $(-\infty, \theta^{\star}]$.$\square$

For Lagrange models one can characterize locally optimal parameters without requiring any stability assumption.

Theorem 5. *(Characterization of Local Optimality for Lagrange Models Relative to Arbitrary Perturbations.) Consider the convex model (P,θ). Assume that*

(P,θ) is a Lagrange model on some nonempty set $S \subset \mathbb{R}^p$. Take $\theta^\star \in S$ and let $x^\star$ be an optimal solution of the program (P,$\theta^\star$). Then $\theta^\star$ is a locally optimal parameter of f^o relative to S if, and only if, there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ such that, for every θ in $N(\theta^\star) \cap S$, there exists an element $u^\circ(\theta)$ in $\mathcal{L}(\theta)$ such that

$$\mathsf{L}(x^\star, u; \theta^\star) \le \mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) \le \mathsf{L}(x, u^\circ(\theta); \theta) \tag{LSP}$$

for all x in $\mathbb{R}^n$ and all $u = (u_i)$, $u_i \ge 0$, $i \in P$.

Proof. Let us first prove the necessity part first. We know that $\theta^\star$ is a locally optimal parameter of $f^{\rm o}$ relative to S, so there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ such that

$$f^{\rm o}(\theta^\star) \le f^{\rm o}(\theta) \quad \text{for all } \theta \in N(\theta^\star) \cap S.$$

Since $\mathcal{L}(\theta^\star) \ne \emptyset$, we have

$$f^{\rm o}(\theta^\star) = f(x^\star, \theta^\star) = \mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) \quad \text{for all } u^\circ(\theta^\star) \in \mathcal{L}(\theta^\star).$$

So

$$\mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) = f^{\rm o}(\theta^\star) \le f^{\rm o}(\theta) \tag{3.1}$$

for all θ in $N(\theta^\star) \cap S$ and all $u^\circ(\theta^\star)$ in $\mathcal{L}(\theta^\star)$. Now, since (P,$\theta$) is a Lagrange model on $N(\theta^\star) \cap S$, we know that, for every $\theta \in N(\theta^\star) \cap S$, there exist an optimal solution $x^\circ(\theta) \in \mathrm{F}^{\rm o}(\theta)$ and an element $u^\circ(\theta)$ in $\mathcal{L}(\theta)$ such that

$$\begin{aligned} f^{\rm o}(\theta) &= f(x^\circ(\theta), \theta) \\ &= \mathsf{L}(x^\circ(\theta), u^\circ(\theta); \theta) \\ &\le \mathsf{L}(x, u^\circ(\theta); \theta) \qquad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

Thus, using (3.1), we conclude that

$$\mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) \le \mathsf{L}(x, u^\circ(\theta); \theta) \quad \text{for all } x \in \mathbb{R}^n.$$

The left-hand side of the "Lagrange saddle point inequality" (LSP) follows by the fact that $x^\star$ is a feasible point of (P,$\theta^\star$).
Now, let us prove the sufficiency part. We are going to prove that $\theta^\star$ is a locally optimal parameter of $f^{\rm o}$ on the set $N(\theta^\star) \cap S$. Take an arbitrary θ in $N(\theta^\star) \cap S$. Take a saddle point $u^\circ(\theta^\star)$ in $\mathcal{L}(\theta^\star)$ and use the left-hand side of (LSP) with $u = 0$ to get $\mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) \ge 0$. By feasibility of the point $x^\star$ we also have $\mathsf{L}(x^\star, u^\circ(\theta^\star); \theta^\star) \le 0$. Thus

$$\sum_{i \in P} u_i^\circ(\theta^\star) f^i(x^\star, \theta^\star) = 0$$

and consequently, using the right-hand side of (LSP), we get

$$f(x^\star, \theta^\star) \le f(x, \theta) + \sum_{i \in P} u_i^\circ(\theta) f^i(x, \theta)$$

for some $u^{\circ}(\theta)$ in $\mathcal{L}(\theta)$ and all x in $\mathbb{R}^n$. In particular

$$f(x^\star,\theta^\star) \le f(x,\theta) + \sum_{i\in P} u_i^{\circ}(\theta) f^i(x,\theta) \le f(x,\theta) \quad \text{for all } x \in \mathrm{F}(\theta).$$

Therefore $f(x^\star,\theta^\star) \le f(x,\theta)$ for all θ in $N(\theta^\star)\cap S$ and all x in $\mathrm{F}(\theta)$. □
The following example illustrates this characterization.

Example 3. *Consider*

$$\min_{(x_1,x_2,x_3)} \ \theta^2 x_1 - 4x_2 + |x_3|$$

$$s.t.$$

$$\begin{aligned} x_1 + (3+\theta^2)x_2 - x_3 &\le 0,\\ -2x_1 - 4x_2 + |x_1| + |x_2| + |x_3| &\le 0,\\ x_2 + x_1 - (1-\theta^2) &\le 0,\\ -x_1 &\le 0,\\ -x_2 &\le 0,\\ \theta^2 x_3 &\le 0. \end{aligned}$$

around $\theta^\star = 0$. We want to know whether $\theta^\star = 0$ is a local optimal parameter for the optimal value function. Here $\mathsf{F} = [-1,1]$ and the model is Lagrange at every $\theta \in [-1,1]$. So we can use Theorem 5 *to characterize optimality of $\theta^\star$. In this model one finds that $f^o(\theta^\star) = -1$ and $f^o(\theta) = 0$ for $\theta \ne \theta^\star$. The right-hand side of (LSP) reduces to*

$$-1 \le \begin{cases} -1 + |x_3| - x_3 + u_2^{\circ}(\theta)[|x_1| - x_1 + |x_2| - x_2 + |x_3| - x_3], & \textit{if } \theta = 0\\ |x_3| - r x_3, & \textit{otherwise.} \end{cases}$$

for all $(x_1,x_2,x_3) \in \mathbb{R}^3$, with the Lagrange multipliers

$$u_1^{\circ}(\theta) = \begin{cases} 1 + u_2^{\circ}(\theta), & \textit{if } \theta = 0\\ r + \theta^2 u_6^{\circ}(\theta), & \textit{otherwise.} \end{cases} \qquad u_2^{\circ}(\theta) = \begin{cases} \ge 0, & \textit{if } \theta = 0\\ 0, & \textit{otherwise.} \end{cases}$$

$$u_3^{\circ}(\theta) = \begin{cases} 1, & \textit{if } \theta = 0\\ 0, & \textit{otherwise.} \end{cases} \qquad u_4^{\circ}(\theta) = \begin{cases} 2, & \textit{if } \theta = 0,\\ r + \theta^2(1 + u_6^{\circ}(\theta)), & \textit{otherwise.} \end{cases}$$

$$u_5^{\circ}(\theta) = \begin{cases} 0, & \textit{if } \theta = 0\\ (3+\theta^2)[r + \theta^2 u_6^{\circ}(\theta)] - 4, & \textit{otherwise.} \end{cases}$$

$$u_6^{\circ}(\theta) = \begin{cases} \ge 0, & \textit{if } \theta = 0\\ \{x \in \mathbb{R} : x \ge \frac{1}{\theta^2}(\frac{4}{3+\theta^2} - r)\}, & \textit{otherwise.} \end{cases}$$

(Here r is an arbitrary number in $[-1,1]$.) This confirms local optimality of $\theta^\star = 0$. Note that the feasible set mapping is not lower semi-continuous at $\theta^\star$. Indeed, one can find that

$$\mathrm{F}(\theta) = \begin{cases} \{(x_1,x_2,x_3) : x_3 = x_1 + 3x_2,\ x_1 + x_2 \le 1,\ x_1, x_2 \ge 0\}, & \textit{if } \theta = 0\\ \{(0,0,0)\}, & \textit{otherwise.} \end{cases}$$

This means that the results from, e.g., [19, 20], are not applicable here.

How to check optimality for the models that are not Lagrange? In particular, *Theorem 5* cannot be applied to the convex model given in *Example 2* to check whether $\theta^\star$ is an optimal parameter relative to the set $(-\infty, \theta^\star]$. For this kind of models one could check optimality of the parameters using a result given in, e.g., [18, 19, 20]. That result can be applied to general convex models (not necessarily Lagrange), but it requires stable perturbations. Let us recall this characterization. We denote the cardinality of $P^<(\theta^\star) = P \setminus P^=(\theta^\star)$ by $c = \mathrm{card}(P^<(\theta^\star))$. Without loss of generality, we assume that $P^<(\theta^\star) = \{1, \ldots, c\}$. In this case, to test local optimality for the convex model (P,θ) on some set S containing $\theta^\star$, we can use the "reduced" Lagrangian function

$$\mathsf{L}_\star^<(x, u, w; \theta) = f(x, \theta) + \sum_{i \in P^<(\theta^\star)} u_i f^i(x, \theta).$$

Such a characterization also requires that objective function of the convex model (P,θ) be "realistic" (see, e.g., [20]):

Definition 5. *Consider the convex model (P,θ) around some $\theta^\star \in \mathsf{F}$. The objective function is said to be realistic at $\theta^\star$ if $\mathrm{F}^o(\theta^\star) \neq \emptyset$ and bounded.*

In the following result, borrowed from [18], we denote $\mathbb{R}^c_+ = \{x \in \mathbb{R}^c : x \geq 0\}$.

Theorem 6. *(Characterization of Local Optimality for Arbitrary Models on a Region of Stability.) Consider the convex model (P,θ) with a realistic objective function f at some $\theta^\star$ in F. Let $x^\star$ be an optimal solution of the program (P,$\theta^\star$) and let S be a subset of F containing $\theta^\star$. Assume that the mapping F is lower semi-continuous at $\theta^\star$ relative to S. Then $\theta^\star$ is a locally optimal parameter of f^o relative to S if, and only if, there exists a neighborhood $N(\theta^\star)$ of $\theta^\star$ and a non-negative vector function $u^\star : N(\theta^\star) \cap S \longrightarrow \mathbb{R}^c_+$ such that*

$$\mathsf{L}_\star^<(x^\star, u; \theta^\star) \leq \mathsf{L}_\star^<(x^\star, u^\star(\theta^\star); \theta^\star) \leq \mathsf{L}_\star^<(x, u^\star(\theta); \theta) \tag{SSP}$$

for all θ in $N(\theta^\star) \cap S$, u in $\mathbb{R}^c_+$ and all x in $\mathrm{F}^=_\star(\theta)$.

Note that (P,θ) is not assumed to be a Lagrange model in this characterization, but the price we have to pay is lower semi-continuity of F at $\theta^\star$. Now, we apply this result to the model given in *Example 2*.

Example 4. *Consider the model given in* Example 2. *The model is not Lagrange around $\theta^\star$ (the root of $\theta^3 + \theta = 1$), but it is stable at $\theta^\star$. Hence* Theorem 6 *applies. We have $P^=(\theta) = \{1, 2\}$ and $\mathrm{F}^=_\star(\theta) = \{(1, 0, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\}$ for every $\theta \in \mathbb{R}$. The right-hand side of the "stable saddle point inequality" (SSP) is*

$$0 \leq x_1 - \theta x_3 + u_3^\star(\theta)[-(1+\theta)x_1 + x_3] + u_4^\star(\theta)[-x_3]$$

for all $(x_1, x_2, x_3) \in \mathrm{F}^=_\star(\theta)$. It holds, for every $\theta \in (-\infty, \theta^\star]$, with

$$u_3^\star(\theta) = \{u_3^\star \in \mathbb{R} : \max\{\theta, 0\} \leq u_3^\star \leq \frac{1}{1+\theta^2}\} \quad \text{and} \quad u_4^\star(\theta) = u_3^\star(\theta) - \theta.$$

Hence $\theta^\star$ is locally optimal for $\theta \leq \theta^\star$.

It is clear that we cannot use *Theorem 6* to check optimality of the parameter $\theta^\star = 0$ for the model given in *Example 3* because the feasible set mapping F, associated to that model, is not lower semi-continuous at $\theta^\star = 0$.

Remark 2. *One can replace the usual Lagrangian* L *in* Theorem 5 *by the reduced Lagrangian* $\mathsf{L}_\star^<$ *from* Theorem 6.
If a convex model is not a Lagrange model and if it is unstable at $\theta^\star$, then none of the above characterizations works. One can see this using the example from, e.g., [1, Example 5.4].

4. Optimality in LFS Models

We have seen in the preceding section that one can characterize locally optimal parameters in convex Lagrange models (regardless of whether perturbations are stable or not). In this section we will show that there is a class of easily identifiable Lagrange models. These are models whose constraint functions have "locally flat surfaces" at every θ. (They are also known as "LFS models".) We recall these functions from [11, 12]. First, given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the cones of *directions of constancy*, *non-ascent* and *descent* of f at a given $x^\star \in \mathbb{R}^n$ are defined, respectively, by

$$\begin{aligned} D_f^=(x^\star) &= \{d \in \mathbb{R}^n : \exists\ \alpha^0 \text{ such that } f(x^\star + \alpha d) = f(x^\star),\ 0 < \alpha < \alpha^0\}, \\ D_f^\leqslant(x^\star) &= \{d \in \mathbb{R}^n : \exists\ \alpha^0 \text{ such that } f(x^\star + \alpha d) \leqslant f(x^\star),\ 0 < \alpha < \alpha^0\} \end{aligned}$$

and

$$D_f^<(x^\star) = \{d \in \mathbb{R}^n : \exists\ \alpha^0 \text{ such that } f(x^\star + \alpha d) < f(x^\star),\ 0 < \alpha < \alpha^0\}.$$

We recall that the directional derivative of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ at $x^\star \in \mathbb{R}^n$ in the direction $d \in \mathbb{R}^n$ is

$$f'(x^\star; d) = \lim_{\lambda \longrightarrow 0^+} \frac{f(x^\star + \lambda d) - f(x^\star)}{\lambda}.$$

It is well known that for a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ the directional derivative exists for any direction d and any point $x^\star \in \mathbb{R}^n$ (see, e.g., [16]). The following definition was introduced in [11, 12].

Definition 6. *A convex function* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is said to have a locally flat surface at* $x^\star \in \mathbb{R}^n$ *if*

$D_f^\leqslant(x^\star)$ *is polyhedral when* $D_f^<(x^\star) \neq \emptyset$*, or if*

$D_f^=(x^\star) = \{d \in \mathbb{R}^n : f'(x^\star; d) = 0\}$ *and polyhedral when* $D_f^<(x^\star) = \emptyset$.

We refer to such a function as an *LFS function* or a function having the *LFS property.* Such functions are, for example, all linear functions $f(x) = a^{\mathrm{T}}x + \alpha$, $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, at every $x \in \mathbb{R}^n$. So is the l_1-norm $\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$, $f(x) = a^{\mathrm{T}}x + \|x\|_1$, at every $x \in \mathbb{R}^n$, the exponential function e^t at every $t \in \mathbb{R}$, etc. Now, we introduce the LFS models.

Definition 7. *Consider the convex model (P,θ) and some subset* S *of* F*. If at every* $\theta \in S$ *there exists an optimal solution* $x^\circ(\theta)$ *such that* $f^i(\cdot, \theta) : \mathbb{R}^n \longrightarrow \mathbb{R}$*,* $i \in P$*, is LFS at* $x^\circ(\theta)$*, then (P,θ) is called an LFS model on* S.

Note that the model given in *Example 3* is LFS at every $\theta \in \mathsf{F} = [-1, 1]$. The following result states that for LFS model we can also use *Theorem 5* to characterize locally optimal parameters.

Theorem 7. *Consider the convex model (P,θ). If (P,θ) is an LFS model on a subset S of F, then (P,θ) is a Lagrange model on S.*

Proof. Take an arbitrary, but fixed, $\theta \in S$. By hypothesis, we know that there exists an optimal solution $x^{\circ}(\theta)$ such that $f^i(\cdot, \theta)$, $i \in P$, is LFS at $x^{\circ}(\theta)$. This implies that $\mathcal{L}(\theta) \neq \emptyset$ (see, e.g., [12, 13]). Therefore (P,θ) is a Lagrange model on S. □

Note that the model given in *Example 2* is not LFS at any $\theta \in \mathbb{R}$. This is true because the function $f(x) = (x_1 - 1)^2 + (x_2 - 1)^2 - 1$ is not LFS at any $x \in \mathbb{R}^3$. Indeed, in this case

$$D_f^{\leqslant}(x^\star) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0,\ x_1, x_3 \in \mathbb{R}\} \cup \{(0,0,0)\}$$

which is not polyhedral.

If a convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable, then it is LFS at $x^\star$ if, and only if, $D_f^{=}(x^\star) = \mathrm{Null}(\nabla f(x^\star))$, e.g., [12]. In the differentiable case we can relate LFS functions to a class of convex functions having globally "proportional" gradients at every point. Let us introduce these functions.

Definition 8. *Consider a differentiable convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and a set M in $\mathbb{R}^n$. We say that f is a function with globally proportional gradients (abbreviated: GPG) on the set M if $\nabla f(x) = \gamma(x)c^T$ at every $x \in M$, for some function $\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}$ and for some vector $c \in \mathbb{R}^n$.*

We can characterize these functions:

Theorem 8. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable faithfully convex function, i.e., $f(x) = \varphi(Ax + b) + a^T x + \beta$ where $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$ is strictly convex, A is an $m \times n$-matrix, b is a vector in $\mathbb{R}^m$, a is a vector in $\mathbb{R}^n$ and β is a scalar. Consider the set of its non-stationary points $M = \{x \in \mathbb{R}^n : \nabla f(x) \neq 0\}$. Then f is LFS on M if, and only if, f is GPG on M.*

Proof. Assume that f is LFS at some $x \in M$. Then $D_f^{=}(x) = \mathrm{Null}(\nabla f(x))$. Using the fact that, for differentiable faithfully convex functions,

$$D_f^{=}(x) = \mathrm{Null}\begin{bmatrix} A \\ a^{\mathrm{T}} \end{bmatrix}, \quad \text{(see [3])}$$

we conclude that $\mathrm{Null}(\nabla f(x))$ is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ *independent* of x. Denote a basic vector of its (one-dimensional) orthogonal complement by c. Then $\nabla f(x) = \gamma(x)c^{\mathrm{T}}$. Hence f is GPG on M. Now suppose that f is GPG on M, i.e., $\nabla f(x) = \gamma(x)c^{\mathrm{T}}$ at every $x \in M$. Take an $x \in M$. We already know that $D_f^{=}(x) \subset \mathrm{Null}(\nabla f(x))$, by definition of the cone of directions of constancy. So, take a direction $d \in \mathrm{Null}(\nabla f(x))$. Then $c^{\mathrm{T}} d = 0$. Note that $x \in M$ implies that $y = x + \alpha d \in M$ for all sufficiently small $\alpha > 0$. (This follows by continuity of the gradient.) So, using convexity of f, we have $f(y) - f(x) \geq \alpha \nabla f(x) d = 0$. Also,

$$\begin{aligned}
f(x) - f(y) &\geq \nabla f(x + \alpha d)(-\alpha d) \quad \text{(again by convexity of } f) \\
&= [-\alpha\gamma(x + \alpha d)c^{\mathrm{T}}]d \quad (x + \alpha d \in M \text{ and } f \text{ is GPG on } M) \\
&= 0, \quad \text{since } c^{\mathrm{T}} d = 0.
\end{aligned}$$

Hence $f(x) = f(y) = f(x + \alpha d)$, i.e., $d \in D_f^{=}(x)$. This means that f is LFS at x. $\square$

5. Conclusions

In view of *Section 2* (*Figure 1*) we have given three equivalent definitions of structurally stable convex systems. Then we have observed that optimality conditions are different in the presence of stability and in its absence. Characterizations of locally optimal parameters relative to stable perturbations do exist in the literature for convex programming models. Here characterizations are given for general perturbations (not necessary stable) but only for Lagrange models. In particular, all LFS and linear models are Lagrange models. It remains an open question how to characterize locally optimal parameters for general convex models relative to general perturbations.

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