On a class of module maps of Hilbert C^*-modules

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Abstract. *The paper describes some basic properties of a class of module maps of Hilbert C^*-modules.*

In Section 1 ideal submodules are considered and the canonical Hilbert C^*-module structure on the quotient of a Hilbert C^*-module over an ideal submodule is described. Given a Hilbert C^*-module V, an ideal submodule $V_{\mathcal{I}}$, and the quotient $V/V_{\mathcal{I}}$, canonical morphisms of the corresponding C^*-algebras of adjointable operators are discussed.

In the second part of the paper a class of module maps of Hilbert C^*-modules is introduced. Given Hilbert C^*-modules V and W and a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ of the underlying C^*-algebras, a map $\Phi : V \to W$ belongs to the class under consideration if it preserves inner products modulo φ: $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ for all $x, y \in V$. It is shown that each morphism Φ of this kind is necessarily a contraction such that the kernel of Φ is an ideal submodule of V. A related class of morphisms of the corresponding linking algebras is also discussed.

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Introduction

A (right) Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module V equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ which is $\mathcal{A}$-linear in the second and conjugate linear in the first variable such that V is a Banach space with the norm $\|v\| = \|\langle v, v \rangle\|^{1/2}$. Hilbert C^*-modules are introduced and initially investigated in [3], [5] and [8].

The present paper is organized as an introduction to a study of extensions of Hilbert C^*-modules.

Section 1 contains a detailed discussion on ideal submodules. As their basic properties are already known (see [10] and [7]), some of the results are stated without proof. The starting point is *Theorem 1.6* which states that the quotient of a

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Hilbert C^*-module over an ideal submodule admits a natural Hilbert C^*-module structure. Considering a Hilbert C^*-module V, an ideal submodule $V_{\mathcal{I}} \subseteq V$, and the quotient $V/V_{\mathcal{I}}$, we describe canonical morphisms of the corresponding C^*-algebras of adjointable operators $\boldsymbol{B}(V), \boldsymbol{B}(V_{\mathcal{I}})$ and $\boldsymbol{B}(V/V_{\mathcal{I}})$. Also, some properties of ideal submodules arising from essential ideals are obtained. In particular, we show in *Theorem 1.12* that the canonical morphism $\alpha : \boldsymbol{B}(V) \to \boldsymbol{B}(V_{\mathcal{I}})$ sending each operator T to its restriction $T|V_{\mathcal{I}}$ is an injection if and only if $\mathcal{I}$ is an essential ideal in the underlying C^*-algebra $\mathcal{A}$.

In *Section 2* a class of module maps of Hilbert C^*-modules over possibly different C^*-algebras is introduced. We consider morphisms of Hilbert C^*-modules which are in a sense supported by morphisms of the underlying C^*-algebras. Their basic properties are collected and a couple of examples is provided. In *Theorem 2.15* we establish a correspondence between the class of module maps under consideration and a class of morphisms of the corresponding linking algebras.

The present material provides a necessary tool for the later study of extensions of Hilbert C^*-modules. A related discussion will appear in our subsequent paper.

Throughout the paper we denote the C^*-algebras of all adjointable and "compact" operators on a Hilbert C^*-module V by $\boldsymbol{B}(V)$ and $\boldsymbol{K}(V)$, respectively. We also use $\boldsymbol{B}(\cdot,\cdot)$ and $\boldsymbol{K}(\cdot,\cdot)$ to denote spaces of all adjointable, resp. "compact" operators acting between different Hilbert C^*-modules.

We denote by $\langle V, V\rangle$ the closed linear span of all elements in the underlying C^*-algebra $\mathcal{A}$ of the form $\langle x, y\rangle$, $x, y \in V$. Obviously, $\langle V, V\rangle$ is an ideal in $\mathcal{A}$. (Throughout the paper, an ideal in a C^*-algebra always means a closed two-sided ideal.) V is said to be a full $\mathcal{A}$-module if $\langle V, V\rangle = \mathcal{A}$.

For this and other general facts concerned with Hilbert C^*-modules we refer to [4], [7] and [9].

1. Ideal submodules and quotients of Hilbert C^*-modules

We begin with the definition of an ideal submodule. A related discussion can be found in [10].

Definition 1.1. *Let V be a Hilbert C^*-module over $\mathcal{A}$, and $\mathcal{I}$ an ideal in $\mathcal{A}$. The associated ideal submodule $V_{\mathcal{I}}$ is defined by*

$$V_{\mathcal{I}} = [V\mathcal{I}]^- = [\{vb : v \in V,\, b \in \mathcal{I}\}]^-$$

(the closed linear span of the action of $\mathcal{I}$ on V).

Clearly, $V_{\mathcal{I}}$ is a closed submodule of V. It can be also regarded as a Hilbert C^*-module over $\mathcal{I}$.

In general, there exist closed submodules which are not ideal submodules. For instance, if a C^*-algebra $\mathcal{A}$ is regarded as a Hilbert $\mathcal{A}$-module (with the inner product $\langle a, b\rangle = a^*b$), then ideal submodules of $\mathcal{A}$ are precisely ideals in $\mathcal{A}$, while closed submodules of $\mathcal{A}$ are closed right ideals in $\mathcal{A}$.

We proceed with a couple of basic properties of ideal submodules. Our first proposition is already known ([10]).

Proposition 1.2. *Let V be a Hilbert C^*-module over $\mathcal{A}$, and let $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then $V_\mathcal{I} = V\mathcal{I} = \{vb : v \in V,\, b \in \mathcal{I}\}$.*

Proof. The associated ideal submodule $V_\mathcal{I}$ is by definition equal to $V_\mathcal{I} = [V\mathcal{I}]^- = [\{vb : v \in V,\, b \in \mathcal{I}\}]^-$. Regarding $V_\mathcal{I}$ as a Hilbert $\mathcal{I}$-module we may apply the Hewitt-Cohen factorization theorem ([6], Theorem 4.1, see also [7], Proposition 2.31): for each $x \in V_\mathcal{I}$ there exist $y \in V_\mathcal{I}$ and $b \in \mathcal{I}$ such that $x = yb$. This shows $V\mathcal{I} \subseteq [V\mathcal{I}]^- = V_\mathcal{I} \subseteq V_\mathcal{I}\mathcal{I} \subseteq V\mathcal{I}$, i.e. $V_\mathcal{I} = V\mathcal{I}$. □

Proposition 1.3. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_\mathcal{I}$ the associated ideal submodule. Then*

$$V_\mathcal{I} = \{x \in V : \langle x, x\rangle \in \mathcal{I}\} = \{x \in V : \langle x, v\rangle \in \mathcal{I},\, \forall v \in V\}.$$

If V is full, then $V_\mathcal{I}$ is full as a Hilbert $\mathcal{I}$-module.

Proof. $\langle vb, vb\rangle = b^*\langle v, v\rangle b \in \mathcal{I}$, $\forall b \in \mathcal{I}, \forall v \in V$. This shows $x = vb \in V_\mathcal{I} \Rightarrow \langle x, x\rangle \in \mathcal{I}$. A well known formula ([9], Lemma 15.2.9)

$$x = \lim_n x\left(\langle x, x\rangle + \frac{1}{n}\right)^{-1}\langle x, x\rangle,\ \forall x \in V$$

implies the converse. The second equality is now an immediate consequence.

Suppose that V is full as a Hilbert C^*-module over $\mathcal{A}$. Then there is an approximate unit (a_λ) for $\mathcal{A}$ such that each a_λ is a finite sum of the form $a_\lambda = \sum_{i=1}^{n(\lambda)}\langle x_i^\lambda, x_i^\lambda\rangle$ ([1], Remark 1.9). Take any positive $b \in \mathcal{I}$, let ε be given.

Since (a_λ) is an approximate unit for $\mathcal{A}$, there exists λ such that $\|b^{1/2} - a_\lambda b^{1/2}\|$ is small enough so that $\|b^{1/2}(b^{1/2} - a_\lambda b^{1/2})\| < \varepsilon$. It remains to observe that the left-hand side of the above inequality can be rewritten in the form

$$\|b - b^{1/2}a_\lambda b^{1/2}\| = \|b - \sum_{i=1}^{n(\lambda)}\langle x_i^\lambda b^{1/2}, x_i^\lambda b^{1/2}\rangle\|.$$

This shows that b can be approximated by inner products of elements from $V_\mathcal{I}$, i.e. $b \in \langle V_\mathcal{I}, V_\mathcal{I}\rangle$. □

Now we introduce a natural Hilbert C^*-module structure on the quotient of a Hilbert C^*-module over an ideal submodule.

Definition 1.4. *Let V be a Hilbert C^*-module over $\mathcal{A}$, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_\mathcal{I}$ the associated ideal submodule. Denote by $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ and $q : V \to V/V_\mathcal{I}$ the quotient maps. A right action of $\mathcal{A}/\mathcal{I}$ on the linear space $V/V_\mathcal{I}$ is defined by $q(v)\pi(a) = q(va)$.*

The action of $\mathcal{A}/\mathcal{I}$ on the quotient $V/V_\mathcal{I}$ given by $q(v)\pi(a) = q(va)$ is well defined precisely because $V_\mathcal{I}$ is an ideal submodule of V. Indeed, if $\pi(a) = \pi(a')$ then $q(v)\pi(a) = q(v)\pi(a')$ is ensured by definition of an ideal submodule: $vb \in V_\mathcal{I}, \forall b \in \mathcal{I}, \forall v \in V$.

If X is an arbitrary closed submodule of V one can also consider the quotient of linear spaces V/X. Further, denote by $\mathcal{I} = \langle X, X\rangle \subseteq \mathcal{A}$ the closed linear span of the set of all $\langle x,y\rangle$, $x,y\in X$. Since X is by assumption a closed submodule of V, $\mathcal{I}$ is an ideal in $\mathcal{A}$.

Now an action of $\mathcal{A}/\mathcal{I}$ on V/X given by $q(x)\pi(a)=q(xa)$ will be unambiguously defined if and only if $vb\in X$ is satisfied for each $b\in\mathcal{I}$ and $v\in V$; i.e. $V\mathcal{I}\subseteq X$. Since X is a closed submodule, this implies $V_{\mathcal{I}}\subseteq X$. Because the reverse inclusion is always satisfied, we conclude: the action of $\mathcal{A}/\mathcal{I}$ on V/X is well defined if and only if X is the ideal submodule $V_{\mathcal{I}}$ associated with $\mathcal{I}=\langle X,X\rangle$.

Remark 1.5. *The role of ideal submodules in the preceding discussion should be compared with Proposition 3.25 in [7]. Recall that each right Hilbert $\mathcal{A}$ -module V is also equipped with a natural left Hilbert $\boldsymbol{K}(V)$-module structure. Moreover, there is a standard Hilbert $\boldsymbol{K}(V)-\mathcal{A}$ bimodule structure on V. Now one easily show the following assertions (which are stated without proofs):*

(1) Each ideal submodule $V_{\mathcal{I}}$ of V is also an ideal submodule of the left Hilbert $\boldsymbol{K}(V)$-module V.

(2) Let X be a closed submodule of a right Hilbert C^*-module V. Then X is an ideal submodule of V if and only if X is a closed subbimodule of the Hilbert $\boldsymbol{K}(V)-\mathcal{A}$ bimodule V.

The following theorem is known ([7], Proposition 3.25, [10], Lemma 3.1). We state it for the sake of completeness.

Theorem 1.6. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_{\mathcal{I}}$ the associated ideal submodule. Then $V/V_{\mathcal{I}}$ equipped with a right $\mathcal{A}/\mathcal{I}$ -action from* Definition 1.4 *is a pre-Hilbert $\mathcal{A}/\mathcal{I}$-module with the inner product given by $\langle q(v),q(w)\rangle=\pi(\langle v,w\rangle)$. The resulting norm $\|q(v)\|=\|\pi(\langle v,v\rangle)\|^{1/2}$ coincides with the quotient norm $d(v,V_{\mathcal{I}})$ defined on the quotient of Banach spaces $V/V_{\mathcal{I}}$. In particular, $V/V_{\mathcal{I}}$ is complete, hence a Hilbert C^*-module over $\mathcal{A}/\mathcal{I}$.*

Remark 1.7. *$V/V_{\mathcal{I}}$ is a full $\mathcal{A}/\mathcal{I}$-module if and only if V is full. This follows at once from the evident equality $\langle V/V_{\mathcal{I}},V/V_{\mathcal{I}}\rangle=\pi(\langle V,V\rangle)$.*

Example 1.8. *Let us briefly describe an application of* Theorem 1.6. *Consider a Hilbert C^*-module V over $\mathcal{A}$ and a surjective morphism of C^*-algebras $\varphi:\mathcal{A}\to\mathcal{B}$. Define*

$$N_\varphi=\{x\in V:\varphi(\langle x,x\rangle)=0\}.$$

One easily shows that N_φ is a closed submodule of V. There is a standard construction ([2], p. 19) which provides a pre-Hilbert $\mathcal{B}$-module structure on V/N_φ: one defines $q(v)\varphi(a)=q(va)$ and $\langle q(x),q(y)\rangle=\varphi(\langle x,y\rangle)$. However, it seems to be overlooked that V/N_φ is already complete with respect to the resulting norm.

To prove this, first observe that $\mathcal{A}/Ker\varphi$ and $\mathcal{B}$ are isomorphic C^*-algebras. This enables us to regard V/N_φ as a Hilbert $\mathcal{A}/Ker\varphi$-module. Now, $N_\varphi=\{x\in V:\langle x,x\rangle\in Ker\varphi\}=$ (by Proposition 1.3*) $=V_{Ker\varphi}$; i.e. N_φ is the ideal submodule associated to the ideal $Ker\varphi$. It remains to apply* Theorem 1.6.

Theorem 1.6 also implies that a property of the Rieffel correspondence is that, assuming that two C^*-algebras are Morita equivalent, the corresponding ideals and quotients are Morita equivalent themselves (Proposition 3.25 in [7]). We shall proceed in a different direction. Our goal is to compare the C^*-algebras of all adjointable and "compact" operators acting on a Hilbert C^*-module V with the corresponding algebras of operators on an ideal submodule $V_{\mathcal{I}}$ and the quotient $V/V_{\mathcal{I}}$, respectively.

To fix our notation, we recall the definition of the ideal of all "compact" operators on a Hilbert C^*-module V. Given $v, w \in V$, let $\theta_{v,w} : V \to V$ denote the operator defined by $\theta_{v,w}(x) = v\langle w, x\rangle$. Each $\theta_{v,w}$ is an adjointable operator on V and the linear span

$$[\{\theta_{v,w} : v, w \in V\}]$$

is a two-sided ideal in $\boldsymbol{B}(V)$. Its closure in the operator norm

$$\boldsymbol{K}(V) = [\{\theta_{v,w} : v, w \in V\}]^- \subseteq \boldsymbol{B}(V)$$

is an ideal in $\boldsymbol{B}(V)$ and elements of $\boldsymbol{K}(V)$ are called "compact" operators.

Let V be a Hilbert $\mathcal{A}$ -module. Assume that $\mathcal{I}$ is an ideal in $\mathcal{A}$, and let $V_{\mathcal{I}}$ be the associated ideal submodule. Observe that $V_{\mathcal{I}}$ is invariant for each $T \in \boldsymbol{B}(V)$; namely $T(vb) = (Tv)b \in V_{\mathcal{I}}$, $\forall b \in \mathcal{I}, \forall v \in V$. Consequently, there is an operator $T|V_{\mathcal{I}}$ on $V_{\mathcal{I}}$ induced by T such that $(T|V_{\mathcal{I}})^* = T^*|V_{\mathcal{I}}$. This gives a well defined map $\alpha : \boldsymbol{B}(V) \to \boldsymbol{B}(V_{\mathcal{I}})$, $\alpha(T) = T|V_{\mathcal{I}}$. Clearly, α is a morphism of C^*-algebras.

We shall prove that the map α is an injection if and only if $\mathcal{I}$ is an essential ideal in $\mathcal{A}$. (An ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{A}$ is said to be essential if its annihilator $\mathcal{I}^\perp = \{a \in \mathcal{A} : a\mathcal{I} = \{0\}\}$ is trivial: $\mathcal{I}^\perp = \{0\}$.)

To do this, we need a few simple results on ideal submodules associated to essential ideals. We start with a property of essential ideals which is certainly known. Since we are unable to provide a reference, the proof is included.

Lemma 1.9. *Let $\mathcal{I}$ be an ideal in a C^*-algebra $\mathcal{A}$. Then $\mathcal{I}$ is an essential ideal in $\mathcal{A}$ if and only if there exists a faithful representation $\rho : \mathcal{A} \to \boldsymbol{B}(H)$ of $\mathcal{A}$ on a Hilbert space H such that $\mathcal{I}$ acts non-degenerately on H.*

Proof. Suppose $\mathcal{I} \subset \mathcal{A} \subseteq \boldsymbol{B}(H)$ such that $\mathcal{I}$ acts non-degenerately on H. Let (u_λ) be an approximate unit for $\mathcal{I}$. Then $\xi = \lim_\lambda u_\lambda \xi$, $\forall \xi \in H$. Now $a \in \mathcal{I}^\perp$ implies $au_\lambda = 0$, $\forall \lambda$, hence $a = 0$.

To prove the converse, suppose that $\mathcal{I}$ is an essential ideal in $\mathcal{A}$. Taking any faithful representation of $\mathcal{A}$ we may write $\mathcal{I} \subset \mathcal{A} \subseteq \boldsymbol{B}(H)$. Define $H_0 = [\mathcal{I} H]^-$. Clearly, $\mathcal{I}$ acts non degenerately on H_0. Since $\mathcal{I}$ is an ideal in $\mathcal{A}$, H_0 reduces $\mathcal{A}$. We shall show that $a \mapsto a|H_0$ is also a faithful representation of $\mathcal{A}$. Let $a|H_0 = 0$. Since H_0 is invariant for each $b \in \mathcal{I}$, this implies $ab|H_0 = 0$, $\forall b \in \mathcal{I}$. On the other hand, $ab \in \mathcal{I}$ shows $ab|H_0^\perp = 0$, $\forall b \in \mathcal{I}$ (observe $H_0^\perp = \cap_{b\in\mathcal{I}} \mathrm{Ker}\, b$). This gives $ab = 0$, $\forall b \in \mathcal{I}$ and, since $\mathcal{I}$ is essential, $a = 0$. □

Lemma 1.10. *Let $\mathcal{I}$ be an ideal in a C^*-algebra $\mathcal{A}$. The following conditions are mutually equivalent:*
(a) $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.
(b) $\|a\| = sup_{b\in\mathcal{I},\|b\|\le 1}\|ab\|$, $\forall a \in \mathcal{A}$.
(c) $\|a\| = sup_{b\in\mathcal{I},\|b\|\le 1}\|ba\|$, $\forall a \in \mathcal{A}$.
(d) $\|a\| = sup_{b\in\mathcal{I},\|b\|\le 1}\|bab^*\|$, $\forall a \in \mathcal{A}^+$.

Proof. $(a) \Rightarrow (b)$: By *Lemma 1.9* we may assume $\mathcal{I} \subset \mathcal{A} \subseteq \boldsymbol{B}(H)$ such that $\mathcal{I}$ acts non-degenerately on H. Given $a \in \mathcal{A}$, we have to show $\|a\| \leq \sup_{b\in\mathcal{I},\|b\|\leq 1}\|ab\|$ (the opposite inequality is trivial). Let (u_λ) be an approximate unit for $\mathcal{I}$. Then $\xi = \lim_\lambda u_\lambda \xi$, $\forall \xi \in H$. Take $\|\xi\| \leq 1$. Then

$$\|a\xi\| = \lim_\lambda \|au_\lambda\xi\| \leq \limsup_\lambda \|au_\lambda\|\|\xi\| \leq \sup_{b\in\mathcal{I},\|b\|\leq 1}\|ab\|.$$

$(b) \Leftrightarrow (c)$ is obvious (by taking adjoints).

$(c) \Rightarrow (d)$: Let a be positive. Then

$$\|a\| = \|a^{1/2}\|^2 = \text{by (c)} = \sup_{b\in\mathcal{I},\|b\|\leq 1}\|ba^{1/2}\|^2 = \sup_{b\in\mathcal{I},\|b\|\leq 1}\|bab^*\|.$$

$(d) \Rightarrow (a)$: Take any $a \in \mathcal{I}^\perp$. Then (d) applied to a^*a gives $a^*a = 0$, thus $\mathcal{I}^\perp = \{0\}$. □

Proposition 1.11. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an essential ideal in $\mathcal{A}$, and $V_\mathcal{I}$ be the associated ideal submodule. Then*
(1) $\|v\| = sup_{b\in\mathcal{I},\|b\|\leq 1}\|vb\|$, $\forall v \in V$ and
(2) $\|v\| = sup_{y\in V_\mathcal{I},\|y\|\leq 1}\|\langle v, y\rangle\|$, $\forall v \in V$.
Conversely, if V is a full $\mathcal{A}$-module in which (1) or (2) is satisfied with respect to (the ideal submodule associated with) some ideal $\mathcal{I}$ in $\mathcal{A}$, then $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.

Proof. Take any $v \in V$. Using *Lemma 1.10(d)* we find

$$\|v\|^2 = \|\langle v, v\rangle\| = \sup_{b\in\mathcal{I},\|b\|\leq 1}\|b^*\langle v, v\rangle b\| = \sup_{b\in\mathcal{I},\|b\|\leq 1}\|vb\|^2.$$

To prove the second formula, take any $v \in V$ such that $\|v\| = 1$. Then

$$\begin{aligned}\|v\| = \|v\|^2 = \|\langle v, v\rangle\| = \text{ (by } \textit{Lemma 1.10(b)}) &= \sup_{b\in\mathcal{I},\|b\|\leq 1}\|\langle v, v\rangle b\| \\ = \sup_{b\in\mathcal{I},\|b\|\leq 1}\|\langle v, vb\rangle\| \leq \sup_{y\in V_\mathcal{I},\|y\|\leq 1}\|\langle v, y\rangle\| &\leq \|v\|.\end{aligned}$$

To prove the converse, suppose that V is a full $\mathcal{A}$-module and $\mathcal{I}$ is not essential so that $\mathcal{I}^\perp \neq \{0\}$. Take any $c \in \mathcal{I}^\perp$, $c \neq 0$. Then there exists $v \in V$ such that $vc \neq 0$. Indeed, $vc = 0$, $\forall v \in V$ would imply $\langle v, vc\rangle = 0$, $\forall v \in V$ or $\langle v, v\rangle c = 0$, $\forall v \in V$. Since V is full, it would follow $c^*c = 0$, thus $c = 0$.

After all, it remains to observe that $x = vc \neq 0$ with $c \in \mathcal{I}^\perp$ contradicts to (1) and (2), respectively. □

Theorem 1.12. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_\mathcal{I}$ the associated ideal submodule. If $\mathcal{I}$ is an essential ideal in $\mathcal{A}$, then the map $\alpha : \boldsymbol{B}(V) \to \boldsymbol{B}(V_\mathcal{I})$, $\alpha(T) = T|V_\mathcal{I}$ is an injection. Conversely, if V is full and if α is injective, then $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.*

Proof. Suppose $\alpha(T) = T|V_{\mathcal{I}} = 0$ for some T. Observe that, since $V_{\mathcal{I}}$ is an ideal submodule, $vb \in V_{\mathcal{I}}$, $\forall b \in \mathcal{I}, \forall v \in V$. Since by assumption T vanishes on $V_{\mathcal{I}}$, this implies $T(vb) = 0$, $\forall b \in \mathcal{I}, \forall v \in V$. Now, taking arbitrary $v \in V$, we find

$$\|Tv\| = (\text{by } \textit{Proposition 1.11}) = \sup_{b\in\mathcal{I}, \|b\|\leq 1} \|(Tv)b\| = \sup_{b\in\mathcal{I}, \|b\|\leq 1} \|T(vb)\| = 0.$$

To prove the converse, let V be full and α injective. Assume that $\mathcal{I}$ is not essential. For $c \in \mathcal{I}^{\perp}$, $c \neq 0$, find $v \in V$ such that $vc \neq 0$ (as in the preceding proof). Then $\theta_{vc,vc} \neq 0$, but $\alpha(\theta_{vc,vc}) = \theta_{vc,vc}|V_{\mathcal{I}} = 0$ - a contradiction. □

Remark 1.13. *In general, α is not surjective, even if $\mathcal{I}$ is an essential ideal in $\mathcal{A}$. As an example, consider a nonunital C^*-algebra $\mathcal{A}$ contained as an essential ideal in a unital C^*-algebra $\mathcal{B}$. Assume further that $\mathcal{B}$ is not the maximal unitization of $\mathcal{A}$, i.e. that $\mathcal{B}$ is properly contained in the multiplier algebra $M(\mathcal{A})$. Consider $\mathcal{B}$ as a Hilbert $\mathcal{B}$-module. It is well known that, since $\mathcal{B}$ is unital, $\boldsymbol{K}(\mathcal{B}) = \boldsymbol{B}(\mathcal{B}) = \mathcal{B}$. Further, $\mathcal{A}$ is an ideal submodule of $\mathcal{B}$ associated with the essential ideal $\mathcal{A}$ of $\mathcal{B}$. We also know $\boldsymbol{K}(\mathcal{A}) = \mathcal{A}$ and $\boldsymbol{B}(\mathcal{A}) = M(\mathcal{A})$. One easily concludes that the map $\alpha : \boldsymbol{B}(\mathcal{B}) = \mathcal{B} \to \boldsymbol{B}(\mathcal{A}) = M(\mathcal{A})$ from* Theorem 1.12 *acts as the inclusion $\mathcal{B} \hookrightarrow M(\mathcal{A})$; thus, by assumption, α is not a surjection.*

Consider again an arbitrary Hilbert $\mathcal{A}$-module and an ideal $\mathcal{I}$ in $\mathcal{A}$. Using the map α one can easily determine $\boldsymbol{K}(V_{\mathcal{I}})$. Our next proposition, in which $\boldsymbol{K}(V_{\mathcal{I}})$ is recognized as an ideal in $\boldsymbol{K}(V)$, is known; hence we state it without proof. For the proof we refer to [7], Theorem 3.22. (Alternatively, it can be deduced from *Theorem 1.12* above after observing that for each ideal $\mathcal{I}$ in $\mathcal{A}$, we have $V_{\mathcal{I}} \oplus V_{\mathcal{I}^{\perp}} = V_{\mathcal{I}\oplus\mathcal{I}^{\perp}}$.)

Proposition 1.14. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_{\mathcal{I}}$ be the associated ideal submodule. Then $\boldsymbol{J} = [\{\theta_{x,y} : x, y \in V_{\mathcal{I}}\}]^- \subseteq \boldsymbol{K}(V)$ is an ideal in $\boldsymbol{K}(V)$ and the restriction $\alpha' = \alpha|\boldsymbol{J} : \boldsymbol{J} \to \boldsymbol{K}(V_{\mathcal{I}})$ is an isomorphism of C^*-algebras.*

Remark 1.15. *Using the same notation as above one easily concludes that $V_{\mathcal{I}}$ is also an ideal submodule of the left $\boldsymbol{K}(V)$-module V (with the inner product $[x, y] = \theta_{x,y}$) associated with the ideal $\boldsymbol{J} = [\{\theta_{x,y} : x, y \in V_{\mathcal{I}}\}]^- \subseteq \boldsymbol{K}(V)$. As in* Proposition 1.3 *one obtains $V_{\mathcal{I}} = \{x \in V : \theta_{x,v} \in \boldsymbol{J}, \forall v \in V\}$.*

Corollary 1.16. *Let V be a full Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, t $V_{\mathcal{I}}$ the associated ideal submodule. Then:*

(i) $\boldsymbol{J} = [\{\theta_{x,y} : x, y \in V_{\mathcal{I}}\}]^- \simeq \boldsymbol{K}(V_{\mathcal{I}})$ *is an essential ideal in* $\boldsymbol{K}(V)$ *if and only if $\mathcal{I}$ is an essential ideal in $\mathcal{A}$.*

(ii) $\boldsymbol{J} = \boldsymbol{K}(V)$ *if and only if* $\mathcal{I} = \mathcal{A}$.

Proof. Assume that $\mathcal{I}$ is an essential ideal in $\mathcal{A}$ and take $T \in \boldsymbol{K}(V)$ such that $T \perp \boldsymbol{J}$. By the preceding remark for each v in V and x in $V_{\mathcal{I}}$ the operator $\theta_{v,x}$ belongs to $\boldsymbol{J}$, hence $T\theta_{v,x} = \theta_{Tv,x} = 0$. In particular, $Tv\langle x, y\rangle = 0, \forall x, y \in V_{\mathcal{I}}$. Since V is full, $V_{\mathcal{I}}$ is a full $\mathcal{I}$-module and now the first assertion of *Proposition 1.11* implies $Tv = 0$.

The proof of the second assertion is similar, hence omitted. □

We end this section with the corresponding result on quotients. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$, and let $V_{\mathcal{I}}$ be the associated ideal submodule. Since $V_{\mathcal{I}}$ is invariant for each $T \in \boldsymbol{B}(V)$, there is a well defined induced operator $\hat{T}$ on $V/V_{\mathcal{I}}$ given by $\hat{T}(q(v)) = q(Tv)$. Moreover, $\hat{T}$ is adjointable because $(\hat{T})^* = \overset{\wedge}{T^*}$. This enables us to define $\beta : \boldsymbol{B}(V) \to \boldsymbol{B}(V/V_{\mathcal{I}})$, $\beta(T) = \hat{T}$. Obviously, β is a morphism of C^*-algebras.

The following proposition is proved by applying β to the ideal $\boldsymbol{K}(V)$ of all "compact" operators on V. However, as the result is already known ([7], Proposition 3.25, see also [10]), we omit the proof.

Proposition 1.17. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, $V_{\mathcal{I}}$ the associated ideal submodule, and let $\boldsymbol{J} = [\{\theta_{x,y} : x, y \in V_{\mathcal{I}}\}]^- \subseteq \boldsymbol{K}(V)$ be as in* Proposition 1.14. *Then $\boldsymbol{K}(V)/\boldsymbol{J}$ and $\boldsymbol{K}(V/V_{\mathcal{I}})$ are isomorphic C^*-algebras.*

Corollary 1.18. *Let V be a Hilbert $\mathcal{A}$-module, $\mathcal{I}$ an ideal in $\mathcal{A}$, and $V_{\mathcal{I}}$ the associated ideal submodule. Then the map $\beta : \boldsymbol{B}(V) \to \boldsymbol{B}(V/V_{\mathcal{I}})$, $\beta(T) = \hat{T}$ is the unique morphism of C^*-algebras satisfying $\beta(\theta_{x,y}) = \theta_{q(x),q(y)}$, $\forall x, y \in V$ and $\beta(\boldsymbol{K}(V)) = \boldsymbol{K}(V/V_{\mathcal{I}})$. If V is countably generated, then β is surjective.*

Proof. The equality $\beta(\theta_{x,y}) = \theta_{q(x),q(y)}$, $\forall x, y \in V$ is verified by a direct calculation. Since β is a morphism of C^*-algebras, this ensures $\beta(\boldsymbol{K}(V)) = \boldsymbol{K}(V/V_{\mathcal{I}})$. Now the small extension theorem applies (see [9], Propositions 2.2.16 and 2.3.7) because $\boldsymbol{B}(V)$ and $\boldsymbol{B}(V/V_{\mathcal{I}})$ are the multiplier algebras of $\beta(\boldsymbol{K}(V))$, resp. $\boldsymbol{K}(V/V_{\mathcal{I}})$. Thus $\beta : \boldsymbol{B}(V) \to \boldsymbol{B}(V/V_{\mathcal{I}})$ is uniquely determined as the extension of $\beta' = \beta|\boldsymbol{K}(V) : \boldsymbol{K}(V) \to \boldsymbol{K}(V/V_{\mathcal{I}})$ by strict continuity.

The last assertion follows from Tietze's extension theorem. First, if V is countably generated, then $\boldsymbol{K}(V)$ is a σ-unital C^*-algebra ([4], Proposition 6.7]). Since $\beta' : \boldsymbol{K}(V) \to \boldsymbol{K}(V/V_{\mathcal{I}})$ is a surjection, Proposition 6.8 from [4] implies that β is also a surjective map. □

2. Morphisms of Hilbert C^*-modules

In this section we introduce a class of module maps of Hilbert C^*-modules, not necessarily over the same C^*-algebra (cf. [2], p. 9, [4], p. 24 and also [7], p. 57). The motivating example is provided by the quotient map $q : V \to V/V_{\mathcal{I}}$ taking values in the quotient module of V over an ideal submodule $V_{\mathcal{I}}$ satisfying $\langle q(x), q(y)\rangle = \pi(\langle x, y\rangle)$.

Definition 2.1. *Let V and W be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras. A map $\Phi : V \to W$ is said to be a φ-morphism of Hilbert C^*-modules if $\langle \Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle)$ is satisfied for all x, y in V.*

Using polarization, one immediately concludes that Φ is a φ-morphism if and only if $\langle \Phi(x), \Phi(x)\rangle = \varphi(\langle x, x\rangle)$ is satisfied for each x in V.

It is also easy to show that each φ-morphism is necessarily a linear operator and a module map in the sense $\Phi(va)\Phi(v)\varphi(a)$, $\forall v \in V, \forall a \in \mathcal{A}$.

Further, let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be morphisms of C^*-algebras and let V, W, Z be Hilbert C^*-modules over $\mathcal{A}, \mathcal{B}, \mathcal{C}$, respectively. If $\Phi : V \to W$ is a φ-morphism and $\Psi : W \to Z$ is a ψ-morphism, then obviously $\Psi\Phi : V \to Z$ is a $\psi\varphi$-morphism of Hilbert C^*-modules.

Example 2.2. *Consider a Hilbert C^*-module V over a C^*-algebra $\mathcal{A}$. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$, and let $V_{\mathcal{I}}$ be the associated ideal submodule. Then we have an exact sequence of C^*-algebras $\mathcal{I} \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \mathcal{A}/\mathcal{I}$ and the corresponding sequence of Hilbert C^*-modules $V_{\mathcal{I}} \xrightarrow{j} V \xrightarrow{q} V/V_{\mathcal{I}}$. (Here i and j denote inclusions while π and q denote canonical quotient maps). Obviously, j is an i-morphism and q is a π-morphism in the sense of the above definition.*

Theorem 2.3. *Let V and W be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras and let $\Phi : V \to W$ be a φ-morphism of Hilbert C^*-modules. Then Φ is a contraction satisfying $Ker\,\Phi = V_{Ker\varphi}$. If φ is an injection, then Φ is an isometry, hence also injective. If V is a full $\mathcal{A}$-module and if Φ is injective, then φ is also an injection.*

Proof. $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle) \Rightarrow \|\Phi(x)\|^2 = \|\langle \Phi(x), \Phi(x) \rangle\| = \|\varphi(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2$, $\forall x \in V$. This proves that Φ is a contraction. The same calculation also shows: if φ is an injection, then the inequality above is replaced by the equality, hence Φ is also an isometry.

Obviously, $\mathrm{Ker}\,\Phi$ is a closed submodule of V such that $V_{\mathrm{Ker}\varphi} \subseteq \mathrm{Ker}\,\Phi$.

Further, $x \in \mathrm{Ker}\,\Phi \Rightarrow \langle \Phi(x), \Phi(x) \rangle = 0 \Rightarrow \varphi(\langle x, x \rangle) = 0$; i.e. $\langle x, x \rangle \in \mathrm{Ker}\,\varphi$. By *Proposition 1.3* we conclude $x \in V_{\mathrm{Ker}\varphi}$ which gives $\mathrm{Ker}\,\Phi \subseteq V_{\mathrm{Ker}\varphi}$.

Finally, suppose that Φ is an injection. Then $\mathrm{Ker}\,\Phi = V_{\mathrm{Ker}\varphi} = \{0\}$. Take any $a \in \mathrm{Ker}\,\varphi$. Then the last equality means $xa = 0$, $\forall x \in V$. In particular, $\langle y, xa \rangle = \langle y, x \rangle a = 0$, $\forall x, y \in V$. Since V is by hypothesis full, this implies $a = 0$. □

Lemma 2.4. *Let V and W be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras and let $\Phi : V \to W$ be a φ-morphism of Hilbert C^*-modules. Denote by $\hat{\varphi}$ and $\hat{\Phi}$ the maps induced on the quotients by φ and Φ, respectively:*

$$\hat{\varphi} : \mathcal{A}/Ker\,\varphi \to \mathcal{B}, \quad \hat{\varphi}(\pi(a)) = \varphi(a), \qquad \hat{\Phi} : V/Ker\,\Phi \to W, \quad \hat{\Phi}(q(v)) = \Phi(v).$$

Then $\hat{\Phi}$ is a well defined $\hat{\varphi}$-morphism of Hilbert C^*-modules $V/Ker\,\Phi$ and W.

Proof. First, by *Theorem 2.3*, $\mathrm{Ker}\,\Phi = V_{\mathrm{Ker}\varphi}$. This ensures that $V/\mathrm{Ker}\,\Phi = V/V_{\mathrm{Ker}\varphi}$ is a Hilbert $\mathcal{A}/\mathrm{Ker}\,\varphi$-module. Both maps are obviously well defined, so we only need to check that $\hat{\Phi}$ is a $\hat{\varphi}$-morphism. Indeed:

$$\langle \hat{\Phi}(q(v)), \hat{\Phi}(q(w)) \rangle \langle \Phi(v), \Phi(w) \rangle = \varphi(\langle v, w \rangle)\ \hat{\varphi}(\pi(\langle v, w \rangle)) = \hat{\varphi}(\langle q(v), q(w) \rangle).$$

□

Proposition 2.5. *Let V and W be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras and let $\Phi : V \to W$ be a φ-morphism of Hilbert C^*-modules. Then $Im\,\Phi$ is a closed subspace of W. It is also a Hilbert C^*-module over the C^*-algebra $Im\,\varphi \subseteq \mathcal{B}$ such that $\langle Im\,\Phi, Im\,\Phi\rangle = \varphi(\langle V, V\rangle)$. If V is a full $\mathcal{A}$-module, then $Im\,\Phi$ is a full $Im\,\varphi$-module. In particular, if Φ is surjective, and if W is a full $\mathcal{B}$-module, then φ is also a surjection.*

Proof. First suppose that φ is injective. Then by *Theorem 2.3* Φ is an isometry which implies that $\mathrm{Im}\,\Phi$ is a closed subspace of W. Also, $\Phi(v)\varphi(a) = \Phi(va) \in \mathrm{Im}\,\Phi$ and $\langle\Phi(v), \Phi(w)\rangle = \varphi(\langle v, w\rangle) \in \mathrm{Im}\,\varphi$. This shows that $\mathrm{Im}\,\Phi$ is a Hilbert $\mathrm{Im}\,\varphi$-module. The last equality also proves $\langle\mathrm{Im}\,\Phi, \mathrm{Im}\,\Phi\rangle = \varphi(\langle V, V\rangle)$.

If V is full, this implies $\langle\mathrm{Im}\,\Phi, \mathrm{Im}\,\Phi\rangle = \varphi(\mathcal{A})$ which means that $\mathrm{Im}\,\Phi$ is a full $\mathrm{Im}\,\varphi$-module. If Φ is a surjection and if W is full, we additionally get $\mathcal{B} = \langle W, W\rangle = \langle\mathrm{Im}\,\Phi, \mathrm{Im}\,\Phi\rangle = \varphi(\langle V, V\rangle)$, hence φ is also a surjection.

To prove the general case, take the maps $\hat{\varphi}$ and $\hat{\Phi}$ from *Lemma 2.4*. Since $\hat{\varphi}$ is an injection, we may apply the first part of the proof.

To do this, one has only to observe $\mathrm{Im}\,\varphi = \mathrm{Im}\,\hat{\varphi}$, $\mathrm{Im}\,\Phi = \mathrm{Im}\,\hat{\Phi}$ and $\langle\mathrm{Im}\,\Phi, \mathrm{Im}\,\Phi\rangle\ \hat{\varphi}$ $(\langle V/V_{\mathrm{Ker}\varphi}, V/V_{\mathrm{Ker}\varphi}\rangle)\ \hat{\varphi}(\pi(\langle V, V\rangle)) = \varphi(\langle V, V\rangle)$.
(The equality $\langle V/V_{\mathrm{Ker}\varphi}, V/V_{\mathrm{Ker}\varphi}\rangle\pi(\langle V, V\rangle)$ is noted in *Remark 1.7.*) □

Remark 2.6. *Let us observe: if V is a full $\mathcal{A}$-module and if φ and Φ are surjective, then W is also a full $\mathcal{B}$-module.*

On the other hand, we cannot conclude that Φ is a surjection if φ is surjective, even if V and W are full. As an example we may take $V = \mathcal{A}, W = \mathcal{A} \oplus \mathcal{A}$, $\varphi = id, \Phi(a) = (a, 0)$.

Example 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras considered as Hilbert C^*-modules over $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of C^*-algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a surjective φ-morphism of Hilbert C^*-modules $\mathcal{A}$ and $\mathcal{B}$. Then there exists an isometry m in the multiplier C^*-algebra of $\mathcal{B}$, $m \in M(\mathcal{B})$, such that $\Phi(a) = m\varphi(a)$, $\forall a \in \mathcal{A}$.*

To prove this, let us take any approximate unit (e_j) for $\mathcal{A}$. We shall show that $(\Phi(e_j))$ is a net in $\mathcal{B}$ strictly convergent in $M(\mathcal{B})$. First observe that $\mathcal{A}$ and $\mathcal{B}$ are full, so φ is also surjective.

For each $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$. Now, $\Phi(e_j)b = \Phi(e_j)\varphi(a) = \Phi(e_ja)$ converges since (e_j) is an approximate unit for $\mathcal{A}$ and Φ is continuous. On the other hand, since Φ is by assumption a surjection, there exists $c \in \mathcal{A}$ such that $(\Phi(c))^* = b$. This implies $b\Phi(e_j) = (\Phi(c))^*\Phi(e_j) = \langle\Phi(c), \Phi(e_j)\rangle = \varphi(\langle c, e_j\rangle) = \varphi(c^*e_j)$, hence $b\Phi(e_j)$ converges too.

Let $m \in M(\mathcal{B})$ be the strict limit: $m = (st.)lim_j\Phi(e_j)$; i.e. $mb = lim_j\Phi(e_j)b$, $bm = lim_jb\Phi(e_j)$, $\forall b \in \mathcal{B}$. Using continuity of Φ we get $\Phi(a) = \Phi(lim_je_ja)lim_j\Phi(e_ja)$ $= lim_j\Phi(e_j)\varphi(a) = m\varphi(a)$, $\forall a \in \mathcal{A}$. It remains to show that m is an isometry. First, $\langle\Phi(x), \Phi(y)\rangle = \langle m\varphi(x), m\varphi(x)\rangle = \varphi(x)^*m^*m\varphi(y)$. On the other hand, $\langle\Phi(x), \Phi(y)\rangle = \varphi(\langle x, y\rangle) = \varphi(x^*y) = \varphi(x)^*\varphi(y)$. Since φ is a surjection, this gives $bm^*mc = bc$, $\forall b, c \in \mathcal{B}$ i.e. $(bm^*m - b)c = 0$, $\forall b, c \in \mathcal{B}$. Taking $c = (bm^*m - b)^*$ we find $bm^*m - b = 0$, $\forall b \in \mathcal{B}$. The last equality can be written in the form

$b(m^*m - 1) = 0, \forall b \in \mathcal{B}$. *Since* $\mathcal{B}$ *in an essential ideal in* $M(\mathcal{B})$*, this implies* $m^*m - 1 = 0$.

Definition 2.8. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras, and let* V *and* W *be Hilbert* C^**-modules over* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. A map* $\Phi : V \to W$ *is said to be a unitary operator if there exists an injective morphism of* C^**-algebras* $\varphi : \mathcal{A} \to \mathcal{B}$ *such that* Φ *is a surjective* φ*-morphism.*

Remark 2.9.

(a) *Each unitary operator of Hilbert* C^**-modules is necessarily (by* Theorem 2.3*) an isometry.*

(b) *Since* Φ *is a surjection,* Proposition 2.5 *implies* $\langle W, W\rangle = \varphi(\langle V, V\rangle) \simeq \langle V, V\rangle$*. If* W *is additionally a full* $\mathcal{B}$*-module, then* φ *is also surjective, hence an isomorphism of* C^**-algebras.*

(c) *If* V *is a Hilbert* C^**-module over a* C^**-algebra* $\mathcal{A}$ *and if* $\varphi : \mathcal{A} \to \mathcal{B}$ *is an isomorphism of* C^**-algebras, then* V *can also be regarded a Hilbert* $\mathcal{B}$*-module and the identity map is obviously a unitary operator between these two versions of* V*.*

Conversely, if V *and* W *are full unitary equivalent Hilbert* C^**-modules over* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively (in the sense that there exists a unitary operator* $\Phi : V \to W$*), then* $\mathcal{A}$ *and* $\mathcal{B}$ *are isomorphic* C^**-algebras.*

(d) *Suppose that* V *and* W *are full Hilbert* C^**-modules over* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively. Let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be an isomorphism of* C^**-algebras. Then a surjective operator* $\Phi : V \to W$ *satisfying* $\Phi(va)\Phi(v)\varphi(a)$*,* $\forall v \in V, \forall a \in \mathcal{A}$ *is a unitary operator of Hilbert* C^**-modules if and only if* Φ *is an isometry.*

To see this, we have to show that Φ*, having the property* $\|\Phi(v)\| = \|v\|, \forall v \in V$*, also satisfies the condition from* Definition 2.1*. This can be done by repeating the nice argument from [4], Theorem 3.5.*

Take $x \in V$ *and* $b \in \mathcal{B}$*. Then there exists* $a \in \mathcal{A}$ *such that* $\varphi(a) = b$ *and*

$$\begin{aligned}
\|\langle \Phi(x), \Phi(x)\rangle^{1/2} b\|^2 &= \|b^*\langle \Phi(x), \Phi(x)\rangle b\| = \|\langle \Phi(x)b, \Phi(x)b\rangle\| \\
&= \|\langle \Phi(x)\varphi(a), \Phi(x)\varphi(a)\rangle\| = \|\langle \Phi(xa), \Phi(xa)\rangle\| \\
&= \|\Phi(xa)\|^2 = \|xa\|^2 = \|\langle xa, xa\rangle\| = \|\varphi(\langle xa, xa\rangle)\| \\
&= \|\varphi(\langle x, x\rangle)^{1/2}\varphi(a)\|^2 = \|\varphi(\langle x, x\rangle)^{1/2} b\|^2.
\end{aligned}$$

By Lemma 3.4 from [4] this implies $\langle \Phi(x), \Phi(x)\rangle^{1/2} = \varphi(\langle x, x\rangle)^{1/2}$*.*

(e) *Unitary equivalence of full Hilbert* C^**-modules is an equivalence relation.*

(f) *Suppose that* V *and* W *are full Hilbert* C^**-modules over* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$*, respectively such that* $\varphi : \mathcal{A} \to \mathcal{B}$ *is an isomorphism and that* $\Phi : V \to W$ *is a unitary* φ*-morphism. Then* $\Phi^{-1} : W \to V$ *is a unitary* φ^{-1}*-morphism. Then we also have*

$$\langle w, \Phi(x)\rangle = \varphi(\langle \Phi^{-1}(w), x\rangle),\ \forall x \in V,\ w \in W.$$

Indeed, putting $w = \Phi(v)$, one obtains

$$\langle w, \Phi(x)\rangle = \langle \Phi(v), \Phi(x)\rangle = \varphi(\langle v, x\rangle) = \varphi(\Phi^{-1}(w), x\rangle).$$

Example 2.10. *Consider an arbitrary C^*-algebra $\mathcal{A}$ regarded as a Hilbert $\mathcal{A}$-module with $\langle a, b\rangle = a^*b$. It is well known that the map $\gamma : \mathcal{A} \to \boldsymbol{K}(\mathcal{A})$, $\gamma(a) = T_a$, $T_a(x) = ax$ is an isomorphism of C^*-algebras. Its unique extension to the corresponding multiplier algebras ([9], Proposition 2.2.16) $\overline{\gamma} : M(\mathcal{A}) \to \boldsymbol{B}(\mathcal{A})$ is also an isomorphism of C^*-algebras and acts in the same way: $\overline{\gamma}(m) = T_m$, $T_m(x) = mx$.*

Let V be a Hilbert $\mathcal{A}$-module, let us denote $V_d = \boldsymbol{B}(\mathcal{A}, V)$. It is well known that V_d is a Hilbert $\boldsymbol{B}(\mathcal{A})$-module with the $\boldsymbol{B}(\mathcal{A})$-valued inner product $\langle r_1, r_2\rangle = r_1^* r_2$ such that the resulting norm coincides with the operator norm on V_d.

Further, each $v \in V$ induces the map $r_v \in V_d$ given by $r_v(a) = va$. It is also known ([7], Lemma 2.32) that $\{r_v : v \in V\} = \boldsymbol{K}(\mathcal{A}, V) \subseteq V_d$.

(Observe that each $v \in V$ also induces the map $l_v : V \to \mathcal{A}$ defined by $l_v(x) = \langle v, x\rangle$. Notice that $l_v^* = r_v$ and $\{l_v : v \in V\} = \boldsymbol{K}(V, \mathcal{A}) \subseteq \boldsymbol{B}(V, \mathcal{A})$.)

Now one can easily verify the following assertions:

(1) $\Gamma : V \to V_d$, $\Gamma(v) = r_v$ is a γ-morphism of Hilbert C^*-modules.

(2) ImΓ is the ideal submodule of V_d associated with the ideal $\boldsymbol{K}(\mathcal{A})$ of $\boldsymbol{B}(\mathcal{A})$.

(3) $\Gamma : V \to$ Im$\Gamma = \boldsymbol{K}(\mathcal{A}, V)$ is a unitary γ-morphism of Hilbert C^*-modules.

Proposition 2.11. *Let V and W be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, let $\varphi : \mathcal{A} \to \mathcal{B}$ be an injective morphism and let $\Phi : V \to W$ be a unitary φ-morphism. Then the map $\Phi^+ : \boldsymbol{B}(V) \to \boldsymbol{B}(W)$, $\Phi^+(T) = \Phi T \Phi^{-1}$ is an isomorphism of C^*-algebras. Moreover, $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$, $\forall x, y \in V$ and $\Phi^+(\boldsymbol{K}(V)) = \boldsymbol{K}(W)$.*

Proof. First observe that $\Phi^+(T) = \Phi T \Phi^{-1}$ is an adjointable operator, in fact we claim $(\Phi T \Phi^{-1})^* = \Phi T^* \Phi^{-1}$. Indeed,

$$\begin{aligned}\langle w_1, \Phi T \Phi^{-1} w_2\rangle &= \text{ (Remark 2.9(f)) } = \varphi(\langle \Phi^{-1} w_1, T\Phi^{-1} w_2\rangle)\\ &= \varphi(\langle T^*\Phi^{-1} w_1, \Phi^{-1} w_2\rangle) = \text{ (Remark 2.9(f))}\\ &= \langle \Phi T^* \Phi^{-1} w_1, w_2\rangle.\end{aligned}$$

Now one easily verifies that Φ^+ is an isomorphism of C^*-algebras. Further,

$$\begin{aligned}\Phi^+(\theta_{x,y})(w) &= \Phi\theta_{x,y}\Phi^{-1}(w) = \text{ (puting } \Phi(v) = w) \ = \Phi(\theta_{x,y}(v))\\ &= \Phi(x\langle y, v\rangle) = \Phi(x)\varphi(\langle y, v\rangle) = \Phi(x)\langle \Phi(y), \Phi(v)\rangle\\ &= \theta_{\Phi(x),\Phi(y)}(w).\end{aligned}$$

The last statement is an immediate consequence. □

Remark 2.12.

(a) Since $\boldsymbol{B}(V)$ and $\boldsymbol{B}(W)$ are the multiplier C^*-algebras of $\boldsymbol{K}(V)$ and $\boldsymbol{K}(W)$, we know that Φ^+, satisfying $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$, $\forall x, y \in V$, is uniquely determined.

(b) *If one applies* Proposition 2.11 *to the case* $V = \mathcal{A}$, $W = \mathcal{B}$, $\Phi = \varphi$, *one obtains the uniquely determined extension of* φ *ensured by the small extension theorem ([9], Proposition 2.2.16):* $\varphi^+ : \boldsymbol{B}(\mathcal{A}) \to \boldsymbol{B}(\mathcal{B})$, $\varphi^+(T)\varphi T\varphi^{-1}$.

(c) Proposition 2.11 *applied to* $V \simeq \Gamma(V) = \boldsymbol{K}(A, V)$ *coincides with (a special case of) Proposition 7.1 in [4].*

Corollary 2.13. *Let* V *and* W *be Hilbert* C^**-modules over* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$, *let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a surjective morphism of* C^**-algebras and let* $\Phi : V \to W$ *be a surjective* φ*-morphism. There exists a morphism of* C^**-algebras* $\Phi^+ : \boldsymbol{B}(V) \to \boldsymbol{B}(W)$ *satisfying* $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$ *and* $\Phi^+(\boldsymbol{K}(V)) = \boldsymbol{K}(W)$.

Proof. Considering the quotient $V/\mathrm{Ker}\,\Phi$ we first apply *Proposition 1.17* and *Corollary 1.18*. The proof is completed by a direct application of *Lemma 2.4* and the preceding proposition. □

Remark 2.14. *Let* V *and* W *be full (right) Hilbert* C^**-modules over* $\mathcal{A}$, *resp.* $\mathcal{B}$, *let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a morphism of* C^**-algebras and let* $\Phi : V \to W$ *be a surjective* φ*-morphism of Hilbert* C^**-modules. We note that* Φ *is also a* Φ^+*-morphism of left Hilbert* C^**-modules* V *and* W *(when* V *and* W *are regarded as the left Hilbert* C^**-modules over* $\boldsymbol{K}(V)$ *and* $\boldsymbol{K}(W)$, *respectively).*

To show this, let us denote by $[\cdot,\cdot]$ *the* $\boldsymbol{K}(V)$*-inner product on* V*; i.e.* $[x, y] = \theta_{x,y}$*; the same notation will be used in* W*. Now the condition from* Definition 2.1 *is an immediate consequence of the preceding corollary:* $[\Phi(x), \Phi(y)] = \theta_{\Phi(x),\Phi(y)} = \Phi^+(\theta_{x,y}) = \Phi^+([x, y])$.

Now we are able to describe morphisms of Hilbert C^*-modules in terms of the corresponding linking algebras.

Recall that, given a Hilbert $\mathcal{A}$-module V, the linking algebra $\mathcal{L}(V)$ may be written as the matrix algebra of the form

$$\mathcal{L}(V) = \begin{bmatrix} \boldsymbol{K}(\mathcal{A}) & \boldsymbol{K}(V, \mathcal{A}) \\ \boldsymbol{K}(\mathcal{A}, V) & \boldsymbol{K}(V) \end{bmatrix}.$$

(cf. [7], Lemma 2.32 and Corollary 3.21). Observe that $\mathcal{L}(V)$ is in fact the C^*-algebra of all "compact" operators acting on $\mathcal{A} \oplus V$. Keeping the notation from *Example 2.10* we may write

$$\mathcal{L}(V) = \begin{bmatrix} \boldsymbol{K}(\mathcal{A}) & \boldsymbol{K}(V, \mathcal{A}) \\ \boldsymbol{K}(\mathcal{A}, V) & \boldsymbol{K}(V) \end{bmatrix} = \left\{ \begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix} : a \in \mathcal{A}, x, y \in V, T \in \boldsymbol{K}(V) \right\}.$$

Accordingly, we shall also identify the C^*-algebras of "compact" operators with the corresponding corners in the linking algebra: $\boldsymbol{K}(\mathcal{A}) = \boldsymbol{K}(\mathcal{A} \oplus 0) \subseteq \boldsymbol{K}(\mathcal{A} \oplus V) = \mathcal{L}(V)$ and $\boldsymbol{K}(V) = \boldsymbol{K}(0 \oplus V) \subseteq \boldsymbol{K}(\mathcal{A} \oplus V) = \mathcal{L}(V)$.

Theorem 2.15. *Let* V *and* W *be full Hilbert* C^**-modules over* $\mathcal{A}$, *resp.* $\mathcal{B}$, *let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a morphism of* C^**-algebras and let* $\Phi : V \to W$ *be a surjective* φ*-morphism of Hilbert* C^**-modules. Then the map* $\rho_{\varphi,\Phi} : \mathcal{L}(V) \to \mathcal{L}(W)$ *defined by*

$$\rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix} \right) = \begin{bmatrix} T_{\varphi(a)} & l_{\Phi(y)} \\ r_{\Phi(x)} & \Phi^+(T) \end{bmatrix}$$

is a morphism of C^*-algebras. Conversely, let $\rho : \mathcal{L}(V) \to \mathcal{L}(W)$ be a morphism of C^*-algebras such that $\rho(\boldsymbol{K}(\mathcal{A})) \subseteq \boldsymbol{K}(\mathcal{B})$ and $\rho(\boldsymbol{K}(V)) \subseteq \boldsymbol{K}(W)$. Then there exist a morphism of C^*-algebras $\varphi : \mathcal{A} \to \mathcal{B}$ and a φ-morphism $\Phi : V \to W$ such that $\rho = \rho_{\varphi,\Phi}$.

Proof. Clearly, $\rho_{\varphi,\Phi}$ is a linear map. Further,

$$\begin{aligned}
\rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_v \\ r_w & T \end{bmatrix}\begin{bmatrix} T_b & l_x \\ r_y & S \end{bmatrix}\right) &= \rho_{\varphi,\Phi}\left(\begin{bmatrix} T_{ab} + T_{\langle v,y\rangle} & l_{xa^*} + l_{S^*v} \\ r_{wb} + r_{Ty} & \theta_{w,x} + TS \end{bmatrix}\right) \\
&= \begin{bmatrix} T_{\varphi(ab+\langle v,y\rangle)} & l_{\Phi(xa^*+S^*v)} \\ r_{\Phi(wb+Ty)} & \Phi^+(\theta_{w,x} + TS) \end{bmatrix} \\
&= \left(\begin{matrix}\text{applying } \textit{Remark 2.14} \text{ to} \\ \text{off-diagonal elements}\end{matrix}\right) \\
&= \begin{bmatrix} T_{\varphi(a)\varphi(b)} + T_{\langle \Phi(v),\Phi(y)\rangle} & l_{\Phi(x)\varphi(a^*)} + l_{\Phi^+(S^*)\Phi(v)} \\ r_{\Phi(w)\varphi(b)} + r_{\Phi^+(T)\Phi(y)} & \theta_{\Phi(w),\Phi(x)} + \Phi^+(TS) \end{bmatrix} \\
&= \begin{bmatrix} T_{\varphi(a)} & l_{\Phi(v)} \\ r_{\Phi(w)} & \Phi^+(T) \end{bmatrix}\begin{bmatrix} T_{\varphi(b)} & l_{\Phi(x)} \\ r_{\Phi(y)} & \Phi^+(S) \end{bmatrix} \\
&= \rho_{\varphi,\Phi}\left(\begin{bmatrix} T_a & l_v \\ r_w & T \end{bmatrix}\right) \rho_{\varphi,\Phi}\left(\begin{bmatrix} T_b & l_x \\ r_y & S \end{bmatrix}\right).
\end{aligned}$$

To prove the converse, first observe that, by assumption, we may write

$$\rho\left(\begin{bmatrix} T_a & 0 \\ 0 & T \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(a)} & 0 \\ 0 & \Psi(T) \end{bmatrix}.$$

It should be noted that the definition of φ actually uses the standard identification $a \leftrightarrow T_a$, $a \in \mathcal{A}, T_a \in \boldsymbol{K}(\mathcal{A})$ denoted by γ in *Example 2.10*. Obviously, both φ and Ψ are morphisms of C^*-algebras.

Take any $x \in V$ and write $\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) = \begin{bmatrix} \rho_{11}(x) & \rho_{12}(x) \\ \rho_{21}(x) & \rho_{22}(x) \end{bmatrix}$. Then

$$\begin{aligned}
\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)^* \rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) &= \begin{bmatrix} \rho_{11}(x)^* & \rho_{21}(x)^* \\ \rho_{12}(x)^* & \rho_{22}(x)^* \end{bmatrix}\begin{bmatrix} \rho_{11}(x) & \rho_{12}(x) \\ \rho_{21}(x) & \rho_{22}(x) \end{bmatrix} \\
= \begin{bmatrix} \rho_{11}(x)^*\rho_{11}(x) + \rho_{21}(x)^*\rho_{21}(x) & \rho_{11}(x)^*\rho_{12}(x) + \rho_{21}(x)^*\rho_{22}(x) \\ \rho_{12}(x)^*\rho_{11}(x) + \rho_{22}(x)^*\rho_{21}(x) & \rho_{12}(x)^*\rho_{12}(x) + \rho_{22}(x)^*\rho_{22}(x) \end{bmatrix}&.
\end{aligned}$$

Observing $\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & l_x \\ 0 & 0 \end{bmatrix}$ and comparing the above result with

$$\rho\left(\begin{bmatrix} 0 & l_x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} T_{\langle x,x\rangle} & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(\langle x,x\rangle)} & 0 \\ 0 & 0 \end{bmatrix}$$

we find $\rho_{12}(x) = \rho_{22}(x) = 0$. Similarly, calculating $\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)^*$, one additionally gets $\rho_{11} = 0$. After all, we conclude that ρ may be written in the form $\rho\left(\begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(a)} & l_{\Phi_2(y)} \\ r_{\Phi_1(x)} & \Psi(T) \end{bmatrix}$. Obviously, the induced maps Φ_1 and Φ_2 are linear.

Let us show $\Phi_1 = \Phi_2$. Indeed, $\rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}\right)^* = \rho\left(\begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}^*\right)$ implies $\begin{bmatrix} 0 & l_{\Phi_1(x)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & l_{\Phi_2(x)} \\ 0 & 0 \end{bmatrix}$, thus $\Phi_1(x) = \Phi_2(x)$ which enables us to write $\Phi_1 = \Phi_2 = \Phi$.

Finally,

$$\rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} T_{\langle y,y\rangle} & 0 \\ 0 & \theta_{x,x} \end{bmatrix}\right) = \begin{bmatrix} T_{\varphi(\langle y,y\rangle)} & 0 \\ 0 & \Psi(\theta_{x,x}) \end{bmatrix}.$$

On the other hand, knowing that ρ is multiplicative, one obtains

$$\rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} 0 & l_y \\ r_x & 0 \end{bmatrix}\right)\rho\left(\begin{bmatrix} 0 & l_x \\ r_y & 0 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & l_{\Phi(y)} \\ r_{\Phi(x)} & 0 \end{bmatrix}\begin{bmatrix} 0 & l_{\Phi(x)} \\ r_{\Phi(y)} & 0 \end{bmatrix} = \begin{bmatrix} T_{\langle\Phi(y),\Phi(y)\rangle} & 0 \\ 0 & \theta_{\Phi(x),\Phi(x)} \end{bmatrix}.$$

This is enough (together with *Corollary 2.13*) to conclude $\rho = \rho_{\varphi,\Phi}$. □

Notice that the assumptions $\rho(\boldsymbol{K}(\mathcal{A})) \subseteq \boldsymbol{K}(\mathcal{B})$ and $\rho(\boldsymbol{K}(V)) \subseteq \boldsymbol{K}(W)$ cannot be dropped from the hypothesis of the second assertion in *Theorem 2.15.*

Let us also note an alternative description of $\rho_{\varphi,\Phi}$. First, define

$$\varphi \oplus \Phi : \mathcal{A} \oplus V \to \mathcal{B} \oplus W, \quad (\varphi \oplus \Phi)(a, v) = (\varphi(a), \Phi(v)).$$

One easily verifies that $\varphi\oplus\Phi$ is a φ-morphism of Hilbert C^*-modules. After applying *Corollary 2.13* it turns out that $(\varphi \oplus \Phi)^+ = \rho_{\varphi,\Phi}$.

At the end let us mention a similar characterization of ideal submodules in terms of linking algebras: there is a natural bijective correspondence between the set of all ideal submodules of a Hilbert C^*-module V and the set of all ideals of the corresponding linking algebra $\mathcal{L}(V)$. Moreover, the ideal submodule associated with an essential ideal corresponds to an essential ideal in $\mathcal{L}(V)$. The proof is an easy calculation similar to the preceding one, hence omitted.

Note added in proof: In *Corollary 2.13* as well as in the subsequent *Remark 2.14* and *Theorem 2.15* the assumption that Φ is surjective is redundant. In fact, the map $\Phi^+ : \boldsymbol{B}(V) \to \boldsymbol{B}(W)$ satisfying $\Phi^+(\theta_{x,y}) = \theta_{\Phi(x),\Phi(y)}$ is always well defined. This can be seen using the identification $\boldsymbol{K}(V) = V \otimes_{h\mathcal{A}} V^*$ (cf. D. BLECHER, *A new approach to Hilbert C^*-modules*, Math. Ann. **307**(1997), 253-290). We thank to the anonymous referee for this observation.

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