Fixed points of fuzzy mappings in Hilbert spaces

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Abstract. *In this paper we work out two fixed point theorems for fuzzy mappings on Hilbert spaces. The proofs rely on the paralellogram law in Hilbert spaces.*

Key words: *fuzzy mapping, Hilbert space, fixed point, approximate quantity*

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1. Introduction

Heilpern [2] introduced the concept of fuzzy mappings as a mapping from an arbitrary set to one subfamily of fuzzy sets in a metric linear space and proved a fixed point theorem for fuzzy mappings. Various authors extended and generalised Heilpern's result [1], [3], [4], [5], [6] and [7]. In the present paper, we prove fixed point theorems of fuzzy mappings as introduced by Heilpern applied to Hilbert spaces.

2. Preliminaries

In the following discussions we mainly follow the definitions and notations due to Heilpern [2].

Let H be a Hilbert space and $F(H)$ be collection of all fuzzy sets in H. Let $A \in F(H)$ and $\alpha \in [0,1]$. The α-level set of A, denoted by A_α is defined as

$$\begin{aligned} A_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0,1] \\ A_0 &= \overline{\{x : A(x) > 0\}}, \end{aligned}$$

where $\overline{B}$ stands for the closure of a set B.

Definition 1. *A fuzzy subset A of H is said to be an approximate quantity iff its α-level set is a nonfuzzy compact convex subset of H for each $\alpha \in [0,1]$ and* $\sup_{x \in H} A(x) = 1$.

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From the collection $F(H)$, the subcollection of all approximate quantities is denoted by $W(H)$.

Definition 2. *Let $A, B \in W(H)$ and $\alpha \in [0,1]$, then*

(i) $P_\alpha(A,B) = \inf_{x\in A_\alpha, y\in B_\alpha} \|x-y\|$

(ii) $D_\alpha(A,B) =$ *dist* (A_α, B_α), *where "dist" denotes the Housdorff metric between A_α and B_α*

(iii) $D(A,B) = \sup_\alpha D_\alpha(A,B)$ *and*

(iv) $P(A,B) = \sup_\alpha P_\alpha(A,B)$.

It is to be noted that for any 'α', P_α is a nondecreasing as well as continuous function.

Definition 3. *Let $A, B \in W(H)$. An approximate quantity A is said to be more accurate than B, denoted by $A \subset B$, iff $A(x) < B(x)$ for each $x \in H$. The relation $\subset$ induces a partial ordering on $W(H)$,*

Definition 4. *A mapping F from the set H onto $W(H)$ is said to be a fuzzy mapping. Any $x \in H$ is called a fixed point of a mapping $F : H \to W(H)$ if*

$$\{x\} \subset Fx$$

where $\{x\}$ is the fuzzy set with a membership function equal to the chracteristic function of the crist set $\{x\}$.

We shall use the following lemma due to Heilpern [2].

Lemma 1. *Let $x \in H$, $A \in W(H)$, then $\{x\} \subset A$ if and only if $P_\alpha(x, A) = 0$ for each $\alpha \in [0,1]$.*

Lemma 2. *$P_\alpha(x,A) \le \|x-y\| + P_\alpha(y,A)$ for any $x, y \in H$.*

Lemma 3. *If $\{x_0\} \subset A$, then $P_\alpha(x_0, B) \le D_\alpha(A,B)$ for each $B \in W(H)$.*

3. Main results

In this section we prove common fixed point theorems for a pair of fuzzy mappings.

Theorem 1. *Let H be a Hilbert space, F and G are fuzzy mappings from H into $W(H)$ satisfying*

$$\begin{aligned} D^2(Fx, Gy) \quad \le \quad & a\|x-y\|^2 + bP_\alpha^2(x, Fx) + cP_\alpha^2(y, Gy) \\ & + \frac{e}{2}\left\{P_\alpha^2(x, Gy) + P_\alpha^2(y, Fx)\right\} \end{aligned} \tag{1}$$

for all x, y in H and for all $\alpha \in [0,1]$ and a, b, c, e are nonnegative numbers satisfying

$$a + b + c + 2e < 1. \tag{2}$$

Then there exists a point z in H such that

$$\{z\} \subset Fz \cap Gz.$$

Proof. Let $x_0 \in H$. We construct the sequence $\{\{x_n\}\}$ as follows.

$$\{x_1\} \subset Fx_0, \{x_2\} \subset Gx_1, \dots, \{x_{2n+1}\} \subset Fx_{2n}, \{x_{2n+2}\} \subset Gx_{2n+1}$$

and

$$\|x_i - x_{i+1}\| \leq D(Fx_{i-1}, Gx_i),\ i = 1, 2, \dots$$

Now,

$$\begin{aligned}
\|x_{2n} - x_{2n+1}\|^2 &\leq D^2(Fx_{2n}, Gx_{2n-1}) \\
&\leq a\|x_{2n} - x_{2n-1}\|^2 + bP_\alpha^2(x_{2n}, Fx_{2n}) + cP_\alpha^2(x_{2n-1}, Gx_{2n-1}) \\
&\quad + \frac{e}{2}\{P_\alpha^2(x_{2n}, Gx_{2n-1}) + P_\alpha^2(x_{2n-1}, Fx_{2n})\} \\
&\leq a\|x_{2n} - x_{2n-1}\|^2 + b\|x_{2n} - x_{2n+1}\|^2 + c\|x_{2n-1} - x_{2n}\|^2 \\
&\quad + \frac{e}{2}\{\|x_{2n} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n+1}\|^2\} \\
&\leq a\|x_{2n} - x_{2n-1}\|^2 + b\|x_{2n} - x_{2n+1}\|^2 + c\|x_{2n-1} - x_{2n}\|^2 \\
&\quad + \frac{e}{2}\{\|(x_{2n-1} - x_{2n}) + (x_{2n} - x_{2n+1})\|^2\} \\
&\leq a\|x_{2n} - x_{2n-1}\|^2 + b\|x_{2n} - x_{2n+1}\|^2 + c\|x_{2n-1} - x_{2n}\|^2 \\
&\quad + e\{\|x_{2n-1} - x_{2n}\|^2 + \|x_{2n} - x_{2n+1}\|^2\}
\end{aligned}$$

which gives

$$\|x_{2n} - x_{2n+1}\|^2 \leq k_1\|x_{2n} - x_{2n-1}\|^2$$

where

$$0 < k_1 = \frac{a + c + e}{1 - b - e} < 1.$$

Again,

$$\begin{aligned}
\|x_{2n-1} - x_{2n}\|^2 &\leq D^2(Fx_{2n-2}, Gx_{2n-1}) \\
&\leq a\|x_{2n-2} - x_{2n-1}\|^2 + bP_\alpha^2(x_{2n-2}, Fx_{2n-2}) + cP_\alpha^2(x_{2n-1}, Gx_{2n-1}) \\
&\quad + \frac{e}{2}\{P_\alpha^2(x_{2n-2}, Gx_{2n-1}) + P_\alpha^2(x_{2n-1}, Fx_{2n-2})\} \\
&\leq a\|x_{2n-2} - x_{2n-1}\|^2 + b\|x_{2n-2} - x_{2n-1}\|^2 + c\|x_{2n-1} - x_{2n}\|^2 \\
&\quad + \frac{e}{2}\{\|x_{2n-2} - x_{2n}\|^2 + \|x_{2n-1} - x_{2n-1}\|^2\} \\
&\leq a\|x_{2n-2} - x_{2n-1}\|^2 + b\|x_{2n-2} - x_{2n-1}\|^2 + c\|x_{2n-1} - x_{2n}\|^2 \\
&\quad + e\{\|x_{2n-2} - x_{2n-1}\|^2 + \|x_{2n-1} - x_{2n}\|^2\}
\end{aligned}$$

which gives

$$\|x_{2n-1} - x_{2n}\|^2 \leq k_2\|x_{2n-2} - x_{2n-1}\|^2$$

where

$$0 < k_2 = \frac{a + b + e}{1 - c - e} < 1.$$

Choosing $k = \max\{k_1, k_2\}$, it follows that

$$\|x_{n+1} - x_n\|^2 \leq k\|x_n - x_{n-1}\|^2$$

where $0 < k < 1$.

Hence $\{x_n\}$ is a Cauchy sequence in H and therefore it converges to a limit in H. We assume

$$\lim_{n\to\infty} x_n = z.$$

Again, using *Lemma 3* and for all $\alpha \in [0,1]$

$$\begin{aligned}
P_\alpha^2(x_{2n+2}, Fz) &\leq D_\alpha^2(Gx_{2n+1}, Fz) \\
&\leq D^2(Gx_{2n+1}, Fz) \\
&\leq a\|x_{2n+1} - z\|^2 + bP_\alpha^2(x_{2n+1}, Gx_{2n+1}) + cP_\alpha^2(z, Fz) \\
&\quad + \frac{e}{2}\{P_\alpha^2(z, Gx_{2n+1}) + P_\alpha^2(x_{2n+1}, Fz)\} \\
&\leq a\|x_{2n+1} - z\|^2 + b\|x_{2n+1} - x_{2n+2}\|^2 + cP_\alpha^2(z, Fz) \\
&\quad + \frac{e}{2}\{P_\alpha^2(z, x_{2n+2}) + P_\alpha^2(x_{2n+1}, Fz)\}
\end{aligned}$$

Making $n \to \infty$ and using the fact that P_α is continuous,

$$P_\alpha^2(z, Fz) \leq \left(c + \frac{e}{2}\right) P_\alpha^2(z, Fz).$$

As $\{c + (e/2)\} < 1$, it follows that $P_\alpha^2(z, Fz) = 0$, hence by *Lemma 1*, $\{z\} \subset Fz$. Similarly, $\{z\} \subset Gz$. Hence, $\{z\} \subset Fz \cap Gz$. □

Theorem 2. *Let H be a Hilbert space and F and G fuzzy mappings from H into $W(H)$ satisfying*

$$\begin{aligned}
D^2(Fx, Gy) \leq\ & q \max\big\{\|x - y\|^2, P_\alpha^2(x, Fx), P_\alpha^2(y, Gy), \\
& 1/2\{P_\alpha^2(x, Gy) + P_\alpha^2(y, Fx))\big\} \qquad (3)
\end{aligned}$$

for all x, y in H and for all $\alpha \in [0,1]$ and $q \in (0, 1/2)$. Then there exists a point z in H such that $\{z\} \subset Fz \cap Gz$.

Proof. Let $x_0 \in H$, we construct the sequence $\{x_n\}$ as in *Theorem 1* and correspondingly

$$\begin{aligned}
\|x_{2n} - x_{2n+1}\|^2 &\leq D^2(Fx_{2n}, Gx_{2n-1}) \\
&\leq q \max\Big[\|x_{2n} - x_{2n-1}\|^2, P_\alpha^2(x_{2n}, Fx_{2n}), P_\alpha^2(x_{2n-1}, Gx_{2n-1}), \\
&\quad \frac{1}{2}\left\{P_\alpha^2(x_{2n}, Gx_{2n-1}) + P_\alpha^2(x_{2n-1}, Fx_{2n})\right\}\Big] \\
&\leq q \max\Big[\|x_{2n} - x_{2n-1}\|^2, \|x_{2n} - x_{2n+1}\|^2, \|x_{2n} - x_{2n-1}\|^2, \\
&\quad \frac{1}{2}\|x_{2n-1} - x_{2n+1}\|^2\Big] \\
&\leq q \max\left[\|x_{2n} - x_{2n-1}\|^2, \frac{1}{2}\|x_{2n-1} - x_{2n+1}\|^2\right] \\
&\leq q \max\left[\|x_{2n} - x_{2n-1}\|^2, \|x_{2n} - x_{2n-1}\|^2 + \|x_{2n} - x_{2n+1}\|^2\right] \\
&\leq q \max\left[\|x_{2n} - x_{2n-1}\|^2 + \|x_{2n} - x_{2n+1}\|^2\right]
\end{aligned}$$

which yields

$$\|x_{2n} - x_{2n+1}\|^2 \leq \frac{q}{q-1}\|x_{2n} - x_{2n-1}\|^2. \qquad (4)$$

Again,

$$\begin{aligned}
\|x_{2n}-x_{2n-1}\|^2 &\le D^2(Fx_{2n-2},Gx_{2n-1})\\
&\le q\max\Big[\|x_{2n-2}-x_{2n-1}\|^2, P_\alpha^2(x_{2n-2},Fx_{2n-2}), P_\alpha^2(x_{2n-1},Gx_{2n-1}),\\
&\quad \frac{1}{2}\left\{P_\alpha^2(x_{2n-2},Gx_{2n-1})+P_\alpha^2(x_{2n-1},Fx_{2n-2})\right\}\Big]\\
&\le q\max\Big[\|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-1}-x_{2n}\|^2,\\
&\quad \frac{1}{2}\|x_{2n-2}-x_{2n}\|^2\Big]\\
&\le q\max\left[\|x_{2n-2}-x_{2n-1}\|^2, \frac{1}{2}\|x_{2n-2}-x_{2n}\|^2\right]\\
&\le q\max\left[\|x_{2n-2}-x_{2n-1}\|^2, \|x_{2n-2}-x_{2n-1}\|^2+\|x_{2n-1}-x_{2n}\|^2\right]\\
&\le q\max\left[\|x_{2n-2}-x_{2n-1}\|^2+\|x_{2n-1}-x_{2n}\|^2\right]
\end{aligned}$$

which yields

$$\|x_{2n}-x_{2n-1}\|^2 \le \frac{q}{1-q}\|x_{2n-2}-x_{2n-1}\|^2 \tag{5}$$

From (4) and (5) it follows that

$$\|x_{n+1}-x_n\|^2 \le k_1\|x_n-x_{n-1}\|^2$$

where

$$0<k_1=\frac{q}{1-q}<1.$$

Hence, $\{x_n\}$ is a Cauchy sequence in H and therefore it converges to a limit in H.

We assume

$$\lim_{n\to\infty} x_n = z.$$

Again, using *Lemma 3*,

$$\begin{aligned}
P_\alpha^2(x_{2n+2},Fz) &\le D_\alpha^2(Gx_{2n+1}Fz)\\
&\le D^2(Gx_{2n+1},Fz)\\
&\le q\max\Big[\|z-x_{2n+1}\|^2, P_\alpha^2(z,Fz), P_\alpha^2(x_{2n+1},Gx_{2n+1}),\\
&\quad \frac{1}{2}\left\{P_\alpha^2(z,Gx_{2n+1})+P_\alpha^2(x_{2n+1},Fz)\right\}\Big]\\
&\le q\max\Big[\|z-x_{2n+1}\|^2,\ P_\alpha^2(z,Fz), \|x_{2n+1}-x_{2n+2}\|,\\
&\quad \frac{1}{2}\left\{\|z-x_{2n+2}\|^2+P_\alpha^2(x_{2n+1},Fz)\right\}\Big].
\end{aligned}$$

Making $n\to\infty$ and using the fact that P_α is continuous,

$$\begin{aligned}
P_\alpha^2(z,Fz) &\le q\max\{P_\alpha^2(z,Fz), \frac{1}{2}P_\alpha^2(z,Fz)\}\\
&\le qP_\alpha^2(z,Fz)
\end{aligned}$$

As $q\in(0,1/2)$, it follows that $P_\alpha^2(z,Fz)=0$. Hence, by *Lemma 1* $\{z\}\subset Fz$. Similarly, $\{z\}\subset Gz$. Hence, $\{z\}\subset Fz\cap Gz$. □

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