Ishikawa iterative process for strongly pseudocontractive operators in arbitrary Banach spaces

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Abstract. *In this note we give a correction to the main result of Zhou in [14] on the convergence of the Ishikawa iteration process to a unique fixed point of a strongly pseudocontractive operator in arbitrary real Banach spaces. Our results extend the recent result of Soltuz [11] to arbitrary strongly pseudocontractive operators.*

Key words: *Ishikawa iteration process, strongly pseudocontractive operator*

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1. Introduction and preliminaries

Let X be a real Banach space and D a nonempty, convex subset of X. Let X^* be the duality space of X and $\langle . \, , \, . \rangle$ be the pairing between X and X^*. The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\}, \quad x \in X$$

is said to be a normalized duality mapping. The Hahn-Banach theorem assures that $J(x) \neq \emptyset$ for each $x \in X$. It is easy to see (c.f. [11]) that

$$\langle x, j(y)\rangle \leq \|x\|\|y\| \tag{1}$$

for all $x, y \in X$ and each $j(y) \in J(y)$.

An operator $T : D \subset X \to X$ is called strongly pseudocontractive if for all $x, y \in D$ there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq k\|x-y\|^2. \tag{2}$$

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One of the effective methods for approximating fixed points of an operator $T : D(T) \subset X \to X$ is the Ishikawa iteration process [5], starting with arbitrary $x_0 \in D(T)$ and for $n \geq 0$ defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T y_n, \end{aligned}$$

where $\alpha_n, \beta_n \in [0,1]$ satisfy suitable conditions (see e.g. [1]-[4], [6]-[9], [11]-[14]). If $\beta_n = 0$ for each $n \geq 0$, then Ishikawa iterations reduce to the Krasnoselski-Mann iterations [6]. In the literature which considers the convergence of the Ishikawa iteration sequence associated with accretive or pseudocontractive operators, one of hypotheses for parameters α_n is, in general, that $\alpha_n \to 0$ as $n \to \infty$. Recently Zhou [14] considered the Ishikawa iteration process with parameters $\alpha_n \geq a > 0$. Osilike in [8] have proved that two assumptions of the main theorem in [14] are contradictory. Recently Soltuz [11] presented a correction for the result of Zhou [14] for a subclass of strongly pseudocontractive operators, namely for operators T which satisfy (2) with $k < \frac{1}{2}$.

The purpose of this note is to extend the result of Soltuz [11] to all strongly pseudocontractive operators which satisfy (2) with $k < 1$ and the parameters α_n in the Ishikawa iteration process satisfy the condition $0 < a \leq \alpha_n \leq b < 2(1-k)$, where $a, b \in (0,1]$ are some constants.

For our result we need the following two lemmas:

Lemma 1 [[7], [12], [13]]. *Let X be a real Banach space and let $J : X \to 2^{X^*}$ be a normalized duality mapping. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle \tag{3}$$

for all $x, y \in X$ and each $j(x+y) \in J(x+y)$.

Lemma 2 [[9], [10]]. *Let $\{\rho_n\}$ be a sequence of non-negative real numbers which satisfy*

$$\rho_{n+1} \leq (1-\omega)\rho_n + \sigma_n, \tag{4}$$

where $\omega \in (0,1)$ is a fixed number and $\sigma_n \geq 0$ is such that $\sigma_n \to 0$ as $n \to \infty$. Then $\rho_n \to 0$ as $n \to \infty$.

2. Main results

Now we prove the following theorems on approximation.

Theorem 1. *Let X be a real Banach space, D a non-empty, convex subset of X and $T : D \to D$ a continuous and strongly pseudocontractive mapping with a pseudocontractive parameter $k \in (0,1)$. Let $x_0 \in D$ be arbitrary and let the Ishikawa iteration sequence $\{x_n\}$ be defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n = 0, 1, 2, \dots, \end{aligned} \tag{5}$$

where $\alpha_n, \beta_n \in [0,1]$ and constants $a, b \in (0,1]$ are such that

$$0 < a \leq \alpha_n \leq b < 2(1-k). \tag{6}$$

If sequences $\{Tx_n\}$ *and* $\{Ty_n\}$ *are bounded and*

$$\|Tx_{n+1} - Ty_n\| \to 0 \quad as \quad n \to \infty, \tag{7}$$

then the sequence $\{x_n\}$ *converges strongly to a unique fixed point of* T *in* D.

Proof. The existence of a fixed point follows from the result of Deimling [3], and the uniqueness from the strongly pseudocontractivity of T. Let x^* be such that $Tx^* = x^*$.
Put

$$\begin{aligned} M = 1 + \|x_0 - x^*\| &+ \sup\{\|Tx_n - x^*\| : x_n \in D\} \\ &+ \sup\{\|Ty_n - x^*\| : y_n \in D\}. \end{aligned} \tag{8}$$

Since $\{Tx_n\}$ and $\{Ty_n\}$ are bounded, we have that $M < +\infty$.
We show that the sequence $\{x_n\}$ is bounded. We shall use the mathematical induction to prove that

$$\|x_n - x^*\| \le M \quad \text{for all} \quad n \ge 0. \tag{9}$$

For $n = 0$, (9) follows from the definition of M. Suppose now that (9) holds for some $n \ge 0$. From (5), (8) and (9) we get

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n M. \end{aligned}$$

Now, by the induction hypothesis we obtain $\|x_{n+1} - x^*\| \le (1-\alpha_n)M + \alpha_n M = M$. Thus, by induction, we conclude that (9) holds for all $n \ge 0$.

From *Lemma 1* and (5) we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Ty_n - x^*, j(x_{n+1} - x^*)\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle Ty_n - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\quad + 2\alpha_n\langle Tx_{n+1} - x*, j(x_{n+1} - x^*)\rangle. \end{aligned}$$

Hence, by strongly pseudocontractivity of T, we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \le (1-\alpha_n)^2\|x_n - x^*\|^2 &+ 2\alpha_n k\|x_{n+1} - x^*\|^2 \\ &+ 2\alpha_n\langle Ty_n - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \end{aligned} \tag{10}$$

for each $j(x_{n+1} - x^*) \in J(x_{n+1} - x^*)$. From (10), (1) and (9) we obtain

$$(1 - 2\alpha_n k)\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)^2\|x_{n+1} - x^*\|^2 + 2\alpha_n\|Ty_n - Tx_{n+1}\|M. \tag{11}$$

From (6) it follows that

$$1 - 2\alpha_n k \ge 1 - 2kb > (2k-1)^2 \ge 0.$$

Thus, from (11) we have

$$\|x_{n+1} - x^*\|^2 \le \frac{(1-\alpha_n)^2}{1 - 2\alpha_n k}\|x_n - x^*\|^2 + \frac{2\alpha_n M}{1 - 2\alpha_n k}\|Tx_{n+1} - Ty_n\|. \tag{12}$$

Since $(1-\alpha_n)^2 < 1 - 2\alpha_n k$ for $0 < a \le \alpha_n \le b < 2(1-k)$, we get

$$\begin{aligned} \frac{(1-\alpha_n)^2}{1-2\alpha_n k} &< (1-\alpha_n)^2 + 2\alpha_n k \\ &\le 1 - 2\alpha_n + \alpha_n b + 2\alpha_n k \\ &= 1 - [2(1-k) - b]\alpha_n \\ &\le 1 - [2(1-k) - b]a; \\ \frac{2\alpha_n}{1-2k\alpha_n} &\le \frac{2}{1-2kb}. \end{aligned}$$

Thus, from (12) we have

$$\|x_{n+1} - x^*\|^2 \le (1-\omega)\|x_n - x^*\|^2 + \sigma_n, \tag{13}$$

where

$$\begin{aligned} \omega &= [2(1-k) - b] \cdot a, \\ \sigma_n &= \frac{2M}{1-2kb}\|Tx_{n+1} - Ty_n\|. \end{aligned}$$

From (7) we have that

$$\lim_{n\to\infty} \sigma_n = 0.$$

Taking $\rho_n = \|x_n - x^*\|^2$, from (13) and *Lemma 2* we get

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

Thus we proved that the sequence $\{x_n\}$ converges strongly to a unique fixed point of T in D. □

Remark 1. *For $k < \frac{1}{2}$ condition (6) in* Theorem 1 *becomes $0 < a \le \alpha_n$, since in this case $2(1-k) > 1$. Thus,* Theorem 1 *contains Theorem 1 of Soltuz [11] as a corollary.*

Let X be a real Banach space and $S : X \to X$ a mapping on X. If for any $x, y \in X$ there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \le k\|x-y\|^2,$$

then S is called a *strongly accretive operator.*

Lemma 3 [[1]]. *If $T : X \to X$ is a strongly accretive operator, then, for any $f \in X$, mapping $S : X \to X$, defined by $Sx = f - Tx + x$ is a strongly pseudocontractive operator, i.e. for any $x, y \in X$:*

$$\langle Sx - Sy, j(x-y)\rangle \le (1-k)\|x-y\|^2,$$

where $k \in (0,1)$ is the strongly accretive constant of T.

Theorem 2. *Let X be a real Banach space and $S : X \to X$ a continuous strongly accretive operator with a strongly accretive constant $k \in (0,1)$. For any given $f \in X$, define a mapping $T : X \to X$ by*

$$Tx = f - Sx + x$$

for all $x \in X$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *be two real sequences in* $[0,1]$ *and* $a, b \in (0,1]$ *be such that*

$$0 < a \leq \alpha_n \leq b < 2k.$$

If the range of $(I - S)$ *is bounded, then for arbitrary* $x_0 \in X$ *the sequence* $\{x_n\}$, *defined by (5) and satisfying (7) in* Theorem 1, *converges strongly to a unique solution of the equation* $Sx = f$.

Proof. Obviously, if $x^* \in X$ is a solution of the equation $Sx = f$, then x^* is a fixed point of T. Also it is easy to prove that T is continuous and strongly pseudocontractive with the strongly pseudocontractivity constant $(1-k)$. Clearly, since the range of $(I - S)$ is bounded, it follows that $\{Tx_n\}$ and $\{Ty_n\}$ are bounded. Thus, *Theorem 2* follows from *Theorem 1*. □

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