DGS–trapezoids in GS–quasigroups

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Abstract. *The concept of the DGS–trapezoid is defined and investigated in any GS–quasigroup and geometrical interpretation in the GS–quasigroup* $C(\frac{1}{2}(1+\sqrt{5}))$ *is also given. The connection of this concept with GS–trapezoids in the general GS–quasigroup is obtained.*

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GS–quasigroups are defined in [1]; in [2] different properties of GS–trapezoids in the GS–quasigroup are explored. In this paper some "geometric" concepts in the general GS–quasigroup will be defined.

A quasigroup $(Q,\cdot)$ is said to be a GS–quasigroup if it is idempotent and if it satisfies the (mutually equivalent) identities

$$a(ab\cdot c)\cdot c=b, \qquad (1) \qquad\qquad a\cdot(a\cdot bc)c=b. \qquad (1)'$$

In a GS–quasigroup we also have the mediality and elasticity

$$ab\cdot cd=ac\cdot bd, \qquad (2)$$

$$a\cdot ba=ab\cdot a, \qquad (3)$$

as well as identities

$$a(ab\cdot c)=b\cdot bc, \qquad (4) \qquad\qquad (c\cdot ba)a=cb\cdot b, \qquad (4)'$$

and equivalencies

$$ab=c\Leftrightarrow a=c\cdot cb, \qquad (5) \qquad\qquad ab=c\Leftrightarrow b=ac\cdot c. \qquad (5)'$$

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If C is the set of all points in Euclidean plane and if groupoid $(C,\cdot)$ is defined so that $aa = a$ for any $a \in C$ and for any two different points a, $b \in C$ we define $ab = c$ if the point b divides the pair a, c in the ratio of golden section. In [1] it is proved that $(C,\cdot)$ is a GS–quasigroup. We shall denote that quasigroup by $C(\frac{1}{2}(1+\sqrt{5}))$ because we have $c = \frac{1}{2}(1+\sqrt{5})$ if $a = 0$ and $b = 1$. Figures in this quasigroup $C(\frac{1}{2}(1+\sqrt{5}))$ can be used for illustration of "geometrical" relations in any GS–quasigroup.

From now on let $(Q,\cdot)$ be any GS–quasigroup. Elements of the set Q are said to be *points*.

Points a, b, c, d successively are said to be the vertices of the *golden section trapezoid* which is denoted by $\mathrm{GST}(a,b,c,d)$ if the identity $a \cdot ab = d \cdot dc$ holds (*Figure 1*). Because of (5), this identity is equivalent to the identity $d = (a \cdot ab)c$.

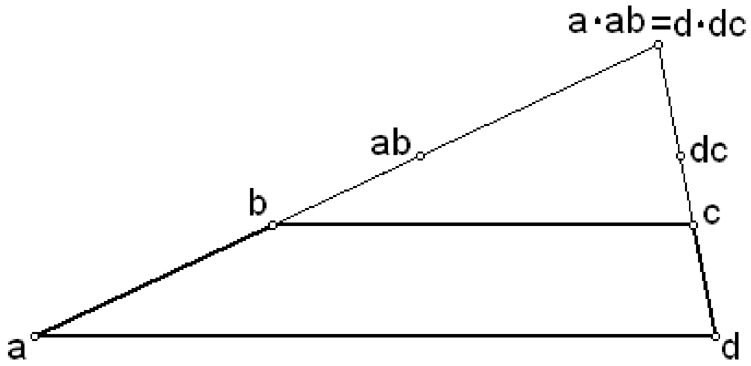

Figure 1.

DGS–trapezoids in GS–quasigroups

Points a, b, c, d are said to be the vertices of a *trapezoid of double golden section* or shorter a *DGS–trapezoid* and we write $DGST(a,b,c,d)$ if the equality $ab = dc$ holds *(Figure 2)*. Namely, because of (5), the equality $d = ab \cdot (ab \cdot c)$.

Obviously the following theorems hold.

Theorem 1. *From $DGST(a,b,c,d)$ there follows $DGST(d,c,b,a)$.*

Theorem 2. *A DGS–trapezoid is uniquely determined with any three of its vertices.*

Based on Theorem 16. from [2] it follows immediately:

Theorem 3. *Any two of the three statements $GST(a,e,f,d)$, $GST(e,b,c,f)$, $DGST(a,b,c,d)$ imply the remaining statement (*Figure 2*).*

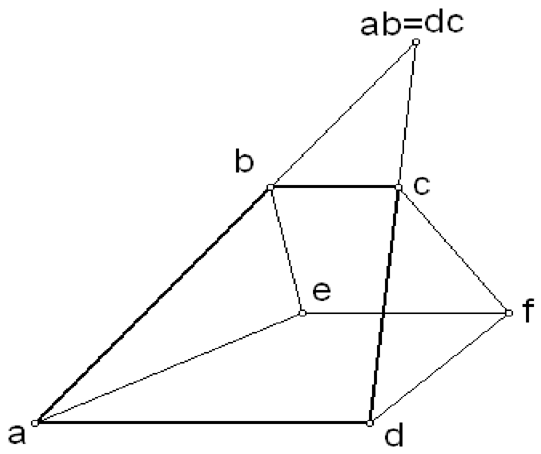

Figure 2.

Corollary 1. *The statement $DGST(a,b,c,d)$ is valid if and only if there are points e,f such that the statements $GST(a,e,f,d)$, $GST(e,b,c,f)$ are valid (*Figure 2*).*

This corollary justifies the name of the trapezoid of double golden section.

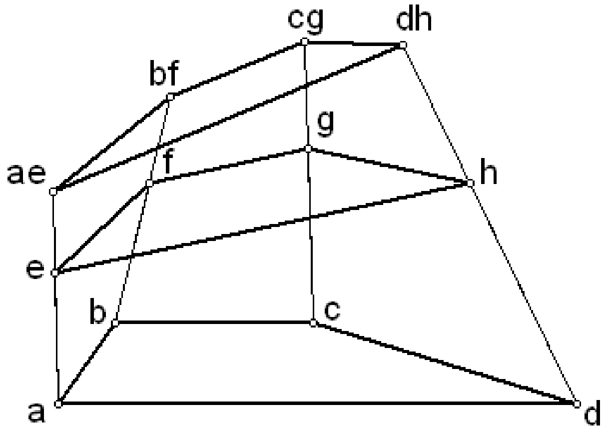

Figure 3.

Theorem 4. *Any two of the three statements $DGST(a,b,c,d)$, $DGST(e,f,g,h)$, $DGST(ae,bf,cg,dh)$ imply the remaining statement (*Figure 3*).*

Proof. We must prove that any two of the three equalities $ab = dc$, $ef = hg$ and $ae \cdot bf = dh \cdot cg$ imply the remaining equality. This is obvious, because of (2) the third equality is equivalent to $ab \cdot ef = dc \cdot hg$. □

For any point p we have obviously $\mathrm{DGST}(p,p,p,p)$ and from *Theorem 4* it follows further:

Corollary 2. *For any point p the statements $DGST(a,b,c,d)$, $DGST(pa,pb,pc,pd)$ and $DGST(ap,bp,cp,dp)$ are mutually equivalent.*

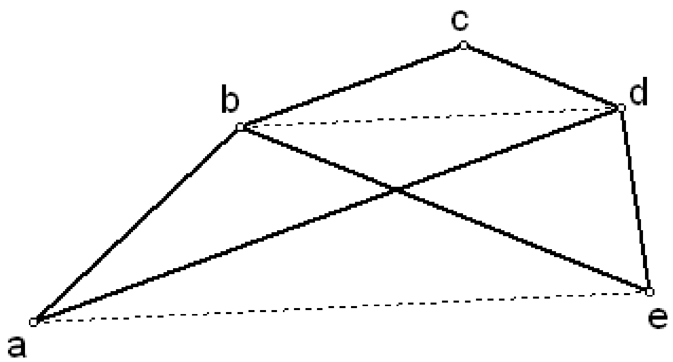

Figure 4.

Theorem 5. *Any two of the three statements $DGST(a,b,c,d)$, $DGST(b,c,d,e)$, $GST(a,b,d,e)$ imply the remaining statement (*Figure 4*).*

Proof. Because of symmetry $a \leftrightarrow e$, $b \leftrightarrow d$, it is sufficient under assumption $DGST(a,b,c,d)$ i.e. $d = ab \cdot (ab \cdot c)$ to prove the equivalency of the statements $DGST(b,c,d,e)$ and $GST(a,b,d,e)$ i.e. $e = bc \cdot (bc \cdot d)$ and $e = (a \cdot ab)d$. However, we have successively

$$\begin{aligned} bc \cdot (bc \cdot d) &= bc \cdot (bc)[ab \cdot (ab \cdot c)] \overset{(2)}{=} bc \cdot (bc)[(a \cdot ab) \cdot bc] \\ &\overset{(3)}{=} bc \cdot [bc \cdot (a \cdot ab)](bc) \overset{(4)}{=} (a \cdot ab) \cdot (a \cdot ab)(bc) \\ &\overset{(2)}{=} (a \cdot ab) \cdot (ab)(ab \cdot c) = (a \cdot ab)d. \end{aligned}$$

□

References

[1] V. Volenec, *GS–quasigroups*, Čas. pěst. mat. **115**(1990), 307–318.

[2] V. Volenec, Z. Kolar, *GS–trapezoids in GS–quasigroups*, Mathematical Communications **7**(2002), 143–158.