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More on unicoherence at subcontinua

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Abstract. *Studies are continued of unicoherence of a continuum X at its subcontinuum Y. Relations are analyzed between unicoherence of X at Y, unicoherence of either X or Y, and structure of components of the complement $X \setminus Y$. The obtained results generalize certain theorems proved in [19]. Further, it is shown that terminality of Y implies unicoherence of X at Y. Applications are shown of this result to compactifications of a ray.*

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1. Introduction

All *spaces* in this paper are assumed to be Hausdorff and *mappings* are continuous. A *continuum* means a compact, connected space.

A *ray* means a one-to-one continuous image of the real half-line $[0, \infty)$. If D is a dense subspace of a compact space C, then C is called a *compactification* of D and $C \setminus D$ is called the *remainder* of D in C (see e.g. [1, p. 34]).

A continuum X is said to be:

– *unicoherent* if the intersection of every two of its subcontinua whose union is X is connected;

– *unicoherent at a subcontinuum* $Y \subset X$ if for each pair of proper subcontinua A and B of X such that $A \cup B = X$ the intersection $A \cap B \cap Y$ is connected.

We denote by $C(X)$ the *hyperspace* of all nonempty subcontinua of a continuum X equipped with the Vietoris topology, see [11, Definition 1.1, p. 3], and we put

$$\mathcal{U}(X) = \{Y \in C(X) : X \text{ is unicoherent at } Y\}.$$

The concept of the unicoherence of a continuum at a subcontinuum is due to Owens [20], and it is related to Bennett's strong unicoherence [2], [3], and

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Maćkowiak's weak hereditary unicoherence [13] (see also [14] and Section 2 of [20]). Some mapping properties of this concept are investigated in [4]. A further study is provided in [22] and in [6]. Relations between local connectedness and unicoherence at subcontinua are studied in [19]. The present paper is — in a sense — a continuation of that study. It extends and generalizes some results obtained there.

After Introduction, in *Section 2* relations are investigated between unicoherence of X, of Y, and of X at Y. The relations are expressed in terms of the structure of components of the complement $X \setminus Y$. The obtained results generalize some theorems from [19]. *Section 3* is devoted to locally connected continua. As an application of results obtained in *Section 2* it is shown that one of the main theorems of [19, Section 3] can be generalized in two directions: one assumption (of local connectedness of the continuum considered) can be omitted, and another one can be essentially reduced. *Section 4*, where metric continua are considered, deals with terminality (in the sense of Wallace). It is shown that terminality of proper subcontinua of a continuum X implies that X is unicoherent at Y. Applications of this result to compactifications of a ray are presented. Some open questions related to the subject are asked indicating lines of future research in the area.

In the whole paper we analyze essentiality of assumptions made in theorems by constructing corresponding examples.

2. Relations to unicoherence

As indicated in [19, p. 223], unicoherence of a continuum X at its subcontinuum Y neither implies nor is implied by unicoherence of Y. To see the former statement consider the following (well-known) example, that will also be used several times for other purposes.

Example 2.1. *There exist a metric continuum X and its subcontinuum Y such that:*
(2.1.1) X is unicoherent;
(2.1.2) Y is not unicoherent;
(2.1.3) X is unicoherent at Y;
(2.1.4) for each subcontinuum K of X the intersection $Y \cap K$ is connected.

Proof. Indeed, let X be the union of a ray approximating a circle (i.e., the compactification of the real half line $[0, \infty)$ with a circle Y as the remainder). Then X and Y have all the mentioned properties. □

To see the latter statement it is enough to take Y as an arc contained in a 2-cell X. Then Y is unicoherent, X is not unicoherent at Y (but X is unicoherent and locally connected). Therefore, some additional conditions have to be assumed to attain either one of the mentioned implications or even both (i.e., the equivalence of the mentioned properties of a continuum). To this aim a concept of a continuum that is strangled by its subcontinuum has been introduced in [19, p. 224]. Recall that a continuum X is said to be *strangled by its subcontinuum* Y provided that the intersection of Y and the closure of each component of $X \setminus Y$ consists of a single point. This concept was studied by G. T. Whyburn, who has proved in [21, Chapter IV, Theorem 3.3, p. 67] that a semi-locally connected continuum X is strangled by its subcontinuum Y if and only if Y is the union of some cyclic elements of X.

It is shown in [19, Theorem 2.4, p. 226] that under this additional assumption unicoherence of Y implies unicoherence of X at Y, while the opposite implication does not hold [19, Example 2.1 and Figure 1, p. 226].

Consider the following conditions that may be satisfied by a continuum X and its subcontinuum Y:

(2.a) For each component M of $X \setminus Y$ the intersection $Y \cap \mathrm{cl}(M)$ is a singleton.

(2.b) For each subcontinuum K of X the intersection $Y \cap K$ is connected.

(2.c) For each component M of $X \setminus Y$ and for each subcontinuum K of $\mathrm{cl}(M)$ the intersection $Y \cap K$ is connected.

(2.d) For each component M of $X \setminus Y$ the intersection $Y \cap \mathrm{cl}(M)$ is connected.

Note that condition (2.a) is just the one used in the definition of "X is strangled by Y".

Example 2.2. *There exist a metric continuum X and its subcontinuum Y such that* (2.d) *is satisfied, while* (2.c) *is not.*

Proof. Let $X = [-1, 1] \times [0, 1]$ and $Y = \{0\} \times [0, 1]$. Then (2.d) holds. Taking $K = ([0, 1] \times \{0\}) \cup (\{1\} \times [0, 1]) \cup ([0, 1] \times \{1\})$ we get $Y \cap K = \{(0, 0), (0, 1)\}$. □

Below we present a 1-dimensional example with similar properties.

Example 2.3. *There exist a metric continuum X and its subcontinuum Z such that:*

(2.3.1) X is 1-dimensional;

(2.3.2) X is unicoherent;

(2.3.3) Z is hereditarily unicoherent;

(2.3.4) X is unicoherent at Z;

(2.3.5) $X \setminus Z$ is a connected and dense subset of X;

(2.3.6) X and Z satisfy (2.d)*;*

(2.3.7) X and Z do not satisfy (2.c).

Proof. Let X and Y be as in *Example 2.1*, and define Z as a semicircle contained in Y. It is easy to verify that the continuum X and its subcontinuum Z have the needed properties. □

It is well-known (see e.g. [17] and [9], where the same results are shown without the use of the continuum hypothesis) that some important properties of metric continua do not hold for nonmetric continua. On the other hand, a substantial number of results, especially the ones related to the theory of irreducible metric continua can be generalized to irreducible Hausdorff continua, either with a new argument, or with the same proofs as in the metric case (compare e.g. [8]).

The following concept will be used in the proof of the next result. Let X be a continuum, and let A and B be nonempty closed subsets of X. We say that a subcontinuum C of X is *irreducible from A to B* provided that $A \cap C \neq \emptyset \neq B \cap C$, and no proper subcontinuum of C intersects both A and B. The existence of such continuum can be shown using the same ideas as in the proofs of [10, Theorems 2-10 and 2-11, p. 44] (for the metric case see [18, Proposition 11.30, p. 212]).

Proposition 2.4. *For each continuum X and for each of its subcontinua Y the following implications hold and neither the first nor the last of them can be reversed.*

$$(2.a) \implies (2.b) \iff (2.c) \implies (2.d).$$

Proof. Indeed, the first implication is shown in [19, Lemma 2.3 *i)*, p. 225] (formulated for metric continua only, but valid in the general case since no metric argument is used in its proof). It cannot be reversed by *Example 2.1.* Implications (2.b) $\Longrightarrow$ (2.c) and (2.c) $\Longrightarrow$ (2.d) are obvious, and the latter cannot be reversed by *Example 2.2.* It remains to prove the implication (2.c) $\Longrightarrow$ (2.b).

So, assume (2.c) and suppose on the contrary that (2.b) does not hold, i.e., that there exist a continuum X, its subcontinuum Y, and a subcontinuum K of X such that $Y \cap K$ is not connected. Let A and B be nonempty disjoint closed subsets such that $Y \cap K = A \cup B$, and let C be a subcontinuum of K which is irreducible from A to B. It follows from [12, Chapter V, §48, VIII, Theorem 5, p. 220] (which also holds for Hausdorff continua, with the same proof as in the metric case) that the set $D = C \setminus (A \cup B)$ is connected and dense in C. Since $D \subset C \setminus Y$, D is contained in a component M of $X \setminus Y$, and therefore $C = \mathrm{cl}(D)$ is a subcontinuum of the continuum $\mathrm{cl}(M)$, whence it follows that the intersection $Y \cap C$ is connected according to the assumption (2.c). On the other hand, $Y \cap C \subset Y \cap K = A \cup B$, the sets A and B are separated, and $Y \cap C$ contains points of both of them. This contradiction finishes the proof. □

In connection with *Proposition 2.4* and *Example 2.1* we will show in the next result that the implication (2.a) $\Longrightarrow$ (2.b) can be reversed under additional assumptions.

A space S is said to be *continuumwise connected* provided that any two points of S can be joined by a continuum contained in S. Obviously each arcwise connected space is continuumwise connected, each continuumwise connected space is connected, and none of these implications can be reversed (even in the metric case).

Proposition 2.5. *Let a continuum X and its subcontinuum Y be such that*
(2.5.1) X is locally connected at each point of $\mathrm{bd}(Y)$, *and*
(2.5.2) each component of $X \setminus Y$ is continuumwise connected.
Then (2.b) *implies* (2.a).

Proof. Let M be a component of $X \setminus Y$. Then $\mathrm{cl}(M)$ is a continuum, and the intersection $L = \mathrm{cl}(M) \cap Y$ is also a continuum, according to (2.b). Note that $L = [(\mathrm{cl}(M) \setminus M) \cup M] \cap Y = [(\mathrm{cl}(M) \setminus M) \cap Y] \cup (M \cap Y) = (\mathrm{cl}(M) \setminus M) \cap Y$ (since $M \cap Y = \emptyset$). Thus $L \subset \mathrm{bd}(Y)$.

Suppose contrary to (2.a) that $L = \mathrm{cl}(M) \cap Y$ is not a singleton. So, it is a nondegenerate subcontinuum of $\mathrm{bd}(Y)$, and therefore it is composed exclusively of points of local connectivity of X. Let p_1 and p_2 be two distinct points of L. Since X is Hausdorff, for each $i \in \{1, 2\}$ there is a closed connected neighborhood $V(p_i)$ such that $V(p_1) \cap V(p_2) = \emptyset$. Take points $p'_i \in M \cap V(p_i)$. Let K be a subcontinuum of M joining the points p'_1 and p'_2. Thus $K \cup V(p_1) \cup V(p_2)$ is a continuum whose intersection with Y is not connected. This contradicts (2.b) and completes the proof. □

Corollary 2.6. *Let a continuum X and its subcontinuum Y be such that* (2.5.1) *and* (2.5.2) *hold. Then conditions* (2.a), (2.b) *and* (2.c) *are equivalent.*

Remark 2.7. *Note that if X and Y are as in* Example 2.2, *then conditions (2.5.1), (2.5.2) and (2.d) are satisfied, while (2.c) is not. This shows that condition (2.d) cannot be added to the ones in the conclusion of* Corollary 2.6.

Remark 2.8. *The conditions (2.5.1) and (2.5.2) are independent in the sense that none of them implies or is implied by the other. Indeed, for the continuum X and its subcontinuum Y as in the proof of* Example 2.1 *condition (2.5.2) is satisfied, while (2.5.1) is not. Thus (2.5.2) does not imply (2.5.1). Since by the same example condition (2.b) holds while (2.a) does not, the example shows that (2.5.1) is an essential assumption in* Proposition 2.5.

To verify that (2.5.1) does not imply (2.5.2) consider the following example.

Example 2.9. *There exists a metric continuum X containing a point p such that if $Y = \{p\}$, then condition* (2.5.1) *is satisfied, while* (2.5.2) *is not.*

Proof. Indeed, let S' be a ray viewed as the noncompact arc component of the topologist's sine-curve located in the half-open interval $(0, 1]$, thus defined in the Cartesian coordinates (x, y) in the plane $\mathbb{R}^2$ by

$$S' = \{(x, y) \in \mathbb{R}^2 : y = \sin \tfrac{1}{x} \text{ and } x \in (0, 1]\}. \tag{2.9.1}$$

Take the straight segment $A = [0, 1] \times \{0\}$ and put $X = A \cup \operatorname{cl}(S')$. Then X is locally connected at $p = (0, 0)$, thus the conclusion holds. □

Below the same phenomenon is presented with a nondegenerate continuum Y.

Example 2.10. *There are a metric continuum X and its nondegenerate subcontinuum Y such that condition* (2.5.1) *is satisfied, while* (2.5.2) *is not.*

Proof. For each $n \in \mathbb{N}$ and each integer $k \in \{1, 2, 3, \ldots, 2^{n-1}\}$ let $A(n, k)$ be the straight line segment in the plane defined by $A(n, k) = [0, \frac{1}{2^{n+1}}] \times \{\frac{2k-1}{2^{n+1}}\}$. Then

$$D' = \{0\} \times [0, \tfrac{1}{2}] \cup \bigcup \Big\{ \bigcup \{A(n, k) : k \in \{1, 2, 3, \ldots, 2^{n-1}\}\} : n \in \mathbb{N} \Big\}$$

is a dendrite homeomorphic to the one pictured in [12, Fig. 6, p. 247]. Denote by D'' the copy of D' under the symmetry with respect to the line $y = 0$. Thus the union $D = ([0, 1] \times \{0\}) \cup D' \cup D''$ is again a dendrite. Recall that the ray S' is defined by (2.9.1), and put

$$X = D \cup \operatorname{cl}(S') = ([-1, 1] \times \{0\}) \cup D \cup S'.$$

Thus X is a continuum whose set of points of non-local connectedness is the union $\{0\} \times ([-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1])$. Define $Y = \{0\} \times [-\frac{1}{4}, \frac{1}{4}]$ and observe that $Y = \operatorname{bd}(Y)$ and that X is locally connected at each point of Y. Thus condition (2.5.1) is satisfied. Note further that $X \setminus Y$ is connected but not continuumwise connected (namely there is no continuum in $X \setminus Y$ joining a point of S' with the point $(0, 1)$). Thus (2.5.2) does not hold. □

Remark 2.11. *In connection with* Remark 2.8 *let us mention that if condition (2.5.1) is strengthened to the local connectedness of the whole continuum X, then (2.5.2) holds, and consequently (2.b) implies (2.a). This will be shown in the next section, namely in* Proposition 3.2 *and* Corollary 3.3.

In the rest of this section we apply *Proposition 2.4* to study relations between unicoherence of either X and Y from one side and unicoherence of X at its subcontinuum Y from the other. We start with the implication from unicoherence of X to unicoherence of X at Y.

Theorem 2.12. *Let a continuum X and its subcontinuum Y be such that*
(2.12.1) X is unicoherent;

(2.c) for each component M of $X \setminus Y$ and for each subcontinuum K of $\mathrm{cl}(M)$ the intersection $Y \cap K$ is connected.

Then
(2.12.2) $Y \in \mathcal{U}(X)$.

Proof. Let H and K be subcontinua of X such that $X = H \cup K$. Since X is unicoherent, the intersection $H \cap K$ is connected. Since condition (2.c) is equivalent to (2.b), according to *Proposition 2.4*, it follows that $Y \cap H \cap K$ is connected, as needed. □

Unicoherence of X is an essential assumption in the above result by [19, Example 3.1, p. 230].

Note that *Theorem 2.12* generalizes the result in [19, Theorem 3.10, p. 231]. Some related results will be discussed in *Section 3*.

Our next result is a consequence of *Proposition 2.4* and of the definition of the unicoherence at a subcontinuum. It is stronger than [19, Theorem 2.4, p. 226] not only by its formulation for Hausdorff continua but also since condition (2.c) is assumed in place of (2.a).

Theorem 2.13. *Let a continuum X and its subcontinuum Y be such that*
(2.13.1) Y is unicoherent;

(2.c) for each component M of $X \setminus Y$ and for each subcontinuum K of $\mathrm{cl}(M)$ the intersection $Y \cap K$ is connected.

Then
(2.12.2) $Y \in \mathcal{U}(X)$.

Since (2.a) implies (2.b) and (2.c), according to *Proposition 2.4*, we get a corollary.

Corollary 2.14. *[19, Theorem 2.4, p. 226]. Let a subcontinuum Y of a continuum X be unicoherent. If X is strangled by Y, then $Y \in \mathcal{U}(X)$.*

Remarks 2.15.

(a) *A continuum X and its proper subcontinuum Y are shown in [19, Example 2.1 and Fig. 1, p. 226] such that (2.a) (and hence each of the other conditions of* Proposition 2.4*) hold, X is unicoherent at Y, and Y is not unicoherent. Thus the reverse to* Theorem 2.13 *(in the considered sense) is not true.*

(b) *Let X and Z be as in* Example 2.3. *Then Z is unicoherent, X is unicoherent at Z, and condition (2.c) does not hold. Thus again the reverse to* Theorem 2.13 *(in this sense) is not true.*

(c) *Let X and Y be as in* Example 2.2. *Then X is unicoherent, Y is hereditarily unicoherent, condition (2.d) is satisfied, and X is not unicoherent at Y since taking C as the union of two straight segments $\overline{(-1,0)(\frac{1}{2},\frac{1}{2})}$ and $\overline{(\frac{1}{2},\frac{1}{2})(-1,1)}$ we see that C is the common boundary of two proper subcontinua A and B with $X = A \cup B$ while $Y \cap A \cap B = Y \cap C$ is not connected, whence X is not unicoherent at Y. This shows that in* Theorem 2.13 *the assumption (2.c) (or (2.b)) cannot be weakened to (2.d), even under two additional assumptions: that X is unicoherent and that Y is hereditarily unicoherent.*

Theorems 2.13 and *2.12* imply the following corollary.

Corollary 2.16. *Let a continuum X and its subcontinuum Y be given. If either X or Y is unicoherent, and condition* (2.c) *is satisfied, then $Y \in \mathcal{U}(X)$.*

The reverse implication to that of the above corollary is not true by the next example.

Example 2.17. *There are a metric continuum X and its subcontinuum Y such that $Y \in \mathcal{U}(X)$, condition* (2.c) *is satisfied, and neither X nor Y is unicoherent.*

Proof. Indeed, take a ray R approximating a circle Y as in *Example 2.1*, and let e be the (only) end point of R. Let a circle C be such that $(Y \cup R) \cap C = \{e\}$. Then $X = Y \cup R \cup C$ is the needed continuum. □

In the light of the above example as well as *Remark 2.15 (a)*, it is interesting to know under what additional conditions about Y unicoherence of X at Y implies unicoherence of Y. This implication has already been studied in [19]. Namely, we have the following result.

Theorem 2.18. *[19, Theorem 2.5, p. 226]. Let a continuum X and its subcontinuum Y be such that*

(2.12.2) $Y \in \mathcal{U}(X)$;

(2.a) for each component M of $X \setminus Y$ the intersection $Y \cap \mathrm{cl}(M)$ is a singleton;

(2.5.1) X is locally connected at each point of $\mathrm{bd}(Y)$.

Then

(2.13.1) Y is unicoherent.

As a consequence of *Corollary 2.6* and *Theorem 2.18* we get the following result.

Theorem 2.19. *Let a continuum X and its subcontinuum Y be such that*

(2.12.2) $Y \in \mathcal{U}(X)$;

(2.c) for each component M of $X \setminus Y$ and for each subcontinuum K of $\mathrm{cl}(M)$ the intersection $Y \cap K$ is connected;

(2.5.1) X is locally connected at each point of $\mathrm{bd}(Y)$;

(2.5.2) each component of $X \setminus Y$ is continuumwise connected.

Then

(2.13.1) Y is unicoherent.

A question can be asked if condition (2.5.2) is essential in the above results, in particular in *Proposition 2.5*. This question can be reformulated as follows.

Question 2.20. *Do there exist a continuum X and its subcontinuum Y such that conditions (2.5.1) and (2.b) are satisfied, while (2.5.2) and (2.a) are not?*

We also do not know if condition (2.c) in *Theorem 2.19* can be weakened to (2.d). In other words, we have the following question.

Question 2.21. *Do there exist a continuum X and its subcontinuum Y such that conditions (2.12.2),* (2.d)*, (2.5.1) and (2.5.2) hold, and Y is not unicoherent?*

3. Relations to local connectedness

Recall that Section 3 of [19] deals with locally connected (Hausdorff) continua. In particular, the following result is shown.

Theorem 3.1. *[19, Theorem 3.8, p. 230]. If a continuum X is locally connected and $Y \in \mathcal{U}(X)$, then Y is unicoherent.*

Therefore, if X is assumed to be locally connected, then in *Theorem 2.19* conditions (2.c) and (2.5.2) can be omitted.

Let us come back to the implication (2.b) $\Longrightarrow$ (2.a) proved in *Proposition 2.5* under the assumptions (2.5.1) and (2.5.2). As indicated in *Remark 2.8* these two conditions are independent, in the sense that none of them implies the other. However, according to *Remark 2.11*, if (2.5.1) is strengthened to the local connectedness of the whole continuum X, then (2.5.2) follows. This is shown in the next result.

Proposition 3.2. *If a continuum X is locally connected, then for each subcontinuum Y of X*

(2.5.2) each component of $X \setminus Y$ is continuumwise connected.

Proof. Let Y be a subcontinuum of a locally connected continuum X, and let M be a component of $X \setminus Y$. Thus, by [10, Theorem 3-2 and Lemma 3-1, p. 106] M is an open and locally connected subset of X. Thus, for each point $x \in M$ there exists a connected open neighborhood $U(x)$ of x that is contained in M. Moreover, by regularity of compact Hausdorff spaces we may assume that $\operatorname{cl} U(x)$ does not intersect $\operatorname{bd}(Y)$ (see e.g. [7, Theorem 3.1.9, p. 125, and Proposition 1.5.5, p. 38]). Given two points a and b of M, there exists a chain $\{U(x_1), \dots, U(x_k)\}$ of the mentioned neighborhoods such that $a \in U(x_1)$ and $b \in U(x_k)$, see [10, Theorem 3-4, p. 108]. Thus, $\operatorname{cl}(U(x_1) \cup \dots \cup U(x_k))$ is a subcontinuum of M joining a and b. So, M is continuumwise connected, as required. □

Propositions 2.5 and *3.2* imply the following corollary.

Corollary 3.3. *If a continuum X is locally connected, then for each subcontinuum Y of X condition* (2.b) *implies* (2.a)*, and consequently, by* Proposition 2.4*, all three conditions* (2.a)*,* (2.b) *and* (2.c) *are equivalent.*

One of the results of Section 3 of [19] is [19, Theorem 3.10, p. 231] saying that the implication from unicoherence of X to unicoherence of X at Y holds if X is unicoherent and strangled by Y. In the proof of this theorem the following lemma was used as a key argument.

Lemma 3.4. *[19, Lemma 3.5, p. 228]. Let a locally connected continuum X and its subcontinuum Y be such that*

(2.a) for each component M of $X \setminus Y$ the intersection $Y \cap \operatorname{cl}(M)$ is a singleton.

Then

(3.4.1) for each connected subset V of X the intersection $V \cap Y$ is also a connected subset of X.

Considering possible generalizations of some results of [19], in particular of [19, Theorem 3.10, p. 231], a natural question arises concerning the possibility of an extension of *Lemma 3.4*. Because of the importance of the lemma in the proof of the mentioned theorem, one can ask whether the assumption of local connectedness of X is essential in the lemma, or whether it can be relaxed to a weaker condition, for example to (2.5.1). The next example shows that this is not the case.

Example 3.5. *There are a metric not locally connected continuum X, its subcontinuum Y and a connected subset V of X such that X is locally connected at each point of Y (thus* (2.5.1) *holds),* (2.a) *is satisfied and the intersection $V \cap Y$ is not connected.*

Proof. Indeed, in the plane equipped with the Cartesian coordinate system, define

$$X = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup \bigcup\{(\{-\tfrac{1}{n}, \tfrac{1}{n}\} \times [0,1]) : n \in \mathbb{N}\},$$

$$Y = [-1,1] \times \{0\} \quad \text{and} \quad V = X \setminus \{(0,0)\}.$$

It is evident that the example satisfies all the needed conditions. □

Another variant of an extension of *Lemma 3.4* is to replace (2.a) by a weaker condition, viz. (2.c). But since for locally connected continua the two conditions are equivalent according to *Corollary 3.3*, the extension is nonessential.

It follows from the above discussion of the assumptions of *Lemma 3.4*, in particular from *Example 3.5* that at the moment we are not able to omit or to relax the assumption of local connectedness of X in [19, Theorem 3.10, p. 231] if a variant of *Lemma 3.4* is applied to a proof of the result. However, as shown in *Theorem 2.12*, the lemma is not needed to prove the theorem.

In the light of the above results and comments, it seems to be both interesting and valuable to verify if the assumption of local connectedness of X is needed in other main results of Section 3 of [19]. This is the subject of remarks below.

Remarks 3.6.

(a) The implication in Corollary 2.16 *can be reversed under an additional assumption of local connectedness of X. Namely, [19, Theorem 3.8, p. 230] says that if the continuum X is locally connected, then the condition $Y \in \mathcal{U}(X)$ implies that Y is unicoherent. Therefore,* Example 2.17 *shows that local connectedness of X is necessary in Theorem 3.8 of [19].*

(b) The same Example 2.17 *as well as* Example 2.1 *show that local connectedness of X is indispensable in [19, Theorem 3.6, p. 229] which says that the same condition $Y \in \mathcal{U}(X)$ implies (2.a).*

(c) Another important result of Section 3 of [19] is [19, Theorem 3.11, p. 231] saying that, under the same assumptions, Y is locally connected. Also in this result local connectedness of X is essential. Really, let X be a compactification of a ray with any hereditarily unicoherent non-locally connected continuum Y as the remainder. Then the continuum X is not locally connected, and since it is hereditarily unicoherent, it is unicoherent at each of its subcontinua. In particular, $Y \in \mathcal{U}(X)$, while Y is not locally connected by construction.

4. Relations to terminality

Since in this section we will use results established for metric continua, e.g. the ones from [15], all continua considered in the present section are assumed to be *metric*.

A subcontinuum Y of a continuum X is called *terminal in the sense of Wallace* (abbreviated *W-terminal*) provided that for each subcontinuum K of X the condition $K \cap Y \neq \emptyset$ implies $K \subset Y$ or $Y \subset K$. Note that the whole continuum X is a W-terminal subcontinuum of itself, and that each singleton is W-terminal. Put

$$\mathcal{T}(X) = \{Y \in C(X) : Y \text{ is a W-terminal subcontinuum of } X\}.$$

The reader is referred to [5, Chapter 4, Section 4D] for more information about this concept. Recall that in some papers (see for example [15] and [16]) the name of a *terminal* continuum is used in the same sense.

Theorem 4.1. *If a proper subcontinuum Y of a continuum X is W-terminal, then X is unicoherent at Y, i.e.,*

$$\mathcal{T}(X) \setminus \{X\} \subset \mathcal{U}(X). \tag{4.1.1}$$

Proof. Take two proper subcontinua A and B of X such that $A \cup B = X$. Then either A or B (or both) intersects Y. Without loss of generality, we may assume that $A \cap Y \neq \emptyset$. Then, by W-terminality of Y, we have either $A \subset Y$ or $Y \subset A$.

If $A \subset Y$, then $\emptyset \neq A \cap B \subset Y \cap B$, so B intersects Y, and again by W-terminality of Y, we have either $B \subset Y$ or $Y \subset B$. In the former case the two inclusions $A \subset Y$ and $B \subset Y$ imply $X = A \cup B \subset Y$, whence $Y = X$, contrary to the assumption. In the latter case we have $A \subset Y \subset B$, whence $A \cap B \cap Y = A$, so the intersection $A \cap B \cap Y$ is a continuum, as needed.

If $Y \subset A$, then $A \cap Y = Y$, whence $A \cap B \cap Y = B \cap Y$. If $B \cap Y = \emptyset$, the intersection $A \cap B \cap Y$ is empty, so connected, and therefore the conclusion follows. Otherwise, $B \cap Y \neq \emptyset$, and again either $B \subset Y$ or $Y \subset B$ by W-terminality of Y. The former inclusion gives $B \subset Y \subset A$, whence $A \cap B \cap Y = B$, so the intersection $A \cap B \cap Y$ is connected; the latter one, together with $Y \subset A$, lead to $A \cap B \cap Y = Y$, so we are done. The proof is complete. □

Remarks 4.2.

(a) *The assumption that Y is a proper subcontinuum of X is essential in* Theorem 4.1. *To verify this it is enough to note that, just by the definition of the unicoherence of X at $Y \subset X$,*

   (∗) *a continuum X is unicoherent at X if and only if X is unicoherent.*

(b) *The converse implication to that of* Theorem 4.1 *is not true, because if $Y = [\frac{1}{3}, \frac{2}{3}] \subset X = [0, 1]$, then X is hereditarily unicoherent, thus it is unicoherent at Y (because a hereditarily unicoherent continuum is unicoherent at each of its subcontinua, see [20, Proposition 1.2, p. 146]), and Y is not a W-terminal subcontinuum of X.*

The inclusion (4.1.1) can be a little bit strengthened under an additional assumption about the continuum X. Recall that a continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing x and for each sequence of points x_n converging to x there exists a sequence of subcontinua K_n of X containing x_n and converging to the continuum K (see e.g. [11, p. 167]).

Theorem 4.3. *Let a continuum X have the property of Kelley. If a proper subcontinuum Y of X is the limit of a sequence of W-terminal subcontinua of X, then X is unicoherent at Y, i.e.,*

$$\mathrm{cl}(\mathcal{T}(X)) \setminus \{X\} \subset \mathcal{U}(X). \tag{4.3.1}$$

Proof. Indeed, if X has the property of Kelley, then $\mathcal{T}(X)$ is a closed subset of $C(X)$, see [15, (1.2), p. 177]. Thus (4.3.1) follows from (4.1.1). □

It is known that if D is a locally compact, noncompact, separable metric space, then each continuum is a remainder of D in some compactification of D, [1, Theorem, p. 35]. Taking as D a ray we obtain the following proposition.

Proposition 4.4. *Each nondegenerate metric continuum Y is a remainder of a ray D in some compactification of D. Then $X = Y \cup D$ is a unicoherent continuum having Y as its W-terminal subcontinuum and D as an arc-component, with $Y = \mathrm{cl}(D) \setminus D$.*

As a consequence of *Theorem 4.1* and *Proposition 4.4* we get a corollary.

Corollary 4.5. *For each metric continuum Y any compactification X of a ray having Y as the remainder is a unicoherent continuum that is unicoherent at Y.*

The property of Kelley is an essential assumption in *Theorem 4.3*. This can be seen by the following example.

Example 4.6. *There exist a metric continuum X without the property of Kelley, a (non-W-terminal) subcontinuum Y of X such that X is not unicoherent at Y, and a sequence of W-terminal subcontinua Y_n of X such that $Y = \mathrm{Lim}\, Y_n$.*

Proof. Let S' be the ray defined by (2.9.1). Denote by S'' the copy of S' under the symmetry with respect to the line $x = 1$. Thus the union $S = S' \cup S''$ is a one-to-one image of the real line lying in the rectangle $[0, 2] \times [-1, 1]$ and having the union of the sides $L = \{0\} \times [-1, 1]$ and $R = \{2\} \times [-1, 1]$ of the rectangle as the remainder in its natural compactification.

For each $n \in \mathbb{N}$ put $Y_n = \{\frac{1}{n}\} \times [-1, 1]$ and let S_n be a homeomorphic copy of S located between Y_{n+1} and Y_n so that Y_{n+1} is its left limit segment (that corresponds to L) and Y_n is its right limit segment (that corresponds to R). Further, put $Y = \{0\} \times [-1, 1] = \mathrm{Lim}\, Y_n$ and $C = Y \cup (\{-1\} \times [-1, 1]) \cup ([-1, 0] \times \{-1, 1\}$. Finally define

$$X = C \cup \bigcup\{(Y_n \cup S_n) : n \in \mathbb{N}\}.$$

Then, by construction, X is a continuum without the property of Kelley, Y is its non-W-terminal subcontinuum, and each Y_n is a W-terminal subcontinuum of X. Putting $A = Y \cup \bigcup\{(Y_n \cup S_n) : n \in \mathbb{N}\}$ and $B = (\{-1\} \times [-1, 1]) \cup ([-1, 0] \times \{-1, 1\}$ we see that $B = \mathrm{cl}(C \setminus Y)$, whence $X = A \cup B$, and that $A \cap B \cap Y = \{(0, -1), (0, 1)\}$. Thus X is not unicoherent at Y. □

Remark 4.7. *Note that* Example 4.6 *also shows that terminality of Y is an essential assumption in* Theorem 4.1.

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