Quartics in E^3 which have a triple point and touch the plane at infinity through the absolute conic

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Abstract. *This paper gives the classification of the 4th order surfaces in E^3 which have a triple point and touch the plane at infinity at the absolute conic. The classification is made according to the type of the tangent cubic cone at a triple point. Three types with sixteen subtypes are obtained. For these surfaces the homogeneous and parametric equations are derived and each type is illustrated with* Mathematica *graphics.*

Key words: *quartic, triple point*

AMS subject classifications: 51N20, 51N35

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1. Introduction

In the real three-dimensional projective space $P^3(\mathbb{R})$, in homogeneous Cartesian coordinates $(x:y:z:w) \neq (0:0:0:0)$, $(x,y,z \in \mathbb{R}, w \in \{0,1\})$, equation

$$F_n(x,y,z,w) = 0, \tag{1}$$

where F_n is a homogeneous algebraic polynomial of degree n, defines a two-dimensional extent of points Φ_n, which is called an nth order surface.

According to [8, p. 268], we can also use the following notation

$$u_n + u_{n-1}w + ... + u_{n-i}w^i + ... + u_1 w^{n-1} + u_0 w^n = 0, \tag{2}$$

where $u_j, j \in \{0,1,...,n\}$ are homogeneous polynomials in x, y and z of degree j.

Some properties of surfaces Φ_n, according to [8], [9], [13], [5], are the following:

- Any straight line meets surface Φ_n at n points or lies entirely on the surface. Any plane cuts surface Φ_n into the nth order plane curve.
- If the polynomial F_n can be factorized

  $$F_n(x,y,z,w) = F_{n_1}(x,y,z,w) \cdot ... \cdot F_{n_k}(x,y,z,w), \quad n_1 + ... + n_k = n, \tag{3}$$

  surface Φ_n splits into the surfaces Φ_{n_1},...,Φ_{n_k}.

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- Every homogeneous equation in x, y and z:

$$F_n(x, y, z) = 0 \quad \texttt{or} \quad u_n = 0, \tag{4}$$

represents the nth order cone whose vertex is the origin $(0 : 0 : 0 : 1)$

- Point $T = (x_T : y_T : z_T : w_T)$ which satisfies conditions

$$F_n(x_T, y_T, z_T, w_T) = 0,$$

$$\frac{\partial F_n}{\partial x}(T) \neq 0 \vee \frac{\partial F_n}{\partial y}(T) \neq 0 \vee \frac{\partial F_n}{\partial z}(T) \neq 0 \vee \frac{\partial F_n}{\partial w}(T) \neq 0, \tag{5}$$

is called the regular point of surface Φ_n. All tangent lines to a surface at its regular point form a pencil of lines (T) in the tangent plane which is given by the following equation:

$$(x - x_T)\frac{\partial F_n}{\partial x}(T) + (y - y_T)\frac{\partial F_n}{\partial y}(T) + (z - z_T)\frac{\partial F_n}{\partial z}(T) + (w - w_T)\frac{\partial F_n}{\partial w}(T) = 0. \tag{6}$$

If the origin $O = (0 : 0 : 0 : 1)$ is the regular point of surface Φ_n, then

$$u_n + ... + u_1 w^{n-1} = 0 \tag{7}$$

is its equation, and

$$u_1 = 0 \tag{8}$$

is the equation of the tangent plane of surface Φ_n at the origin O.

- Point $S = (x_S : y_S : z_S : w_S)$ which satisfies conditions

$$F_n(x_S, y_S, z_S, w_S) = 0,$$

$$\frac{\partial F_n}{\partial x}(S) = \frac{\partial F_n}{\partial y}(S) = \frac{\partial F_n}{\partial z}(S) = \frac{\partial F_n}{\partial w}(S) = 0, \tag{9}$$

is called the singular point of surface Φ_n. All tangent lines to a surface at its singular point form an algebraic cone (called the tangent cone) with the vertex S. If $k(1 < k < n)$ is the order of a tangent cone, a singular point S is the k-ple point of surface Φ_n.

If the origin $O = (0 : 0 : 0 : 1)$ is the k-ple point of surface Φ_n, then

$$u_n + ... + u_k w^{n-k} = 0 \tag{10}$$

is its equation, and

$$u_k = 0 \tag{11}$$

is the equation of a tangent cone at a singular point O.

2. Quartics with a triple point

In this paper we use the term *"quartic"* for the 4th order algebraic surfaces. Quartics with singular lines can be classified ([9, pp. 200-252], [6, pp. 1537-1787]) in the following way:

- quartics with a triple straight line (the class contains only ruled quartics);
- quartics with a double twisted cubic (the class contains only ruled quartics);
- quartics with a double conic section (the class contains cyclides);
- quartics with a double conic section and a double line (the class contains only ruled quartics);
- quartics with three double lines (the class contains ruled and STEINER'S quartics);
- quartics with two double straight lines (the class contains only ruled quartics);
- quartics with one double straight line (the class contains ruled quartics and the surfaces considered in [2] and [3]).

In addition to quartics with singular lines, there are quartics with isolated singularities [9, pp. 238-251]:

- quartics with one triple point;
- quartics with double points (at most sixteen double points - conical points, binodes or unodes).

Quartics with a triple point are studied in detail by ROHN [7]. If the origin $O = (0:0:0:1)$ is the triple point of quartic Φ_4, it is defined by the following equation:

$$u_4 + wu_3 = 0, \tag{12}$$

where u_4, u_3 are homogeneous polynomials in x, y and z with degrees 4 and 3, respectively. $u_3 = 0$ is the equation of the tangent cone at the triple point O and $u_4 = 0, w = 0$ are equations of the curve at infinity.

Some properties of these surfaces according to [7] are:

- There is only one triple point on surface Φ_4. (Quartics with two triple points must possess a singular line, which joins triple points.)
- There are 12 straight lines $(g_1, g_2, ..., g_{12})$ through the triple point, which entirely lie on surface Φ_4. They are the intersection of the cones which are given by equations $u_4 = 0$ and $u_3 =$. Some of those lines can coincide or be imaginary in pairs.
- There are $66 = \binom{12}{2}$ planes (determined by the pairs of lines g_i) which cut surface Φ_4 into the conics through the triple point.

- There are $792 = \binom{12}{5}$ 2-order cones (each cone is determined by five lines g_i) which cut surface Φ_4 into the cubics through the triple point.
- In addition to the triple point, surface Φ_4 can possess at most 6 real double points. Those points lie on coinciding lines g_i. Double points can be of type C_2 (conical points) or B_k (binodes), where k is the number of coinciding lines g_i. The points of type U (unodes) exist only in the case when coinciding lines g_i are the singular generator of the tangent cone ($u_3 = 0$) at the triple point.

3. Classification of cubic cones

According to the Newton's classification of plane cubics [10, pp. 162-179], [11, pp. 51-61], there are five types of *divergent parabolas*, which in Cartesian coordinates (x, y), $(x, y \in \mathbb{R})$, can be represented by the equation

$$y^2 = ax^3 + bx^2 + cx + d. \tag{13}$$

The classification of those parabolas corresponds to the roots of the following equation:

$$ax^3 + bx^2 + cx + d = 0. \tag{14}$$

1. If equation (14) has three real and different roots, then a curve has an oval and a parabolic branch (*Figure 1.1*).
2. If equation (14) has one real and two imaginary roots, then a curve has a parabolic branch (*Figure 1.2*).
3. If equation (14) has two equal real roots which are greater than another real root, then a curve has a self-intersecting parabolic branch (crunodal cubic) (*Figure 1.3*).
4. If equation (14) has two equal real roots which are less than another real root, then a curve has a parabolic branch and an isolated singular point (acnodal cubic) (*Figure 1.4*).
5. If equation (14) has three equal real roots, then a curve has a parabolic branch with a cusp (cuspidal cubic) (*Figure 1.5*).

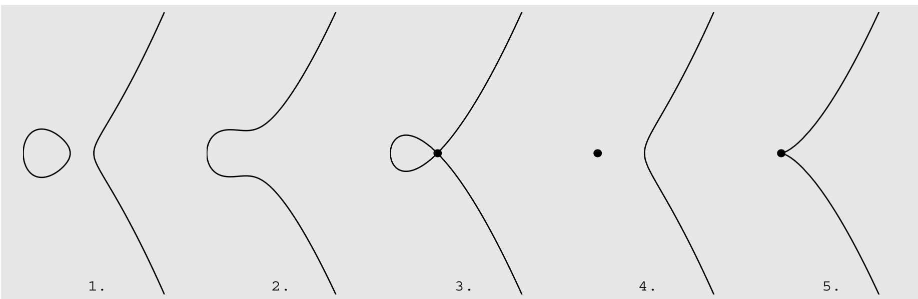

Figure 1.

According to the NEWTON'S theorem [10, p. 163], every cubic may be projected into one of the five divergent parabolas. Therefore, every cubic cone can be regarded as the system of lines which join its vertex with points of a divergent parabola and its equation may be brought to the following form:

$$zy^2 = ax^3 + bx^2z + cxz^2 + dz^3. \tag{15}$$

Now we have the following classification of cubic cones given by equation (15):

1. If equation (14) has three real and different roots, then a cone has a twin-pair sheet and a single sheet (*Figure 2.1*).
2. If equation (14) has one real and two imaginary roots, then a cone has a single sheet only (*Figure 2.2*)
3. If equation (14) has two equal real roots which are greater than another real root, then a cone has a crunodal singular generator - a crunodal cubic cone (*Figure 2.3*).
4. If equation (14) has two equal real roots which are less than another real root, then a cone has an acnodal singular generator - an acnodal cubic cone (*Figure 2.4*).
5. If equation (14) has three equal real roots, then a cone has a cuspidal singular generator - a cuspidal cubic cone (*Figure 2.5*).

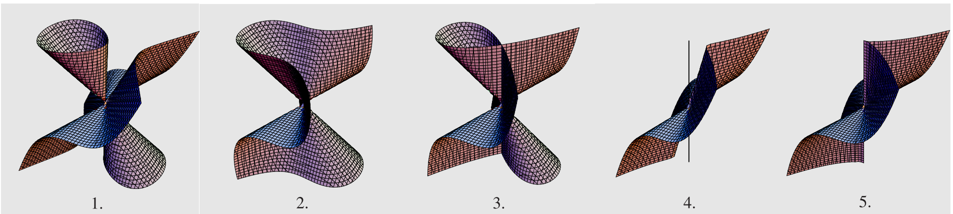

Figure 2.

4. Quartics in E^3 which have a triple point and touch the plane at infinity through the absolute conic

In real projective space $P^3(\mathbb{R})$ the Euclidean metric defines the absolute conic in the plane at infinity and it is given by the formulas: $x^2 + y^2 + z^2 = 0$, $w = 0$.

Theorem 1. *In the 3-dimensional Euclidean space the following equation*

$$F(x, y, z, w) = (x^2 + y^2 + z^2)^2 + wu_3 = 0, \tag{16}$$

where $u_3 \neq (x^2+y^2+z^2)u_1$*, and* u_3*,* u_1 *are homogeneous polynomials in* x*,* y *and* z *of degree 3 and 1 respectively, defines a quartic which has a triple point and touches the plane at infinity through the absolute conic.*

Proof. Since $u_3 \neq (x^2 + y^2 + z^2)u_1$, the polynomial $F(x, y, z, w)$ is irreducible over the field $\mathbb{R}$, then surface Φ_4 (defined by equation (16)) is a proper quartic. (If

$u_3 = (x^2 + y^2 + z^2)u_1$, a quartic splits into the isotropic cone and a sphere through the origin.)

According to equations (10) and (11), equation (16) defines a quartic with a triple point at the origin and the equation $u_3 = 0$ defines a tangent cone at a triple point.

In the plane at infinity quartic Φ_4 has the absolute conic $((x^2 + y^2 + z^2)^2 = 0, w = 0)$ counted twice. It is not a singular line, since for the points of the absolute conic it holds that $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$, $\frac{\partial F}{\partial w} = u_3 \neq 0$, because we assume $u_3 \neq (x^2 + y^2 + z^2)u_1$. Therefore, according to equation (6), the tangent plane at points on the absolute conic is defined by the equation $w = 0$. □

About straight lines and double points on surfaces Φ_4

On surface Φ_4 there are 3 pairs of coinciding isotropic lines which lie in 3 planes. They are the intersection of the isotropic cone and the 3-degree tangent cone with vertex O $(x^2 + y^2 + z^2 = 0,\ u_3 = 0)$. Since double points always lie on coinciding lines, we can conclude that there are no real double points on the surfaces given by equation (16).

4.1. Classification of surfaces Φ_4

According to the type of the tangent cone $\mathcal{T}_3$ $(u_3 = 0)$ at the triple point, surfaces given by equation (16) are classified in the following way:

Type I $\mathcal{T}$ is a proper 3-order cone.

- $\mathbf{I}_1$ $\mathcal{T}$ has a twin-pair sheet and a single sheet (*Figure 3*).
- $\mathbf{I}_2$ $\mathcal{T}$ has a single sheet only (*Figure 4*).
- $\mathbf{I}_3$ $\mathcal{T}$ is a crunodal cubic cone (*Figure 5*).
- $\mathbf{I}_4$ $\mathcal{T}$ is an acnodal cubic cone (*Figure 6*).
- $\mathbf{I}_5$ $\mathcal{T}$ a cuspidal cubic cone (*Figure 7*).

Type II $\mathcal{T}$ splits into a 2-order cone $\mathcal{C}$ and a real plane $\mathcal{P}$.

- $\mathbf{II}_1$ $\mathcal{C}$ is a real cone and $\mathcal{P}$ cuts it into two real and different lines (*Figure 8*).
- $\mathbf{II}_2$ $\mathcal{C}$ is a real cone and $\mathcal{P}$ is its tangent plane (*Figure 9*).
- $\mathbf{II}_3$ $\mathcal{C}$ is a real cone and $\mathcal{P}$ cuts it into a pair of imaginary lines (*Figure 10*).
- $\mathbf{II}_4$ $\mathcal{C}$ is an imaginary cone (*Figure 11*).

Type III $\mathcal{T}$ splits into three planes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$.

- $\mathbf{III}_1$ $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are real and different planes with one common point (*Figure 12*).
- $\mathbf{III}_2$ $\mathcal{P}_1$ is real and $\mathcal{P}_2$, $\mathcal{P}_3$ are a pair of imaginary planes and they have one common point (*Figure 13*).
- $\mathbf{III}_3$ $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are real and diferent planes with one common line (*Figure 14*).

III$_4$ $\mathcal{P}_1$ is real and $\mathcal{P}_2$, $\mathcal{P}_3$ are a pair of imaginary planes and they have one common line (*Figure 15*).

III$_5$ Two of the planes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ coincide (*Figure 16*).

III$_6$ The planes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ coincide (*Figure 17*).

4.2. Parametric equations of surfaces Φ_4

For $w = 1$ equation (16) takes the following form:

$$u_4 + u_3 = 0. \tag{17}$$

If we write $u_3 = T_3(x, y, z)$ and use spherical coordinates (ρ, u, v):

$$x = \rho \cos u \sin v, \qquad y = \rho \sin u \sin v, \qquad z = \rho \cos v,$$

equation (17) takes the form $\rho^3(\rho + T_3(\cos u \sin v, \sin u \sin v, \cos v))$.

For every point on surface Φ_4, which is not its triple point O ($\rho = 0$), the following relation holds:

$$\rho = -T_3(\cos u \sin v, \sin u \sin v, \cos v). \tag{18}$$

Therefore, the parametric equations of surface Φ_4 are:

$$\begin{aligned} x(u,v) &= -T_3(\cos u \sin v, \sin u \sin v, \cos v) \cos u \sin v \\ y(u,v) &= -T_3(\cos u \sin v, \sin u \sin v, \cos v) \sin u \sin v \\ z(u,v) &= -T_3(\cos u \sin v, \sin u \sin v, \cos v) \cos v, \qquad u, v \in [0,\pi] \times [0,\pi]. \end{aligned} \tag{19}$$

In a general case ten real numbers define the tangent cone ($T_3(x, y, z) = 0$) at triple point $O(0:0:0:1)$.

$$\begin{aligned} T_3(x,y,z) &= a_1x^3 + a_2x^2y + a_3x^2z + a_4xy^2 + a_5xz^2 + a_6xyz + a_7y^3 \\ &\quad + a_8y^2z + a_9yz^2 + a_{10}z^3, \qquad a_i \in \mathbb{R}, \exists a_i \neq 0. \end{aligned} \tag{20}$$

4.3. Drawing of surfaces Φ_4 with *Mathematica*

The following *Mathematica* graphics have been created by using formula (20) and parametric equations (19).

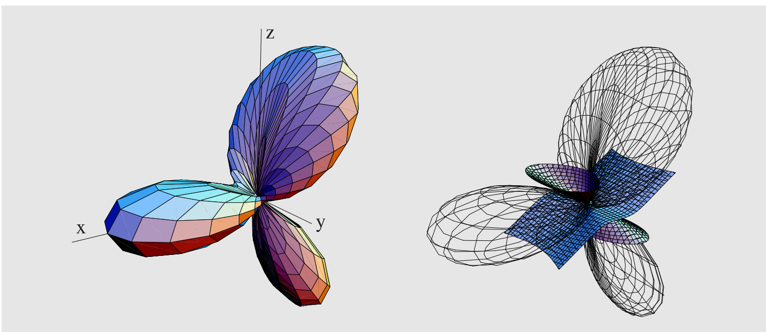

Figure 3. *An example of the type* $\mathbf{I_1}$ $\quad (x^2 + y^2 + z^2)^2 + (-x^3 + 3xz^2 - y^2z)w = 0$

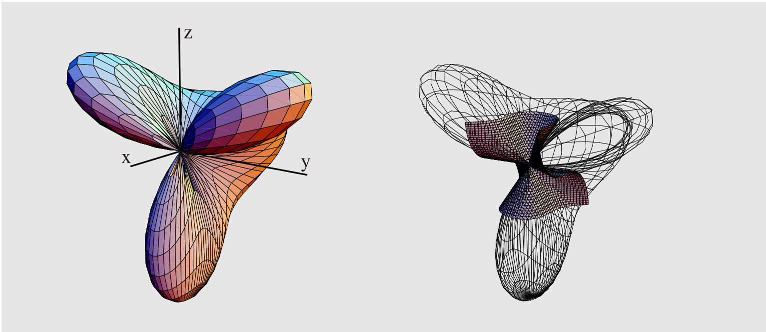

Figure 4. *An example of the type* $\mathbf{I_2}$ $(x^2+y^2+z^2)^2+(x^3-3y^2z+z^3)w=0$

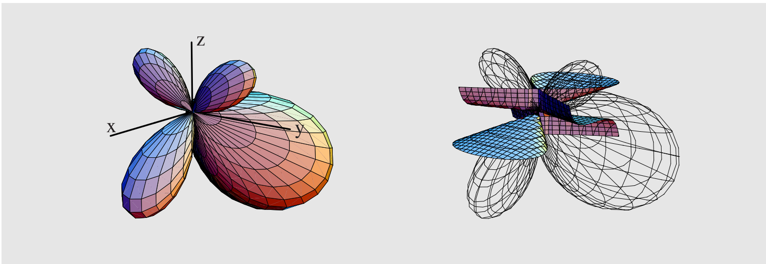

Figure 5. *An example of the type* $\mathbf{I_3}$ $(x^2+y^2+z^2)^2+(x^3+4x^2z-2y^2z)w=0.$

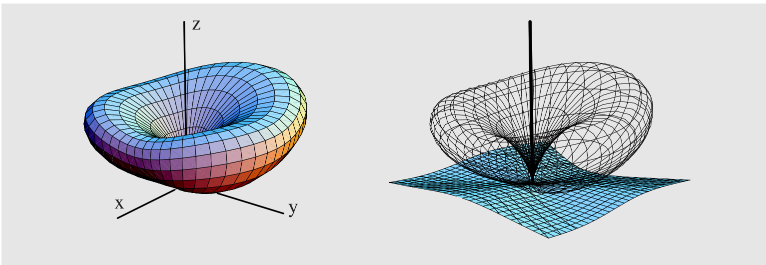

Figure 6. *An example of the type* $\mathbf{I_4}$ $(x^2+y^2+z^2)^2+(x^3-5x^2z-5y^2z)w=0$

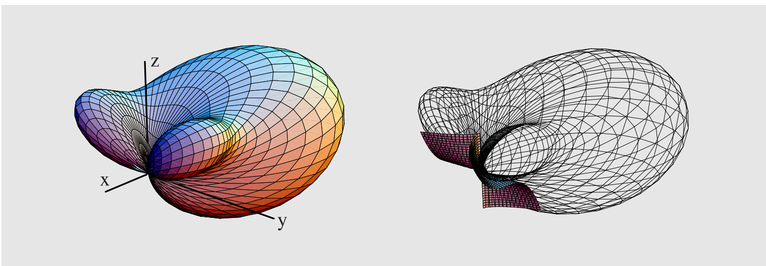

Figure 7. *An example of the type* $\mathbf{I_5}$ $(x^2+y^2+z^2)^2+(x^3-y^2z)w=0$

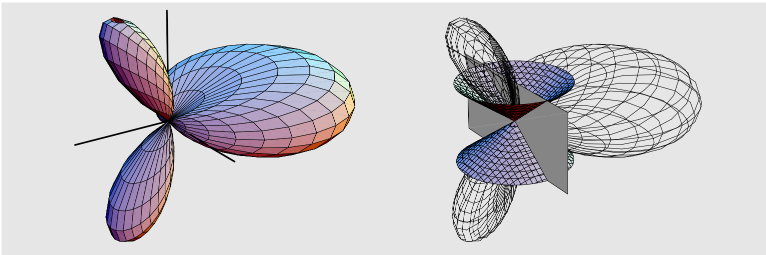

Figure 8. *An example of the type* $\mathbf{II_1}$ $(x^2+y^2+z^2)^2+x(x^2+y^2-2z^2)w=0$

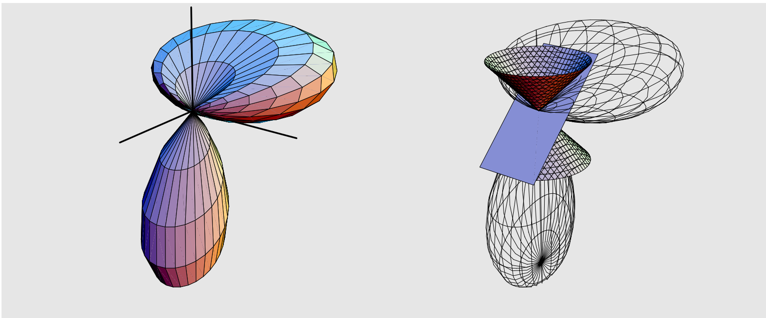

Figure 9. *An example of the type* $\mathbf{II_2}$ $(x^2+y^2+z^2)^2+(x-z)(x^2+y^2-z^2)w=0$

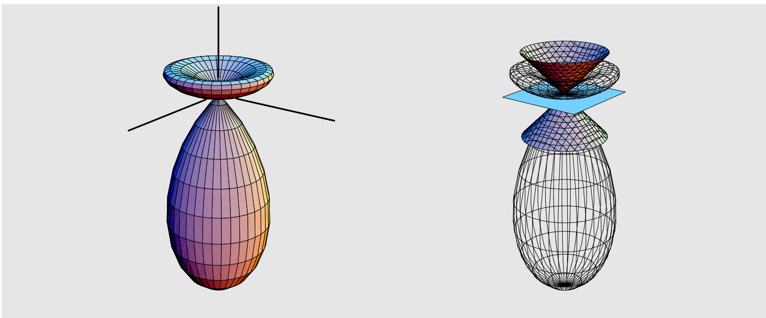

Figure 10. *An example of the type* $\mathbf{II_3}$ $(x^2+y^2+z^2)^2-z(x^2+y^2-z^2)w=0$

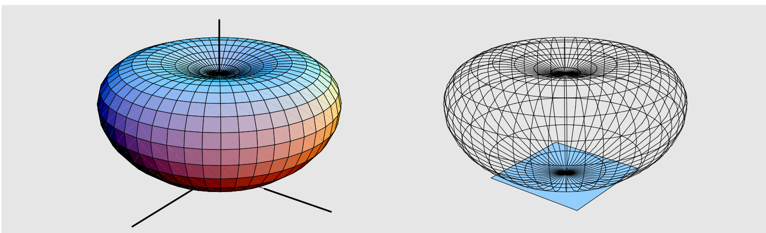

Figure 11. *An example of the type* $\mathbf{II_4}$ $(x^2+y^2+z^2)^2-z(3x^2+3y^2+z^2)w=0$

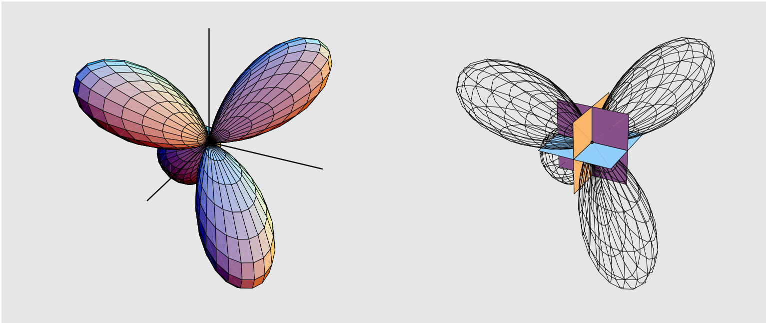

Figure 12. *An example of the type* $\mathbf{III_1}$ $(x^2+y^2+z^2)^2+xyzw=0$

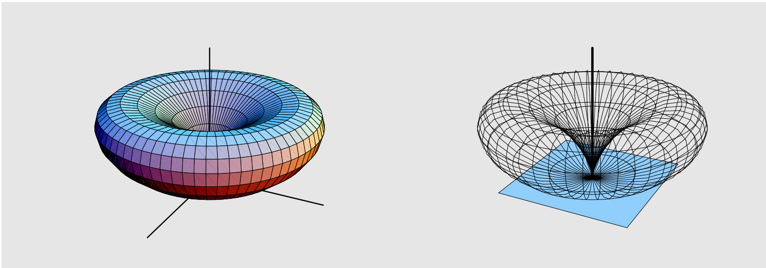

Figure 13. *An example of the type* $\mathbf{III_2}$ $(x^2+y^2+z^2)^2-z(x^2+y^2)w=0$

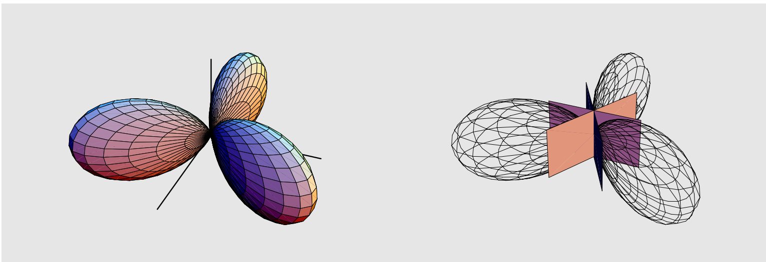

Figure 14. *An example of the type* $\mathbf{III_3}$ $(x^2+y^2+z^2)^2+x(x^2-3y^2)w=0$

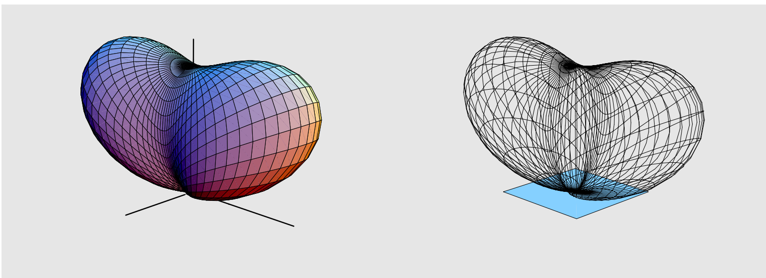

Figure 15. *An example of the type* $\mathbf{III_4}$ $(x^2+y^2+z^2)^2-z(3y^2+z^2)w=0$

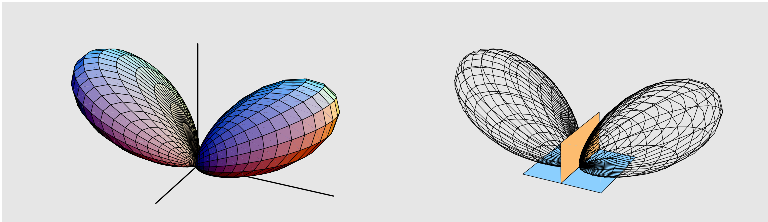

Figure 16. *An example of the type* $\mathbf{III_5}$ $(x^2+y^2+z^2)^2 - y^2zw = 0$

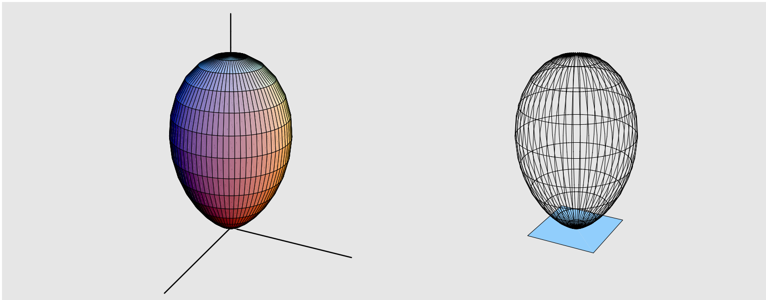

Figure 17. *An example of the type* $\mathbf{III_6}$ $(x^2+y^2+z^2)^2 - z^3w = 0$

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