A note on the root subspaces of real semisimple Lie algebras

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Abstract. *In this note we prove that for any two restricted roots* α, β *of a real semisimple Lie algebra* $\mathfrak{g}$, *such that* $\alpha+\beta\neq 0$, *the corresponding root subspaces satisfy* $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$.

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Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and R the (restricted) root system of the pair $(\mathfrak{g},\mathfrak{a})$ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. For $\alpha\in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root subspace of $\mathfrak{g}$:

$$\mathfrak{g}_\alpha=\{x\in\mathfrak{g};\ [h,x]=\alpha(h)x\ \ \forall h\in\mathfrak{a}\}.$$

The aim of this note is to prove the following theorem:

Theorem. *Let* $\alpha,\beta\in R$ *be such that* $\alpha+\beta\neq 0$. *Then either* $[x,\mathfrak{g}_\alpha]=\mathfrak{g}_{\alpha+\beta}$ $\forall x\in\mathfrak{g}_\beta\setminus\{0\}$ *or* $[x,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$ $\forall x\in\mathfrak{g}_\alpha\setminus\{0\}$.

Although the proof is very simple and elementary, the assertion does not seem to appear anywhere in the literature. The argument for the proof is from [2], where it is used to prove $[\mathfrak{g}_\alpha,\mathfrak{g}_\alpha]=\mathfrak{g}_{2\alpha}$ (a fact which is also proved in [3], 8.10.12), as well as that the nilpotent constituent in an Iwasawa decomposition is generated by the root subspaces corresponding to simple roots.

Let B be the Killing form of $\mathfrak{g}$:

$$B(x,y)=\mathrm{tr}\,(\mathrm{ad}\,x\,\mathrm{ad}\,y),\qquad x,y\in\mathfrak{g}.$$

Choose a Cartan involution ϑ of $\mathfrak{g}$ in accordance with $\mathfrak{a}$, i.e. such that $\vartheta(h)=-h$, $\forall h\in\mathfrak{a}$. Denote by $(\cdot|\cdot)$ the inner product on $\mathfrak{g}$ defined by

$$(x|y)=-B(x,\vartheta(y)),\qquad x,y\in\mathfrak{g}.$$

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We shall use the same notation $(\cdot|\cdot)$ for the induced inner product on the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$. Let $\|\cdot\|$ denote the corresponding norms on $\mathfrak{g}$ and on $\mathfrak{a}^*$. For $\alpha \in R$ let h_α be the unique element of $\mathfrak{a}$ such that

$$B(h, h_\alpha) = \alpha(h) \qquad \forall h \in \mathfrak{a}.$$

Lemma. *Let $\alpha, \beta \in R$ be such that $(\alpha|\alpha+\beta) > 0$. Then*

$$[x, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta} \qquad \forall x \in \mathfrak{g}_\alpha \setminus \{0\}.$$

Proof. Take $x \in \mathfrak{g}_\alpha$, $x \neq 0$. We can suppose that $\|x\|^2\|\alpha\|^2 = 2$. Put

$$h = \frac{2}{\|\alpha\|^2} h_\alpha \qquad \text{and} \qquad y = -\vartheta(x).$$

Then

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h$$

([3], 8.10.12). Therefore, the subspace $\mathfrak{s}$ of $\mathfrak{g}$ spanned by $\{x, y, h\}$ is a simple Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. From the representation theory of $\mathfrak{sl}(2, \mathbb{R})$ ([1],1.8) we know that if π is any representation of $\mathfrak{s}$ on a real finite dimensional vector space V, then $\pi(h)$ is diagonalizable, all eigenvalues of the operator $\pi(h)$ are integers, and if for $n \in \mathbb{Z}$ V_n denotes the $n-$eigenspace of $\pi(h)$, then

$$n \geq -1 \qquad \Longrightarrow \qquad \pi(x)V_n = V_{n+2}.$$

Put

$$V = \sum_{j \in \mathbb{Z}} \mathfrak{g}_{\beta + j\alpha}.$$

Then V is an $\mathfrak{s}-$module for the adjoint action and

$$\mathfrak{g}_{\beta+j\alpha} = V_{n+2j} \qquad \text{where} \qquad n = 2\frac{(\beta|\alpha)}{\|\alpha\|^2} \in \mathbb{Z}.$$

Especially,

$$V_n = \mathfrak{g}_\beta, \qquad V_{n+2} = \mathfrak{g}_{\alpha+\beta}.$$

Now

$$n + 2 = 2\frac{(\alpha|\alpha+\beta)}{\|\alpha\|^2} > 0 \quad \Longrightarrow \quad n \geq -1 \quad \Longrightarrow \quad (\mathrm{ad}\, x)V_n = V_{n+2}.$$

□

Proof of Theorem. It is enough to notice that if $\alpha + \beta \neq 0$ then

$$0 < (\alpha+\beta|\alpha+\beta) = (\alpha|\alpha+\beta) + (\beta|\alpha+\beta),$$

hence, either $(\alpha|\alpha+\beta) > 0$ or $(\beta|\alpha+\beta) > 0$. □

Let m_α denote the multiplicity of $\alpha \in R$ ($m_\alpha = \dim\, \mathfrak{g}_\alpha$). An immediate consequence of the Theorem is:

Corollary. *If $\alpha, \beta \in R$, $\alpha + \beta \neq 0$, then $m_{\alpha+\beta} \leq \max(m_\alpha, m_\beta)$.*

References

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