Some diophantine quadruples in the ring $\mathbf{Z}[\sqrt{-2}]$

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Abstract. *A complex diophantine quadruple with the property $D(z)$, where $z \in Z[\sqrt{-2}]$, is a subset of $Z[\sqrt{-2}]$ of four elements such that the product of its any two distinct elements increased by z is a perfect square in $Z[\sqrt{-2}]$. In the present paper we prove that if b is an odd integer, then there does not exist a diophantine quadruple with the property $D(a+b\sqrt{-2})$. For $z = a+b\sqrt{-2}$, where b is even, we prove that there exist at least two distinct complex diophantine quadruples if a and b satisfy some congruence conditions.*

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1. Introduction

Let n be an integer. A set of positive integers $\{a_i : i = 1, \ldots, m\}$ is said to have the property $D(n)$, if $a_i a_j + n$ is a perfect square for all $1 \leq i \neq j \leq m$. Such a set is called a diophantine m-tuple. It is interesting to point out that although there are infinitely many diophantine quadruples, no algorithm for generating all of them has been found. For more details see [2] and [3].

In 1997, A. Dujella [6] extended the definition of the diophantine quadruples to the ring $\mathrm{Z}[\sqrt{-2}]$. He defined the set $\{a_i + b_i\sqrt{-1} : i = 1,2,3,4\}$ of non-zero elements of $\mathrm{Z}[\sqrt{-2}]$ to be with the property $\mathrm{D}(a+b\sqrt{-1})$ if the product of its any two distinct elements increased by $a+b\sqrt{-1}$ is a square of an element in the ring $\mathrm{Z}[\sqrt{-2}]$. Such a set is called a complex diophantine quadruple. He proved that if b is odd or $a \equiv b \equiv 2 (\mathrm{mod}\, 4)$, then there does not exist a complex diophantine quadruple, and if $z = a+b\sqrt{-1}$ is not of that form and $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exist at least two distinct complex diophantine quadruples with the property $\mathrm{D}(z)$.

In this paper we consider an analogous problem for the ring $\mathrm{Z}[\sqrt{-2}]$. We prove that for any complex diophantine quadruple with the property $\mathrm{D}(a+b\sqrt{-2})$ b should be even and then we consider many different forms for the element z in the

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ring $Z[\sqrt{-2}]$ and prove that for these forms there are at least two distinct complex diophantine quadruples with the property $D(z)$. To prove the theorems we solve many polynomial equations of fourth degree and use software (Matlab -5.3).

2. Auxiliary lemmas

We start by stating some known results, which will be used later on.

Lemma 1 [1, 7, 8]. *If n is an integer of the form $4k+2$, $k \in Z$, then there is no set of four natural numbers with the property $D(n)$.*

Lemma 2 [5, Theorem 5]. *If an integer n does not have the form $4k+2$ and $n \notin S = \{3, 5, 8, 12, 20, -1, -3, -4\}$, then there exists a set of four natural numbers with the property $D(n)$.*

Lemma 3 [4, 5]. *If an integer n does not have the form $4k+2$ and $n \notin S \cup T$ where $T = \{7, 13, 15, 21, 24, 28, 32, 48, 60, 84, -7, -12, -15\}$, then there exist at least two different sets of four natural numbers with the property $D(n)$.*

The proof of the last two lemmas depends on the fact that the sets

$$\{m,\, m(3k+1)^2 + 2k,\, m(3k+2)^2 + 2k + 2,\, 9m(2k+1)^2 + 8k + 4\} \tag{1}$$

$$\{m,\, mk^2 - 2k - 2,\, m(k+1)^2 - 2k,\, m(2k+1)^2 - 8k - 4\} \tag{2}$$

have the property $D(4km + 2m + 1)$. In the proof of our results we will use the following formulas which are direct consequences of formulas (1) and (2). The two sets

$$\{4,\, 9k^2 - 5k,\, 9k^2 + 7k + 2,\, 36k^2 + 4k\} \tag{3}$$

$$\{4,\, k^2 - 3k,\, k^2 + k + 2,\, 4k^2 - 4k\} \tag{4}$$

have the property $D(8k+1)$, the two sets

$$\{2,\, 18k^2 + 14k + 2,\, 18k^2 + 26k + 10,\, 72k^2 + 80k + 22\} \tag{5}$$

$$\{2,\, 2k^2 - 2k - 2,\, 2k^2 + 2k + 2,\, 8k^2 - 2\} \tag{6}$$

have the property $D(8k+5)$, the two sets

$$\{1,\, 9k^2 - 8k,\, 9k^2 - 2k + 1,\, 36k^2 - 20k + 1\} \tag{7}$$

$$\{1,\, k^2 - 6k + 1,\, k^2 - 4k + 4, 4k^2 - 20k + 9\} \tag{8}$$

have the property $D(8k)$, and the two sets

$$\{4,\, 9k^2 - 4k - 1,\, 9k^2 + 8k + 3,\, 36k^2 + 8k\} \tag{9}$$

$$\{4,\, k^2 - 4k - 1,\, k^2 + 3,\, 4k^2 - 8k\} \tag{10}$$

have the property $D(16k+4)$.

3. The case when b is odd

Proposition 1. *If b is an odd integer, then there does not exist a complex diophantine quadruple with the property $D(a+b\sqrt{-2})$.*

Proof. Suppose that b is an odd integer and the set $\{a_i+b_i\sqrt{-2} : i=1,2,3,4\}$ has the property $D(a+b\sqrt{-2})$. Then for all $i \neq j$ there exist c_{ij} and d_{ij} such that

$$a_ia_j - 2b_ib_j + a + (a_ib_j + a_jb_i + b)\sqrt{-2} = c_{ij}^2 - 2d_{ij}^2 + 2c_{ij}d_{ij}\sqrt{-2}$$

so $a_ib_j + a_jb_i \equiv 1 (\mathrm{mod}\, 2)$. That is, at most one of a_j is even. Assume that a_1, a_2 and a_3 are odd. Then

$$b_1 + b_2 \equiv 1 (\mathrm{mod}\ 2),\ b_2 + b_3 \equiv 1 (\mathrm{mod}\ 2),\ b_1 + b_3 \equiv 1 (\mathrm{mod}\, 2).$$

This implies that

$$2(b_1 + b_2 + b_3) \equiv 3 (\mathrm{mod}\, 2),$$

which is a contradiction. □

4. The case when b is even

From *Proposition 1* we get that for any complex diophantine quadruple with the property $D(a+b\sqrt{-2})$, b should be even. Now we prove some results by taking different forms for a and b. We will consider sets $\{a_1, a_2, a_3, a_4\}$ and $\{-a_1, -a_2, -a_3, -a_4\}$ as one diophantine quadruple.

Theorem 1. *If $z = (4a+3) + 4b\sqrt{-2}$ and $z \notin \{-1, 3, 7\}$, then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.*

Proof. The sets

$$\begin{aligned} A &= \Big\{1,\ (9a^2+8a-18b^2+1) + (18ab+8b)\sqrt{-2}, \\ &\quad (9a^2+14a-18b^2+6) + (18ab+14b)\sqrt{-2}, \\ &\quad (36a^2+44a-72b^2+13) + (72ab+44b)\sqrt{-2}\Big\} \\ B &= \Big\{1,\ (a^2-2a-2b^2-2) + (2ab-2b)\sqrt{-2},\ (a^2-2b^2+1) + 2ab\sqrt{-2}, \\ &\quad (4a^2-4a-8b^2-3) + (8ab-4b)\sqrt{-2}\Big\} \end{aligned}$$

have the property $D((4a+3)+4b\sqrt{-2})$. These sets are obtained from formulas (1) and (2) where $m=1$ and $k = a+b\sqrt{-2}$. It remains to determine the pairs (a,b) for which the above sets have at least two equal elements or some elements equal to zero, and the pairs for which the corresponding sets are coincident. It is easy to check that the above cases appear iff

$$(a,b) \in \{(0,0),\ (2,0),\ (1,0),\ (-1,0),\ (3,0)\}.$$

Consequently, for all complex numbers of the form $z = (4a+3) + 4b\sqrt{-2}$ and $z \notin \{-1, 3, 7, 11, 15\}$ there exist at least two distinct diophantine quadruples satisfying the property $D(z)$. But from *Lemma 2* we see that there exist at least two (positive integer) diophantine quadruples with the property D(11). And since the two sets $\{1,\ 106,\ 129,\ 469\}$ and $\left\{1,\ -19+8\sqrt{-2},\ -22+4\sqrt{-2},\ -83+24\sqrt{-2}\right\}$ have the property D(15), we conclude that $z \notin \{-1,\ 3,\ 7\}$. □

Remark 1. *If* $\{a_1, a_2, ..., a_m\} \subset Z[\sqrt{-2}]$ *has the property* $D(z)$*, then*

1. *The set* $\{z_1a_1, a_2z_1, ..., a_mz_1\}$*has the property* $D(zz_1^2)$ *where* $z_1 \in Z[\sqrt{-2}]$.
2. *The set* $\{\bar{a}_1, \bar{a}_2, ..., \bar{a}_m\}$*has the property* $D(\bar{z})$.

Corollary 1. *If* $z = (8c+2) + 8d\sqrt{-2}$ *and* $z \neq -14,\ -6,\ -2$*, then there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Proof. Multiplying the elements of the sets in *Theorem 1* by $\sqrt{-2}$, which satisfy the property D(z), where $z = (4a+3) + 4b\sqrt{-2}$ and $z \notin \{-1, 3, 7\}$, by *Remark 1*, we get two distinct sets with the property D(z), where $z = (-8a-6) - 8b\sqrt{-2}$ and $z \notin \{-14,\ -6,\ 2\}$. We can write this z in the form $z = (8c+2) + 8d\sqrt{-2}$. □

Similarly, multiplying the elements of the sets in *Theorem 1* by 2, we get

Corollary 2. *If* $z = (16a+12) + 16b\sqrt{-2}$ *and* $z \neq -4,\ 12,\ 28$*, then there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Theorem 2. *For all* $z = (8a+1) + 8b\sqrt{-2}$ *there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Proof. The two sets

$$\begin{aligned} A &= \Big\{4,\ (9a^2-18b^2-5a)+(18ab-5b)\sqrt{-2},\\ &\quad (9a^2-18b^2+7a+2)+(18ab+7b)\sqrt{-2},\\ &\quad (36a^2-72b^2+4a)+(72ab+4b)\sqrt{-2}\Big\}\\ B &= \Big\{4,\ (a^2-2b^2-3a)+(2ab-3b)\sqrt{-2},\\ &\quad (a^2-2b^2+a+2)+(2ab+b)\sqrt{-2},\\ &\quad (4a^2-8b^2-4a)+(8ab-4b)\sqrt{-2}\Big\} \end{aligned}$$

have the property $D((8a+1)+8b\sqrt{-2})$. These sets are obtained from formulas (3) and (4), by putting $k = a + b\sqrt{-2}$. It remains to determine the pairs (a, b) for which the above sets have at least two equal elements or some elements equal to zero, and the pairs for which the corresponding sets are coincident. It is easy to check that the above cases appear iff

$$(a, b) \in \{(0,0),\ (4,0),\ (1,0)\ (-1,0),\ (3,0),\ (-2,0)\}.$$

Consequently for all complex numbers of the form $z = (8a+1) + 8b\sqrt{-2}$ and $z \notin \{-15,\ -7,\ 1,\ 9,\ 25,\ 33\}$ there exist at least two distinct diophantine quadruples satisfying the property D(z).

From *Lemma 3* there exist at least two (positive integer) diophantine quadruples with the property D(33). Also there exist an infinite number of (positive integer) diophantine quadruples with properties D(1), D(9) and D(25) [5]. Finally, the two sets $\{1,\ 8,\ 11,\ 16\}$ and $\left\{1,\ 3+8\sqrt{-2},\ 4\sqrt{-2},\ 5+24\sqrt{-2}\right\}$ have the property D(−7), while the two sets $\{4,\ 24,\ 46,\ 136\}$ and $\left\{1,\ 7,\ 8+4\sqrt{-2},\ 24+8\sqrt{-2}\right\}$ have the property D(−15). □

It is easy to prove:

Corollary 3. *For all* $z = (16c+14)+16d\sqrt{-2}$ *there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Theorem 3. *If* $z = (8a+5)+8b\sqrt{-2}$ *and* $z \neq -3$*, then there exist at least two distinct complex diophantine quadruples with the property D(z).*

Proof. The sets

$$\begin{aligned} A &= \Big\{2,\ (18a^2-36b^2+14a+2)+(36ab+14b)\sqrt{-2},\\ &\quad (18a^2-36b^2+26a+10)+(36ab+26b)\sqrt{-2},\\ &\quad (72a^2-144b^2+80a+22)+(144ab+80b)\sqrt{-2}\Big\}\\ B &= \Big\{2,\ (2a^2-4b^2-2a-2)+(4ab-2b)\sqrt{-2},\\ &\quad (2a^2-4b^2+2a+2)+(4ab+2b)\sqrt{-2},\\ &\quad (8a^2-16b^2-2)+16ab\sqrt{-2}\Big\} \end{aligned}$$

have the property $D((8a+5)+8b\sqrt{-2})$. These sets are obtained from formulas (5) and (6) by putting $k = a+b\sqrt{-2}$. It is easy to check that these two sets for all complex numbers of the form $z = (8a+5)+8b\sqrt{-2}$, $z \notin \{-3, 5, 13, 21\}$ give two distinct diophantine quadruples satisfying the property $D(z)$. But from *Lemma 2* there exists at least one diophantine quadruple with properties D(13) and D(21). Also the sets $\{1, -22+18\sqrt{-2}, -15+24\sqrt{-2}, -75+84\sqrt{-2}\}$ and $\{1, -14+6\sqrt{-2}, -19+4\sqrt{-2}, -67+20\sqrt{-2}\}$ have properties D(13) and D(21), respectively. Finally the two sets $\{-\sqrt{-2}, 7\sqrt{-2}, 4+6\sqrt{-2}, -4+6\sqrt{-2}\}$ and $\{1, -6+2\sqrt{-2}, -3+4\sqrt{-2}, -19+12\sqrt{-2}\}$ have the property D(5). □

Corollary 4. *For all* $z = (16c+6)+16d\sqrt{-2}$ *and* $z \neq 6$ *there exist at least two distinct diophantine quadruples with the property* $D(z)$.

Theorem 4. *If* $z = (8a+1)+(4b+2)\sqrt{-2}$ *and*

$$z \notin \left\{1 \pm 2\sqrt{-2}, 9 \pm 6\sqrt{-2}, 9 \pm 2\sqrt{-2}\right\},$$

then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.

Proof. The sets

$$\begin{aligned} A &= \Big\{\sqrt{-2},\ (36ab+12a+2b)-(18a^2+2a-9b^2-6b-1)\sqrt{-2},\\ &\quad (36ab+24a+2b+2)-(18a^2+2a-9b^2-12b-4)\sqrt{-2},\\ &\quad (144ab+72a+8b+4)-(72a^2+8a-36b^2-36b-9)\sqrt{-2}\Big\}\\ B &= \Big\{\sqrt{-2},\ (4ab-2b-2)-(2a^2-b^2-2a)\sqrt{-2},\\ &\quad (4ab+4a-2b)-(2a^2-b^2-2a-2b-1)\sqrt{-2},\\ &\quad (16ab+8a-8b-4)-(8a^2-4b^2-8a-4b-1)\sqrt{-2}\Big\} \end{aligned}$$

have the property $D((8a+1)+(4b+2)\sqrt{-2})$. These sets are obtained from formulas (1) and (2) for $m = \sqrt{-2}$ and $k = a-b\sqrt{-2}$. It is easy to check that these two sets for all complex numbers of the form $z = (8a+1) + (4b + 2\sqrt{-2}), z \notin \{1 \pm 2\sqrt{-2},\ 9 \pm 6\sqrt{-2},\ 9 \pm 2\sqrt{-2}\}$ give two distinct diophantine quadruples satisfying the property D(z). □

Corollary 5. *If* $z = (16c+14) + (8d+4)\sqrt{-2}$ *and*

$$z \notin \left\{-2 \pm 4\sqrt{-2}, -18 \pm 12\sqrt{-2}, -18 \pm 4\sqrt{-2}\right\},$$

then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.

Theorem 5. *For all* $z = 8a + 8b\sqrt{-2}$ *and* $z \neq 8$*, then there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Proof. The sets

$$\begin{aligned} A &= \Big\{1,\ (9a^2 - 8a - 18b^2) + (18ab - 8b)\sqrt{-2}, \\ &\quad (9a^2 - 2a - 18b^2 + 1) + (18ab - 2b)\sqrt{-2}, \\ &\quad (36a^2 - 20a - 72b^2 + 1) + (72ab - 20b)\sqrt{-2}\Big\} \\ B &= \Big\{1,\ (a^2 - 6a - 2b^2 + 1) + (2ab - 6b)\sqrt{-2}, \\ &\quad (a^2 - 4a - 2b^2 + 4) + (2ab - 4b)\sqrt{-2}, \\ &\quad (4a^2 - 20a - 8b^2 + 9) + (8ab - 20b)\sqrt{-2}\Big\} \end{aligned}$$

have the property D($8a + 8b\sqrt{-2}$). These sets are obtained from formulas (7) and (8) by putting $k = a + b\sqrt{-2}$. It is easy to check that these two sets for all complex numbers of the form $z = 8a+8b\sqrt{-2}$, $z \notin \{0, 8, 16, 24, 32, 40, 48\}$ give two distinct diophantine quadruples satisfying the property D(z). But for numbers 0 and 16, the assertion of the theorem is valid since they are a perfect square [5]. Now, by *Lemma 2*, there exists at least one set of four positive integers with properties D(24) and D(-12). Multiplying elements of the set with the property D(-12) by $\sqrt{-2}$, we get a complex diophantine quadruple with the property D(24). Similarly, from *Lemma 3* it follows that there exist at least two (positive integer) diophantine quadruples with properties D(40), D(-16) and D(-24), so we also get two distinct sets with properties D(32) and D(48), which completes the proof. □

Theorem 6. *For all* $z = (16a+4) + 16b\sqrt{-2}$ *there exist at least two distinct complex diophantine quadruples with the property* $D(z)$.

Proof. The sets

$$\begin{aligned} A &= \Big\{4, (9a^2 - 4a - 18b^2 - 1) + (18ab - 4b)\sqrt{-2}, \\ &\quad (9a^2 + 8a - 18b^2 + 3) + (18ab + 8b)\sqrt{-2}, \\ &\quad (36a^2 + 8a - 72b^2) + (72ab + 8b)\sqrt{-2}\Big\} \\ B &= \Big\{4, (a^2 - 4a - 2b^2 - 1) + (2ab - 4b)\sqrt{-2}, \\ &\quad (a^2 - 2b^2 + 3) + 2ab\sqrt{-2}, \\ &\quad (4a^2 - 8a - 8b^2) + (8ab - 8b)\sqrt{-2}\Big\} \end{aligned}$$

have the property $\mathrm{D}((16a+4)+16b\sqrt{-2})$. These sets are obtained from formulas (9) and (10) by putting $k=a+b\sqrt{-2}$. It is easy to check that these two sets for all complex numbers of the form $z=(16a+4)+16b\sqrt{-2}$, $z\notin\{4,\ 36,\ 20,\ 52,\ 84,\ -12\}$ give two distinct diophantine quadruples satisfying the property $\mathrm{D}(z)$. But for numbers 4 and 36, the assertion of the theorem is valid since they are a perfect squares [5]. Now from *Lemma 3* there exist at least two (positive integer) diophantine quadruples with the property D(52). Also the sets $\left\{-2,\ 14,\ 12-4\sqrt{-2},\ 12+4\sqrt{-2}\right\}$ and $\left\{2,\ -6+8\sqrt{-2},\ -12+4\sqrt{-2},\ -8+24\sqrt{-2}\right\}$ with the property D(20). And the sets $\{4,\ 204,\ 268,\ 940\}$ and $\left\{4,\ -38+6\sqrt{-2},\ -30+18\sqrt{-2},\ -140+48\sqrt{-2}\right\}$ have the property D(84). Finally, from *Lemma 2* there exists at least one diophantine quadruple with the property $\mathrm{D}(-12)$ and since the set

$$\left\{2+2\sqrt{-2},\ -1+\sqrt{-2},\ 1-3\sqrt{-2},\ -2-6\sqrt{-2}\right\}$$

has the same property we get our result. □

Theorem 7. *If $z=(24c+18)+(12d+10)\sqrt{-2}$ and $z\neq-6-2\sqrt{-2}$, then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.*

Proof. The result is obtained by multiplying the elements of the two sets in formulas (1) and (2) by $\sqrt{-2}$ and putting $k=(2a+b)+(b-a)\sqrt{-2}$ and $m=1+\frac{\sqrt{-2}}{2}$
. □

From *Remark 1* and *Theorem 7* we get the following:

Corollary 6. *If $z=(24c+18)+(12d+2)\sqrt{-2}$ and $z\neq 4-6\sqrt{-2}$, then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.*

Theorem 8. *If $z=(24c+2)+(12d+14)\sqrt{-2}$ and $z\neq 2+2\sqrt{-2}$, then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.*

Proof. The result is obtained by multiplying the elements of the two sets in formulas (1) and (2) by $\sqrt{-2}$ and putting $k=(2a+b)+(b-a+1)\sqrt{-2}$ and $m=1+\sqrt{-2}/2$. □

From *Remark 1* and *Theorem 8* we get the following

Corollary 7. *If $z=(24a+2)+(12b+10)\sqrt{-2}$ and $z\neq 2-2\sqrt{-2}$, then there exist at least two distinct complex diophantine quadruples with the property $D(z)$.*

Remark 2. *For the exception elements from our theorems the following holds:*

1. *(*Theorem 1*) the set $\{1,\ -2,\ -3,\ -11\}$ has the property $D(3)$ whereas the set $\{1,\ 18,\ 29,\ 93\}$ has the property $D(7)$.*

2. *(*Theorem 4*) the sets*

$$\left\{-\sqrt{-2},\ 26+16\sqrt{-2},\ 12+19\sqrt{-2},\ 76+71\sqrt{-2}\right\},$$
$$\left\{\sqrt{-2},\ 26-16\sqrt{-2},\ 12-19\sqrt{-2},\ 76-71\sqrt{-2}\right\},$$
$$\left\{\sqrt{-2},\ 50-4\sqrt{-2},\ 64+5\sqrt{-2},\ 228+\sqrt{-2}\right\},$$
$$\left\{\sqrt{-2},\ -50-4\sqrt{-2},\ -64+5\sqrt{-2},\ -228+\sqrt{-2}\right\}$$

have properties $D(9-2\sqrt{-2})$, $D(9+2\sqrt{-2})$, $D(9+6\sqrt{-2})$ and $D(9-6\sqrt{-2})$, respectively.

3. *(*Theorem 5*) the set $\{-4,\ -2,\ 2,\ 4\}$ has the property $D(8)$.*

5. Conclusion

From our results proved in *Proposition 1*, *Theorems 1-8* and the corresponding corollaries we can get the following:

If there is a complex diophantine quadruple with the property $D(a+b\sqrt{-2})$, then b should be even, and we have 8 possibilities for a and b modulo 4. Let $a' \equiv a \pmod 4$, $b' \equiv b \pmod 4$ with $a', b' \in \{0, 1, 2, 3\}$. The case $(a', b') = (3, 0)$ is solved completely. The case $(a', b') = (0, 0)$ is considered modulo 16, and we solved 6 of 16 possible subcases, while for $(a', b') = (2, 0)$ we solved 8 of 16 possible subcases. The cases $(a', b') = (1, 0)$ and $(a', b') = (1, 2)$ are considered modulo 8, and in both cases we solved 2 of 4 possible cases. The case $(a', b') = (2, 2)$ is considered modulo 24, and we solved 8 of 36 possible cases. We did not obtain any result in the cases $(a', b') = (0, 2)$ and $(a', b') = (3, 2)$.

References

[1] E. Brown, *Sets in which $xy + k$ is always a square*, Math. Comp. **45**(1985), 613–620.

[2] L. E. Dickson, *History of the Theory of Numbers*, Chelsea, New York, 1952.

[3] A. Dujella, *Diophantine $m-$tuples*, `http://www.math.hr/~duje/dtuples.html`

[4] A. Dujella, *Diophantine quadruples for squares of Fibonacci and Lucas numbers*, Portugaliae Math. **52**(1995), 305–318.

[5] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65**(1993),15–27.

[6] A. Dujella, *The problem of Diophantus and Davenport for Gaussian integers*, Glas. Mat. **32**(**52**)(1997), 1–10.

[7] H. Gupta, K. Singh, *On k-triad sequences*, Internat. J. Math. Math. Sci. **5**(1985), 799–804.

[8] S. P. Mohanty, A. M. S. Ramasamy, *On $P_{r,k}$ sequences*, Fibonacci Quart. **23**(1985), 36–44.