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On path graphs of incidence graphs

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Abstract. *For a given graph G and a positive integer k the P_k-path graph $P_k(G)$ has for vertices the set of paths of length k in G. Two vertices are connected in $P_k(G)$ when the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$. Path graphs were proposed as a generalization of line graphs. In this article we investigate some properties of path graphs of bipartite graphs, especially path graphs of incidence graphs of configurations.*

Key words: *path graph, bipartite graph, perfect graph, incidence graph*

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1. Introduction and preliminaries

In this article all graphs are simple and finite. The set of vertices of graph G is denoted by $V(G)$ and the set of edges by $E(G)$. A clique is a set of vertices every pair of which are adjacent. The cardinality of a largest clique in graph G is denoted by $\omega(G)$.

A vertex colouring of a graph $G = (V(G), E(G))$ is a mapping $c : V(G) \to K$, such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set K are called the available colours. The set of vertices which are mapped to one colour is called a colour class. A chromatic number of G, denoted by $\chi(G)$, is the smallest integer k, such that G has a colouring $c : V(G) \to \{1, \ldots, k\}$. Obviously, $\omega(G) \leq \chi(G)$, since each two vertices of a clique are adjacent and therefore must be in distinct colour classes.

For $S \subseteq V(G)$ an induced subgraph $G(S)$ of a graph G is a graph with vertex set S and the edge set consisting of all the edges of G with both ends in S. A graph G is perfect if $\omega(H) = \chi(H)$ for every induced subgraph H of G. Many problems of interest in practice but intractable in general can be solved efficiently when restricted to the class of perfect graphs. For example, the question of when a certain class of linear programs always has an integer solution can be answered

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in terms of perfection of an associated graph. The following theorem is proved by Lovász in 1972 (see [7]):

Theorem 1 [Perfect graph theorem]. *Graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

The perfect graph theorem was conjectured in 1960 by Berge. Until its proof, it was known as the weak perfect graph conjecture. In 1960 Berge proposed the conjecture, known as the strong perfect graph conjecture. This conjecture was recently proved by Chudnovsky, Robertson, Seymour and Thomas (see [3]).

Theorem 2 [Strong perfect graph theorem]. *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.*

The line graph $L(G)$ of a graph G is the graph with the set of vertices $V(L(G)) = E(G)$ in which two vertices are adjacent if and only if the corresponding edges of G are adjacent. Bipartite graphs and line graphs of bipartite graphs are perfect (see [4]). The perfect graph theorem implies that the complements of bipartite graphs and the complements of line graphs of bipartite graphs are perfect.

For a given graph G and a positive integer k the P_k-path graph $P_k(G)$ has for vertices the set of paths of length k in G. Two vertices are connected in $P_k(G)$ when the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$. Path graphs are introduced as a generalization of line graphs (see [2]). Obviously, $P_1(G)$ coincides with $L(G)$. In fact, in [2] the authors restrict themselves to the $P_2(G)$ case and consider some of the properties of line graphs, such as sufficient conditions for the Hamiltonicity. A number of articles dealing with path graphs have appeared since then, see for example [6] and [8].

Definition 1. *A λ-configuration $(v_r, b_k)_\lambda$ is an incidence structure of v points and b blocks such that*

1. *each block is incident with exactly k points,*
2. *each point is incident with exactly r blocks,*
3. *every pair of points is incident with at most λ blocks.*

If $v = b$ and hence $r = k$, then λ-configuration is symmetric.

Definition 2. *Let $\mathcal{I}$ be an incidence structure with the set of points $\mathcal{P} = \{P_1, P_2, \ldots, P_v\}$ and the set of blocks $\mathcal{B} = \{x_1, x_2, \ldots, x_b\}$. The incidence matrix of $\mathcal{I}$ is a $b \times v$ matrix $M = (m_{ij})$ defined by*

$$m_{ij} = \begin{cases} 1 & \textit{if } P_j \textit{ is incident with } x_i, \\ 0 & \textit{otherwise}. \end{cases}$$

Definition 3. *Let M be the incidence matrix of an incidence structure $\mathcal{I}$. Denote by M^t the transpose of M. The graph with adjacency matrix*

$$\begin{bmatrix} 0 & M \\ M^t & 0 \end{bmatrix}$$

is called the incidence graph of $\mathcal{I}$.

Definition 4. *A $2-(v,k,\lambda)$ design is a λ-configuration $(v_r, b_k)_\lambda$ such that every pair of points is incident with exactly λ blocks.*

If $v = b$, then a $2-(v,k,\lambda)$ design is called a symmetric (v,k,λ) design.

For further basic definitions and properties of configurations and designs we refer the reader to [1], [5] and [9].

2. Path graphs of bipartite graphs

Theorem 3. *Let G be a graph. The path graph $P_{n+1}(G)$ is isomorphic to a subgraph of the graph $P_1(P_n(G))$.*

Proof. First we will prove that each vertex of the graph $P_{n+1}(G)$ corresponds to one vertex of $P_1(P_n(G))$.

A vertex of the graph $P_{n+1}(G)$ corresponds to a path $v_1v_2\dots v_{n+1}v_{n+2}$, where $v_1, v_2, \dots, v_{n+1}, v_{n+2}$ are vertices of the graph G. This path corresponds to paths $v_1v_2\dots v_{n+1}$ and $v_2\dots v_{n+1}v_{n+2}$, which determine two adjacent vertices of the graph $P_n(G)$. These two adjacent vertices of the graph $P_n(G)$ give rise to one vertex of the graph $P_1(P_n(G))$. Obviously, two distinct vertices of the graph $P_{n+1}(G)$ correspond to distinct vertices of the $P_1(P_n(G))$.

Now we will prove that two vertices from $P_1(P_n(G))$ are adjacent if the corresponding vertices from the path graph $P_{n+1}(G)$ are adjacent.

Two adjacent vertices of the graph $P_{n+1}(G)$ correspond to paths $v_1v_2v_3\dots v_{n+1}v_{n+2}$ and $v_2v_3\dots v_{n+1}v_{n+2}v_{n+3}$. The path $v_1v_2v_3\dots v_{n+1}v_{n+2}$ corresponds to the paths $v_1v_2v_3\dots v_{n+1}$ and $v_2v_3\dots v_{n+1}v_{n+2}$, which determine two adjacent vertices of the graph $P_n(G)$, corresponding to a vertex X of the graph $P_1(P_n(G))$. Similarly, the path $v_2v_3\dots v_{n+1}v_{n+2}v_{n+3}$ corresponds to paths $v_2v_3\dots v_{n+1}v_{n+2}$ and $v_3\dots v_{n+1}v_{n+2}v_{n+3}$, corresponding to two vertices of the path graph $P_n(G)$, determining a vertex Y of the graph $P_1(P_n(G))$. Vertices X and Y are adjacent in $P_1(P_n(G))$. $\square$

Lemma 1. *Let G be a bipartite graph and k a positive integer. Then $P_{2k}(G)$ is also a bipartite graph.*

Proof. Let $\{X, Y\}$ be a bipartition of the set $V(G)$. Let us define sets $\overline{X}$ and $\overline{Y}$:

$$\overline{X} \textit{ is the set of all paths of length } 2k \textit{ in } G \textit{ with the first vertex in } X,$$

$$\overline{Y} \textit{ is the set of all paths of length } 2k \textit{ in } G \textit{ with the first vertex in } Y.$$

Two paths from the set $\overline{X}$ (or $\overline{Y}$) cannot intersect each other in a path of length $2k-1$ in such a way that their union forms a path or a cycle of length $2k+1$. Therefore, $\{\overline{X}, \overline{Y}\}$ is a bipartition of the graph $P_{2k}(G)$. $\square$

A direct consequence of *Lemma 1* and the perfect graph theorem is

Theorem 4. *Let G be a bipartite graph and k a positive integer. Then $P_{2k}(G)$ and $P_1(P_{2k})$, and their complements are perfect graphs.*

Remark 1. *Let G be a bipartite graph and k a positive integer. Then $P_{2k+1}(G)$ is not necessarily a perfect graph.*

Example 1. *Let $V = \{v_1, \dots, v_7\}$ be the set of vertices, and $E = \{e_1, \dots, e_7\}$ the set of edges of a graph G, where $e_1 = v_1v_2$, $e_2 = v_2v_3$, $e_3 = v_3v_4$, $e_4 = v_4v_5$,*

$e_5 = v_5v_6$, $e_6 = v_6v_7$, and $e_7 = v_7v_4$. G is a bipartite graph with bipartition $\{\{v_1, v_3, v_5, v_7\}, \{v_2, v_4, v_6\}\}$. The graph $P_3(G)$ consists of 9 vertices $Z_1, \ldots, Z_9$, where Z_1 corresponds to the path $v_1v_2v_3v_4$, $Z_2 = v_2v_3v_4v_5$, $Z_3 = v_3v_4v_5v_6$, $Z_4 = v_4v_5v_6v_7$, $Z_5 = v_5v_6v_7v_4$, $Z_6 = v_6v_7v_4v_3$, $Z_7 = v_7v_4v_3v_2$, $Z_8 = v_5v_4v_7v_6$, $Z_9 = v_6v_5v_4v_7$. A subgraph of $P_3(G)$ induced by vertices Z_1, Z_2, Z_3, Z_4, Z_5, Z_6 and Z_7 is a cycle. The strong perfect graph theorem implies that $P_3(G)$ is not a perfect graph.

Since incidence graphs are bipartite, we will investigate the path graphs of some incidence graphs. We will especially take into consideration the path graphs of incidence graphs of λ-configurations and 2-designs, and incidence structures corresponding to these path graphs. It turns out that some of these path graphs are incidence graphs of λ-configurations.

3. Path graphs of incidence graphs

Theorem 5. *Let G be a $k-$regular graph, $k \geq 2$. Then $P_1(G)$ and $P_2(G)$ are $(2k-2)-$regular graphs.*

Proof. Let us consider the vertex Z of $P_1(G)$ corresponding to an edge $e = v_1v_2$. Neighbors of Z correspond to the edges xv_1 or v_2y, where for both x and y we have $k-1$ possibilities.

Let us now take into consideration the graph $P_2(G)$ and its vertex $W = v_1v_2v_3$. Neighbors of W are of the type xv_1v_2 or v_2v_3y. We have $k-1$ posibilities for choosing both x and y. □

Remark 2. *Let G be a $k-$regular graph, $k \geq 2$, and $n \geq 3$. Then $P_n(G)$ is not necessarily a regular graph.*

Theorem 6. *Let G be a $k-$regular bipartite graph, $k \geq 2$. Then $P_1(G)$ and $P_3(G)$ are $(2k-2)-$regular graphs and $P_2(G)$ is a $(2k-2)-$regular bipartite graph.*

Proof. We have to prove that $P_3(G)$ is a $(2k-2)-$regular graph. If $Z = v_1v_2v_3v_4$ is a vertex of $P_3(G)$, then the neighbors of Z are of the type $xv_1v_2v_3$ or $v_2v_3v_4y$. Since G is bipartite, v_1 cannot be adjacent to v_3, so we have $k-1$ possibilities for choosing the vertex x. Similarly, v_4 cannot be adjacent to v_2, so we have $k-1$ possibilities for choosing y. □

Theorem 7. *Let G be the incidence graph of a λ-configuration $(v_r, b_k)_\lambda$. Then $P_1(G)$ and $P_3(G)$ are $(k+r-2)-$regular graphs.*

Proof. G is a bipartite graph with a bipartition $\{X, Y\}$, where X corresponds to the first b rows (and columns) of the adjacency matrix of G, and Y corresponds to the other v rows (and columns). In other words, the vertices from the set X correspond to the blocks, and the vertices from the Y correspond to the points of the λ-configuration.

Let $e = v_1v_2$ be an edge in G, $v_1 \in X$ and $v_2 \in Y$. Then v_1 has $k-1$ neighbors other than v_2 and v_2 has $r-1$ neighbors other than v_1. There are $k+r-2$ edges adjacent to e in the graph G, so the vertex $Z = v_1v_2$ has $k+r-2$ neighbors in $P_1(G)$.

In a similar way one can prove that $P_3(G)$ is a $(k+r-2)-$regular graph. □

Theorem 8. *Let G be the incidence graph of a λ-configuration $(v_r, b_k)_\lambda$. Then $P_2(G)$ is a bipartite graph. Further, the graph $P_2(G)$ is an incidence graph of a*

$(r-1)$-configuration $(v'_{r'}, b'_{k'})_{r-1}$ with the following properties:

1. $v' = \binom{k}{2} b$,
2. $b' = \binom{r}{2} v$,
3. $k' = 2k - 2$,
4. $r' = 2r - 2$,
5. *every pair of points is incident with $r-1, 2, 1$ or 0 blocks,*
6. *every pair of blocks is incident with $k-1, 2, 1$ or 0 points.*

Proof. G is a bipartite graph with the bipartition $\{X, Y\}$, where X corresponds to the set of blocks and Y corresponds to the set of points of the λ-configuration. $P_2(G)$ is also a bipartite graph with the bipartition $\{\overline{X}, \overline{Y}\}$, such that:

$$\overline{X} \textit{ is the set of all paths of length } 2 \textit{ in } G \textit{ with the first vertex in } X,$$

$$\overline{Y} \textit{ is the set of all paths of length } 2 \textit{ in } G \textit{ with the first vertex in } Y.$$

The graph $P_2(G)$ is an incidence graph of the incidence structure $\mathcal{I}$, where $\overline{X}$ corresponds to the set of blocks and $\overline{Y}$ corresponds to the set of points of $\mathcal{I}$. Let $Z = v_1v_2v_3$ be a vertex from $\overline{Y}$. Then $v_1, v_3 \in Y$ and $v_2 \in X$. We have $|X| = b$ possibilities for choosing v_2, and for fixed v_2 we have $\binom{k}{2}$ possibilities for choosing the set $\{v_1, v_3\}$, which proves that $v' = \binom{k}{2} b$. In a similar way one can prove that $b' = \binom{r}{2} v$.

The neighbors of the vertex $Z \in \overline{Y}$, $Z = v_1v_2v_3$, are of the type xv_1v_2 or v_2v_3y, where $x, y \in X$. Since the vertices v_1 and v_3 have degree r and they are both adjacent to v_2, we have $r-1$ possibilities for x and y. Therefore Z has $2r-2$ neighbors, i.e. $r' = 2r - 2$. Similarly, $k' = 2k - 2$.

The vertex $Z = v_1v_2v_3$ from $\overline{Y}$ and the vertex $Z1 = v_3v_4v_5$ have one common neighbor, namely the vertex which corresponds to the path $v_2v_3v_4$. Vertices $Z = v_1v_2v_3$ and $Z_2 = v_1v_6v_3$ have two common neighbors which correspond to the paths $v_6v_1v_2$ and $v_2v_3v_6$. The vertex Z and the vertex $Z_3 = v_0v_2v_3$ have $r-1$ common neighbors which correspond to the paths v_2v_3y, where y can be chosen from the set of $r-1$ vertices. Z and the vertex $Z_4 = v_1v_2v_7$ have also $r-1$ common neighbors corresponding to the paths xv_1v_2, where x can be chosen from the set of $r-1$ neighbors of v_1 other than v_2. Two vertices from $\overline{Y}$ corresponding to the paths which have no common points do not have common neighbors. So every pair of points is incident with $r-1, 2, 1$ or 0 blocks.

In a similar way one can prove that every pair of blocks is incident with $k-1, 2, 1$ or 0 points. □

Corollary 1. *Let G be the incidence graph of a $2-(v, k, \lambda)$ design. Then the graph $P_2(G)$ is the incidence graph of a $(r-1)$-configuration $(v'_{r'}, b'_{k'})_{r-1}$ with the following properties:*

1. $v' = \binom{v}{2} \lambda$,

2. $b' = \binom{r}{2}v$,

3. $k' = 2k-2$,

4. $r' = 2r-2$,

5. *every pair of points is incident with* $r-1, 2, 1$ *or* 0 *blocks,*

6. *every pair of blocks is incident with* $k-1, 2, 1$ *or* 0 *points.*

Proof. Let the sets $\overline{X}$ and $\overline{Y}$ be defined as in the proof of *Theorem 8*, and let $Z = v_1v_2v_3$ be the vertex from $\overline{Y}$. There are $\binom{v}{2}$ possibilities for choosing a subset $\{v_1, v_3\}$ from the set Y. The vertices v_1 and v_3 have λ common neighbors, so we have λ possibilities for choosing v_2. □

Corollary 2. *Let G be the incidence graph of a symmetric (v,k,λ) design. Then the graph $P_2(G)$ is the incidence graph of a $(k-1)$-configuration $(v'_{r'}, b'_{k'})_{k-1}$ with the following properties:*

1. $v' = b' = \binom{v}{2}\lambda$,

2. $k' = r' = 2k-2$,

3. *every pair of points is incident with* $k-1, 2, 1$ *or* 0 *blocks,*

4. *every pair of blocks is incident with* $k-1, 2, 1$ *or* 0 *points.*

Proof. For a symmetric (v,k,λ) design $b = v$ and $r = k$. □

Theorem 9. *Let G be the incidence graph of a $2-(v,k,\lambda)$ design $\mathcal{D}$ and P a point of $\mathcal{D}$. Further, let v be a vertex of G which corresponds to the point P and H a subgraph of $P_2(G)$ which has for vertices the set of paths of length 2 containing the vertex v. Then H is the incidence graph of a $(r-1)$-configuration $(v'_{r'}, b'_{k'})_{r-1}$ with the following properties:*

1. $v' = (v-1)\lambda$,

2. $b' = \binom{r}{2}$,

3. $k' = 2k-2$,

4. $r' = r-1$,

5. *every pair of points is incident with* 1 *or* $r-1$ *blocks,*

6. *every pair of blocks is incident with* 0 *or* $k-1$ *points.*

Proof. Let the sets $X, Y, \overline{X}$ and $\overline{Y}$ be defined as in the proof of *Theorem 8*. The graph H is a bipartite graph with the bipartition $\{\overline{X}_1, \overline{Y}_1\}$, $\overline{X}_1 \subseteq \overline{X}$, $\overline{Y}_1 \subseteq \overline{Y}$, such that

$$\overline{X}_1 = \{xvy|\ x,y \in X\}, \qquad \overline{Y}_1 = \{vxy|\ x \in X,\ y \in Y\}.$$

Let us count the elements of $\overline{Y}_1$. We have $v-1$ possibilities for choosing the vertex y from the set Y, since $y \neq v$. Vertices v and y have λ common neighbors, hence $v' = |\overline{Y}_1| = (v-1)\lambda$.

The vertex v has r neighbors, all of them from the set X. Because of this $b' = |\overline{X}_1| = \binom{r}{2}$.

Let us determine k'. In other words, we have to determine the degree of a vertex $Z = v_1vv_2$ in H. The neighbors of Z correspond to paths xv_1v and vv_2y, $x, y \in X$. Since $v_1, v_2 \in Y$, they have degree k in G. So v_1 and v_2 have $k-1$ neighbors other than v, hence $k' = 2k-2$.

Let us count the vertices of H which are neighbors of $W \in \overline{Y}_1$, $W = vv_3v_4$. These vertices correspond to paths xvv_3, $x \in X$. The vertex v has $r-1$ neighbors other than v_3, so $r' = r-1$.

Two vertices from $\overline{Y}_1$, $W_1 = v_5v_6v$ and $W_2 = vv_7v_8$, have one common neighbor, namely the vertex from $\overline{X}_1$ corresponding to the path v_6vv_7. The vertices W_1 and $W_3 = v_9v_6v$ have $r-1$ neighbors of type v_6vx, $x \in X$. That proves that every pair of points is incident with 1 or $r-1$ blocks.

Vertices $Z_1 = v_{10}vv_{11}$ and $Z_2 = v_{12}vv_{11}$ from $\overline{X}_1$ have $k-1$ common neighbors of type $vv_{11}y$, $y \in Y$. If two vertices from $\overline{X}_1$ correspond to paths which do not share an edge, then they do not have common neighbors. So every pair of blocks is incident with 0 or $k-1$ points. □

We have constructed λ-configurations using the path graphs of incidence graphs of configurations and designs. These λ-configurations have at most four possibilities for intersection of two blocks, and λ-configurations from *Theorem 9* have only two possibilities for intersection of two blocks. Dually, the same condition is valid for points. Some of the described λ-configurations, e.g. the ones from *Corollary 2*, are symmetric.

The method described in this article can be further investigated. We can expect that using similar techniques one can construct other incidence structures interesting from a combinatorial point of view.

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