Total forcing number of the triangular grid

DAMIR VUKIČEVIĆ* AND JELENA SEDLAR†

Abstract. *Let T be a square triangular grid with n rows and columns of vertices and n an even number. A set of edges $E \subset E(T)$ completely determines perfect matchings on T if there are no two different matchings on T coinciding on E. We establish the upper and the lower bound for the smallest value of $|E|$, i.e. we show that*

$$\frac{5}{4}n^2 - \frac{21}{2}n + \frac{41}{4} \leq |E| \leq \frac{5}{4}n^2 + n - 2 \tag{1}$$

and show that $|E| / |E(T)|$ tends to $5/12$ when n tends to infinity.

Key words: *forcing matchings, total forcing matchings, grid, triangular grid, extremal problem*

AMS subject classifications: 05C35, 05C69

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1. Introduction

The connection between graph theory and chemistry is very important (see [7],[8]). Especially, the concept of perfect matchings [5] from graph theory is related to study of benzenoids [2]. Therefore, most of the study of perfect matchings has been restricted to hexagonal systems [4],[10].

Recently, other classes of graphs have been considered too. For example, the notion of forcing number was introduced by Harary et al. in [4] as follows.

Let G be a graph that admits a perfect matching. The forcing number of a perfect matching M of graph G is defined as the smallest number of edges in a subset $S \subset M$, such that S is in no other perfect matching.

In [6], the lower and the upper bound for forcing number of a perfect matching on a $2n \times 2n$ square grid.

A similar problem is considered in this paper and the attention is restricted to square triangular grid T with n rows and columns of vertices and n even number. We say that the set of edges $E \subset E(T)$ completely determines perfect matchings

*Department of Mathematics, University of Split, Nikole Tesle 12, HR-21 000 Split, Croatia, e-mail: vukicevi@pmfst.hr

†Faculty of Civil Engineering, University of Split, Matice Hrvatske 15, HR-21 000 Split, Croatia, e-mail: Jelena.Sedlar@gradst.hr

on T if there are no two different matchings on T coinciding on E. Our main result establishes the upper and the lower bound for $|E| / |E(T)|$ which read

$$\frac{5}{4}n^2 - \frac{21}{2}n + \frac{41}{4} \leq |E| \leq \frac{5}{4}n^2 + n - 2 \tag{2}$$

and shows that $|E| / |E(T)|$ tends to $5/12$ when n tends to infinity.

2. Preliminaries

In this paper we use standard graph theoretical terminology [1],[9].

Let T be a square triangular grid with n rows and columns of vertices and n an even number. The figure below shows a triangular grid for $n = 4$.

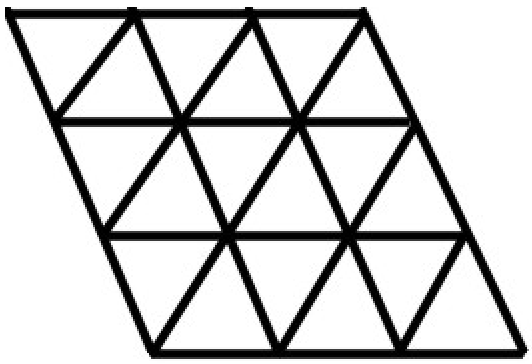

Figure 1.

Obviously, this graph is isomorphic to graph in the following figure.

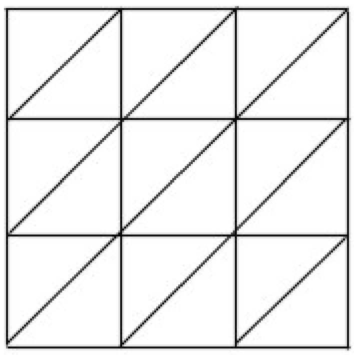

Figure 2.

The set of edges E *completely determines* perfect matchings on T if there are no two different matchings on T coinciding on E. Our goal is to determine the set of edges E which completely determines perfect matchings on T and has a minimal number of edges, i.e. $|E|$ is minimal.

Let G denote a subgraph of T for which $V(G) = V(T)\setminus E$, with v, e, f denoting the number of vertices, edges and faces in G, respectively. Obviously, if $|E|$ is minimal, then e is maximal. Since G is planar, $v - e + f = 2$. Consequently, f must be maximal as well.

Let D be a dual graph of the graph T without a vertex corresponding to an unbounded face of T. The figure below shows a dual graph on the triangular grid for $n = 4$.

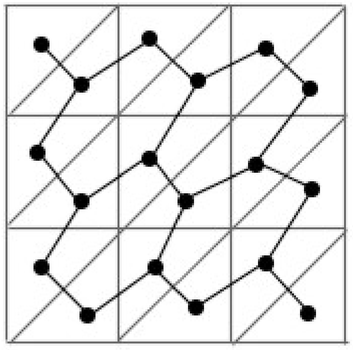

Figure 3.

Let d_t be a vertex in D corresponding to face t in T, then d_t *belongs* to face f of graph G if $t \subseteq f$. The following figure shows arbitrary graph G on triangular grid ($n = 4$). Not all vertices from D are shown, but only those which belong to the same face f.

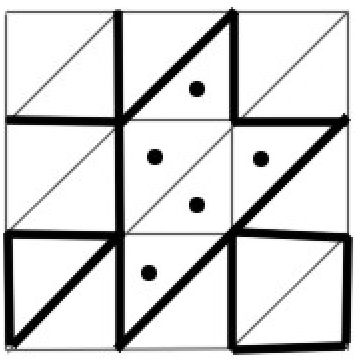

Figure 4.

For each face f in G, the *dual graph of face* f (denoted by D_f) is defined as a subgraph of D induced by all vertices $d_t \in V(D)$ which belong to face f. For each face f in G, let v_f denote the number of vertices in D_f and e_f the number of edges in D_f. The figure shows dual graph D_f of the same face f chosen on the previous figure (note that $v_f = 5$, $e_f = 4$).

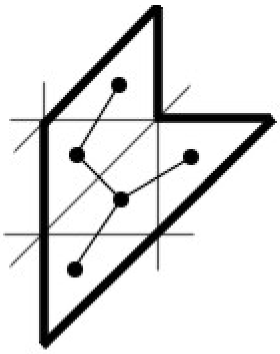

Figure 5.

Let vertices in $V(T)$ be denoted as follows, $v \equiv v_{i,j}$ if v is in the $i-$th row and the $j-$th column. Vertices $\{v_{i,j} : i = 1, n \text{ or } j = 1, n\}$ shall be called *border vertices* in T. All other vertices in T shall be called *interior vertices* of T.

3. Upper bound

Let us consider subgraph G_1 of square triangular grid T which is shown in the following figure.

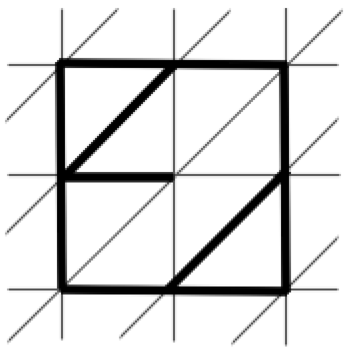

Figure 6.

Since the number of vertices is odd, there is no perfect matching of its vertices, i.e. at least one of the vertices has to be matched with a vertex outside of $V(G_1)$. It is readily seen that there are no two different perfect matchings of the rest of vertices in G_1.

Let us define $R_{k,l} = \{v_{i,j} : 1 + 2k \leq i \leq 3 + 2k, \quad 1 + 2l \leq j \leq 3 + 2l\}$ for $k, l = 0, \ldots, (n-2)/2 - 1$. Now, let G_n be defined as a subgraph of T such that $V(G_n) =$

$V(T)$ and each of the subgraphs of G_n induced by the set of vertices $R_{k,l}$ is isomorphic to G_1. The figure shows such G_n on the square triangular grid for $(n = 6)$.

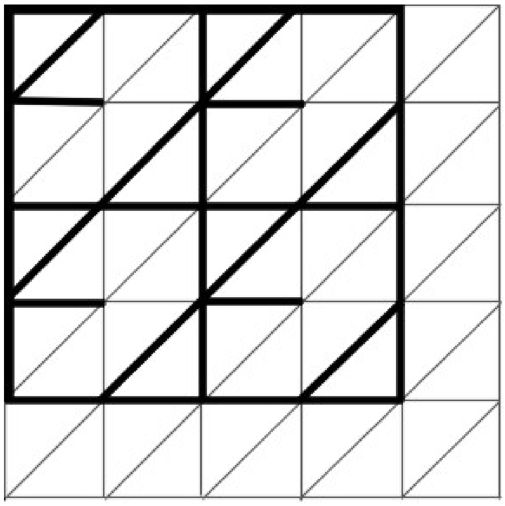

Figure 7.

Since there are no two different matchings on G_1, the set of edges $E = E(T)\backslash E(G_n)$ completely determines perfect matchings on T.

So, graph G_n can serve as a lower bound for G.

Theorem 1. $|E| \leq \frac{5}{4}n^2 + n - 2$

Proof. First note that

$$|E(G_n)| = 2\frac{n}{2}(n-2) + 3\left(\frac{n-2}{2}\right)^2 = \frac{7}{4}n^2 - 5n + 3 \tag{3}$$

and

$$|E(T)| = 2n(n-1) + (n-1)^2 = 3n^2 - 4n + 1. \tag{4}$$

Since G is such graph that $E = E(T)\backslash E(G)$ completely determines perfect matchings on T and $|E|$ is minimal, it obviously follows that $|E(G)| \geq |E(G_n)|$. Hence,

$$|E| = |E(T)| - |E(G)| \leq |E(T)| - |E(G_n)| = \frac{5}{4}n^2 + n - 2. \tag{5}$$

$\square$

Herefrom it readily follows that:

Corollary 1. *When n tends to infinity then $|E| / |E(T)| \leq 5/12$.*

4. Lower bound

We start with two arbitrary results.

Lemma 1. *In G there are no cycles of length 4 with all 4 vertices in interior of T.*

Proof. Let us suppose there is cycle C of length 4 in G with all vertices in interior of T. There are three possibilities for such cycle:

a) $(v_{i,j+1}v_{i+1,j+1}v_{i+2,j}v_{i+1,j})$, $\quad i = 2, \ldots, n-3, \; j = 2, \ldots, n-2,$

b) $(v_{i,j+1}v_{i,j+2}v_{i+1,j+1}v_{i+1,j})$, $\quad i = 2, \ldots, n-2, \; j = 2, \ldots, n-3,$

c) $(v_{i,j}v_{i,j+1}v_{i+1,j+1}v_{i+1,j})$, $\quad i = 2, \ldots, n-2, \; j = 2, \ldots, n-2.$

In each of these cases, vertices from $V(T)\backslash V(C)$ can be perfectly matched in T as shown in following figures.

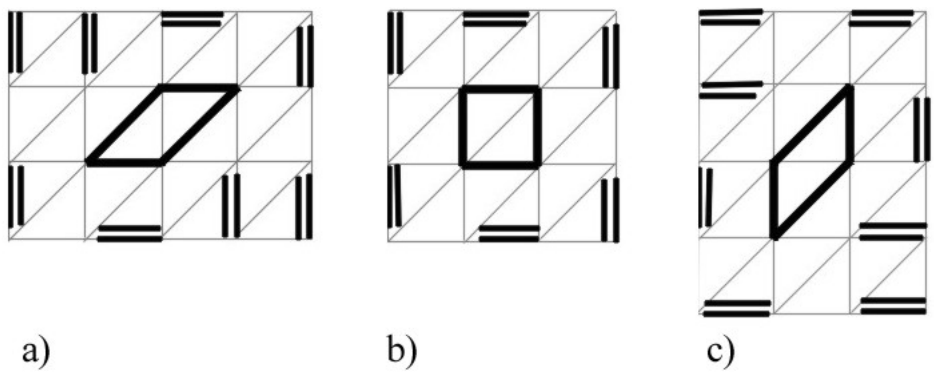

Figure 8.

Since vertices in $V(C)$ can be perfectly matched in two different ways, there are two different perfect matchings in T which coincide on E and that is a contradiction. □

Lemma 2. *Let C be any cycle in T and G_C a corresponding dual graph. The number of vertices in C and the number of vertices in G_C have the same parity.*

Proof. The proof is by induction on $v(G_C)$. If $v(G_C) = 1$, then $v(C) = 3$, so they are obviously of the same parity. Suppose that $v(G_C)$ and $v(C)$ are of the same parity for every cycle C such that $v(G_C) = n$.

If a vertex is added to G_C in order to obtain $G_{C'}$ where $G_{C'}$ is a new cycle in T, then $v(G_{C'}) = n + 1$ and thus parities of $v(G_C)$ and $v(G_{C'})$ are different. It is sufficient to prove that parity of $v(C)$ and $v(C')$ are different too.

A vertex can be added to G_C in two different ways:

a) new vertex is of degree 1 in $G_{C'}$ and

b) new vertex is of degree 2 in $G_{C'}$.

In case a) cycle C' has one vertex more than C, and in case b) one vertex less, and so in both cases parities of $v(C)$ and $v(C')$ are different. □

Face f in G will be called an *$i-$face* if $v_f = i$. $1-$faces will be called *triangles* and $i-$faces for $i > 2$ will be called *multifaces*. *Lemma 1* immediately yields that there are no $2-$faces in G (i.e. each face in G is a triangle or a multiface) and also that two triangles cannot be neighboring (i.e. each triangle is neighboring to three multifaces in G). Multiface f is said to be *cyclic* (*acyclic*) if D_f is a cyclic (acyclic) graph.

Let $F(G)$ denote the set of faces in G. Let

$$v_{avg}(G) = \frac{\sum_{f\in F(G)} v_f}{|F(G)|}. \tag{6}$$

Since

$$|F(G)| = \frac{\sum_{f\in F(G)} v_f}{v_{avg}(G)} = \frac{2(n-1)^2}{v_{avg}(G)}, \tag{7}$$

if $v_{avg(G)} is$ determined, number $|F(G)|$ which is of our interest is also determined.

Lemma 3. *Let G_1 be a subgraph of G induced by vertices*

$$\{v_{i,j} : 2 \leq i \leq n-1, \quad 2 \leq j \leq n-1\}.$$

Then $v_{avg}(G_1) \geq 8/3$.

Proof. To prove the lemma, partition of $F(G)$ will be made, but prior to that the following notions will be needed. Let f be an arbitrary acyclic multiface and t a triangle neighboring to f. Let $D_{f,t}$ denote subgraph of dual graph D induced by vertices d_t which belong to f or t. Triangle t is said to *make face f cyclic* if $D_{f,t}$ is cyclic.

Set $F(G_1)$ is then divided in the following subsets:

1. set F_1' of all triangles in G_1,
2. set F_2' of all cyclic multifaces in G_1,
3. set F_3' of all acyclic multifaces in G_1 with even v_f,
4. set F_4' of all acyclic multifaces in G_1 with odd v_f and no neighboring triangles,
5. set F_5' of all acyclic multifaces f in G_1 with odd v_f with neighboring triangles at least one of which does not make f cyclic,
6. set F_6' of all acyclic multifaces f in G_1 with odd v_f with neighboring triangles all of which make f cyclic.

The following redistribution from F_1' to F_5' and F_6' is then made: for each multiface f in F_5' we add to F_5' one triangle from F_1' which makes it acyclic, and for each multiface f in F_6' we add to F_6' one triangle from F_1' which neighbors f. This results in new sets $F_1, \ldots, F_6$.

Since $F_1, F_2, F_3, F_4, F_5, F_6$ is a partition of $F(G_1)$, we have$|F(G)|$

$$v_{avg}(G_1) = \frac{\sum_{i=1}^{6} \sum_{f \in F_i} v_f}{|F(G_1)|}. \tag{8}$$

For each face f in G_1 let t_f be the number of triangles from F_1 neighboring f. Since each triangle in F_1 neighbors three multifaces it follows

$$v_{avg}(G_1) = \frac{\sum_{i=2}^{6} \sum_{f \in F_i} \left(v_f + \frac{1}{3} t_f\right)}{|F(G_1)|}. \tag{9}$$

Let c be an arbitrary constant. If $\sum_{f \in F_i} \left(v_f + \frac{1}{3} t_f\right) \geq c \sum_{f \in F_i} \left(1 + \frac{1}{3} t_f\right)$ for every $i = 2, \dots, 6$ or $v_f + \frac{1}{3} t_f \geq c\left(1 + \frac{1}{3} t_f\right)$ for every $i = 2, \dots, 6$ and every $f \in F_i$, then

$$\begin{aligned} v_{avg}(G_1) &= \frac{\sum_{i=2}^{6} \sum_{f \in F_i} \left(v_f + \frac{1}{3} t_f\right)}{|F(G_1)|} \geq \frac{\sum_{i=2}^{6} \sum_{f \in F_i} c\left(1 + \frac{1}{3} t_f\right)}{|F(G_1)|} \\ &= \frac{c \sum_{i=2}^{6} \sum_{f \in F_i} \left(1 + \frac{1}{3} t_f\right)}{|F(G_1)|} = \frac{c \sum_{i=2}^{6} \left(|F_i| + \sum_{f \in F_i} \frac{1}{3} t_f\right)}{|F(G_1)|} \\ &= \frac{c\,|F(G_1)|}{|F(G_1)|} = c \end{aligned}$$

So it is sufficient to prove $v_f + \frac{1}{3} t_f \geq \frac{8}{3}(1 + \frac{1}{3} t_f)$ or simplified $9v_f \geq 24 + 5t_f$ for every $i = 2, \dots, 6$ and every $f \in F_i$. Note that for every multiface f in G_1, $t_f \leq 3v_f - 2e_f$.

Case i=2:

Suppose contrary, i.e. $9v_f < 24 + 5t_f$. Since $t_f \leq 3v_f - 2e_f$ for every $f \in F_2$ it follows

$$9v_f < 24 + 5t_f \leq 24 + 5(3v_f - 2e_f) = 24 + 15v_f - 10e_f, \tag{10}$$

$$6v_f > 10e_f - 24. \tag{11}$$

Since each $f \in F_2$ is cyclic we have $e_f \geq v_f$ so

$$6v_f > 10e_f - 24 \geq 10v_f - 24, \tag{12}$$

$$v_f > 6, \tag{13}$$

which is a contradiction because only cycles in D are of length ≥ 6, so in order for f to be cyclic it has to be $v_f \geq 6$.

Case i=3:

Suppose contrary, i.e. $9v_f < 24 + 5t_f$. Since $t_f \leq 3v_f - 2e_f$ for every $f \in F_3$ it follows $6v_f > 10e_f - 24$. Every multiface $f \in F_3$ is acyclic so $e_f \geq v_f - 1$ and thus it follows $v_f \leq 8$.

Let C_f denote a cycle consisting of edges and vertices on the border of the face f. Since v_f is even, according to *Lemma 2* so is $|V(C_f)|$, and therefore vertices in C_f can be perfectly matched in two different ways. If vertices in $V(T) \backslash V(C_f)$ can be perfectly matched, then there are two different perfect matchings on T which coincide on E (those two perfect matchings differ only on the cycle C_f which is in G and thus not in E).

To prove that all vertices from $V(T)\backslash V(C_f)$ can be perfectly matched first note: if $v_{i,j} \notin V(C_f)$, then, since $v_f \leq 8$, one of the following statements is true:

a) there is no l, m such that $l < i < m$ and $v_{i,j}, v_{m,j} \in V(C_f)$, $v_{i,j} \notin V(C_f)$ or

b) there is no l, m such that $l < i < m$ and $v_{i,l}, v_{i,m} \in V(C_f)$, $v_{i,j} \notin V(C_f)$.

Without loss of generality suppose a). Subgraph of T induced by set of vertices $\{v_{i,j} : i_1 \leq i \leq i_2, \quad j_1 \leq j \leq j_2\}$ will be called a *rectangle*. Furthermore, the set of vertices $\{v_{i,j} : i = i_1, \quad j_1 \leq j \leq j_2\}$ will be called the *upper border* of a rectangle, and the set of vertices $\{v_{i,j} : i = i_2, \quad j_1 \leq j \leq j_2\}$ will be called the *lower border* of a rectangle.

Let

$$\begin{aligned} i_{\min} &= \min\{i : v_{i,j} \in V(C_f)\}, \\ i_{\max} &= \max\{i : v_{i,j} \in V(C_f)\}, \\ j_{\min} &= \min\{j : v_{i,j} \in V(C_f)\}, \\ j_{\max} &= \max\{j : v_{i,j} \in V(C_f)\}. \end{aligned}$$

If $j_{\max} - j_{\min}$ is odd, let $j'_{\min} = j_{\min} - 1$, else let $j'_{\min} = j_{\min}$. Let R_1 denote the rectangle induced by $\{v_{i,j} : i_{\min} \leq i \leq i_{\max}, \quad j'_{\min} \leq j \leq j_{\max} + 1\}$ and R_2 denote the rectangle induced by $\{v_{i,j} : i_{\min} - 1 \leq i \leq i_{\max} + 1, \quad j'_{\min} \leq j \leq j_{\max} + 1\}$. Note that the number of vertices in R_1 is even and that vertices from $V(R_1)\backslash V(C_f)$ can be matched so that only unmatched vertices remain on the upper and the lower border of R_1.

Let us prove that all unmatched vertices on the upper border of R_1 and vertices on the upper border of R_2 can be perfectly matched. The proof is by induction by number of pairs of matched vertices on upper border of R_1.

If there is only one pair of matched vertices, i.e. vertices $v_{i,k}$ and $v_{i,m}$, then vertices $v_{i,l}$, for $k < l < m$, are matched by $v_{i,l}v_{i-1,l+1}$, vertices $v_{i-1,l}$ and $v_{i-1,l+1}$ are matched with each other, and all other unmatched vertices on the upper border of R_1 can be matched by $v_{i,l}v_{i-1,l}$.

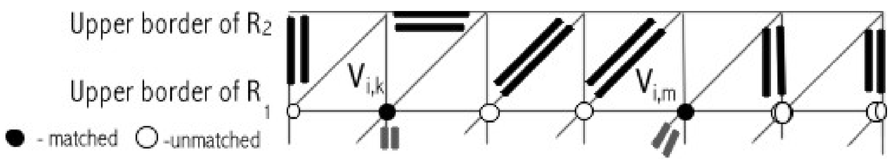

Figure 9.

Suppose perfect matching is possible when there are n unmatched pairs on the upper border of R_1.

If there are $n + 1$ matched pairs on the upper border of R_1, one pair can be matched like in the base of the induction, and the rest by supposition.

Analogously, a perfect matching of all unmatched vertices on the lower border of R_1 and vertices on the lower border of R_2 can be obtained. Thus, we have a perfect matching of vertices from $V(R_2)\backslash V(C_f)$. Since the rest of the vertices in T is readily matched, a perfect matching of $V(T)\backslash V(C_f)$ is obtained.

Case i=4:
Note that $t_f = 0$ for every $f \in F_4$. So what has to be proved is $9v_f \geq 24$. Since each $f \in F_4$ is multiface $v_f \geq 3$, so it has to be $9v_f \geq 27 \geq 24$.

Case i=5:
Set F_5 consists of all acyclic multifaces f in G_1 with odd v_f which have neighboring triangles at least one of which makes f acyclic and that one triangle is added to F_5 for each multiface F_5. So each pair of multiface f and the corresponding triangle can be considered as a new face f' which is acyclic in G_1 with $v_{f'} \leq 8$ and $v_{f'}$ even. Then this case reduces to case $i = 3$.

Case i=6:
In this case $\sum_{f\in F_6}\left(v_f + \frac{1}{3}t_f\right) \geq \frac{8}{3}\sum_{f\in F_6}\left(1 + \frac{1}{3}t_f\right)$ will be proved. Since $v_f = 1$ and $t_f = 0$, when f is a triangle, this inequality can be written as

$$\sum_{\substack{f\in F_6\\ f-multiface}}\left(v_f + 1 + \frac{1}{3}t_f\right) \geq \frac{8}{3}\sum_{\substack{f\in F_6\\ f-multiface}}\left(1 + 1 + \frac{1}{3}t_f\right) \tag{14}$$

so it is sufficient to prove $\left(v_f + 1 + \frac{1}{3}t_f\right) \geq \frac{8}{3}\left(1 + 1 + \frac{1}{3}t_f\right)$ for each multiface $f \in F_6$ or simplified $9v_f \geq 39 + 5t_f$.

Suppose contrary, i.e. $9v_f < 39 + 5t_f$. Note that in this case a better upper bound for t_f can be obtained, i.e. $t_f \leq (3v_f - 2e_f - 2)/2$, since every triangle neighboring multiface $f \in F_6$ makes f cyclic. Thus,

$$9v_f < 39 + 5t_f \leq 39 + 5\frac{3v_f - 2e_f - 2}{2}, \tag{15}$$

$$3v_f < 68 - 10e_f. \tag{16}$$

Since every multiface $f \in F_6$ is acyclic, we have $e_f \geq v_f - 1$ and therefore

$$3v_f < 68 - 10e_f \leq 68 - 10(v_f - 1) = 78 - 10v_f, \tag{17}$$

$$v_f < \frac{78}{13} = 6. \tag{18}$$

Since v_f is odd and there must be at least one triangle neighboring to f which makes f cyclic, only possibility is $v_f = 5$. But in that case $t_f = 0$, so it must be $9v_f < 39$ which is a contradiction because $9v_f = 9 \cdot 5 = 45 > 39$. □

Now the upper bound for $|E(G)|/|E(T)|$ in the asymptotic case can be determined.

Theorem 2. $|E| \geq \frac{5}{4}n^2 - \frac{21}{2}n + \frac{41}{4}$

Proof. Since $v_{avg}(G_1) \geq 8/3$, and $\sum_{f\in F(G_1)} v_f \leq 2(n-1)^2$, it follows

$$|F(G_1)| = \frac{\sum_{f\in F(G_1)} v_f}{v_{avg}(G_1)} \leq \frac{2(n-1)^2}{8/3}. \tag{19}$$

Furthermore, since $|F(G)\backslash F(G_1)| \leq 8 + 8(n-2)$, it follows

$$|F(G_1)| \leq \frac{2(n-1)^2}{8/3} + 8 + 8(n-2) = \frac{3}{4}n^2 + \frac{13}{2}n - \frac{29}{4}. \tag{20}$$

Since G is planar, $|V(G)| - |E(G)| + |F(G)| = 2$. Consequently,

$$|E(G)| = |V(G)| + |F(G)| - 2 \leq \frac{7}{4}n^2 + \frac{13}{2}n - \frac{37}{4}. \tag{21}$$

Therefore, $|E| = |E(T)| - |E(G)| \geq \frac{5}{4}n^2 - \frac{21}{2}n + \frac{41}{4}$. $\square$

Herefrom it readily follows that:

Corollary 2. *When n tends to infinity then $|E|\,/\,|E(T)| \geq 5/12$.*

We can conclude from *Corollary 1* and *Corollary 2* that $|E|\,/\,|E(T)|$ tends to $5/12$ when n tends to infinity.

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