Existence and approximation of solutions of a system of differential equations of Volterra type*

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Abstract. *The present paper deals with the nonlinear systems of differential equations of Volterra type regarding the existence, behaviour, approximation and stability of their definite solutions, all solutions in a corresponding region or parametric classes of solutions on an unbounded interval. The approximate solutions with precise error estimates are determined. The theory of qualitative analysis of differential equations and the topological retraction method are used.*

Key words: *Volterra system, existence and behaviour of solutions, approximation of solutions*

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1. Introduction

We shall consider the systems of differential equations of Volterra type in the form

$$\dot{x}_i = f_i(x,t)\, x_i, \quad i = 1, \cdots, n, \tag{1}$$

or in the special forms

$$\dot{x}_i = [p_i(t) + h_i(x,t)]\, x_i, \quad i = 1, \cdots, n, \tag{2}$$

$$\dot{x}_i = [q_i + g_i(x)]\, x_i, \quad i = 1, \cdots, n, \tag{3}$$

where $x(t) = (x_1(t), \cdots, x_n(t))^\tau$, $f_i, h_i \in C(\Omega, \mathbb{R})$, $g_i \in C(D, \mathbb{R})$, $p_i \in C(I, \mathbb{R})$, $q_i \in \mathbb{R}$, $i = 1, \cdots, n$, $D \subset \mathbb{R}^n$ is an open set, $\Omega = D \times I$, $I = \langle a, \infty \rangle$, $a \in \mathbb{R}$. Functions f_i, h_i, g_i satisfy the Lipschitz's condition with respect to the variable x on D.

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In his paper ([8]), Volterra considers an ecosystem model

$$\begin{aligned} \frac{dx_1}{dt} &= (\rho - \eta x_1 - \theta x_2)\, x_1, \\ \frac{dx_2}{dt} &= (-\mu + \nu x_1)\, x_2, \quad \rho, \eta, \theta, \mu, \nu \in \mathbb{R}^+. \end{aligned} \tag{4}$$

Many authors considered the systems of differential equations of Volterra type, for example, K. Sigmund, Y. Takeuchi, N. Adachi, A. Tineo ([5], [6], [7]). The systems of type (3) are considered very often. This model is used in physics, biophysics, chemistry, biochemistry and economy.

Let

$$\Gamma = \{(x,t) \in \Omega : x = \varphi(t),\, t \in I\}$$

be a curve in Ω, for some $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t))$, $\varphi_i \in C^1(I, \mathbb{R})$. We shall consider the behaviour of integral curves $(x(t), t)$, $t \in I$, of systems $(1), (2)$ and (3), with respect to the sets $\omega, \sigma \subset \Omega$, which are the appropriate neighbourhoods of curve Γ, in the forms

$$\omega = \{(x,t) \in \Omega : \|x - \varphi(t)\| < r(t)\}, \tag{5}$$

$$\sigma = \{(x,t) \in \Omega : |x_i - \varphi_i(t)| < r_i(t),\, i = 1, \cdots n\} \tag{6}$$

($\|\cdot\|$ is a Euclidian norm on $\mathbb{R}^n$), where $r, r_i \in C^1(I, \mathbb{R}^+)$, $i = 1, \cdots, n$, $\mathbb{R}^+ = \langle 0, \infty \rangle$.

The boundary surfaces of ω and σ are, respectively,

$$W = \left\{ (x,t) \in Cl\omega \cap \Omega : B(x,t) := \sum_{i=1}^{n} (x_i - \varphi_i(t))^2 - r^2(t) = 0 \right\}, \tag{7}$$

$$\begin{aligned} W_i^k &= \left\{ (x,t) \in Cl\sigma \cap \Omega : B_i^k(x,t) := (-1)^k (x_i - \varphi_i(t)) - r_i(t) = 0 \right\}, \\ & k = 1, 2, \;\; i = 1, \cdots, n. \end{aligned} \tag{8}$$

Let us denote the tangent vector field to an integral curve $(x(t), t)$, $t \in I$, of $(1), (2)$ and (3) by T. For example, for system (1) we have

$$T(x,t) = (f_1(x,t)\, x_1, \cdots, f_i(x,t)\, x_i, \cdots, f_n(x,t)\, x_n, 1). \tag{9}$$

The vectors ∇B and ∇B_i^k are the external normals on surfaces W and W_i^k, respectively,

$$\frac{1}{2} \nabla B(x,t) = \begin{pmatrix} x_1 - \varphi_1(t), \cdots, x_i - \varphi_i(t), \cdots, x_n - \varphi_n(t), \\ -\sum\limits_{i=1}^{n} (x_i - \varphi_i(t))\, \varphi_i'(t) - r(t)\, r'(t) \end{pmatrix}, \tag{10}$$

$$\nabla B_i^k(x,t) = (-1)^k \left(\delta_{1i}, \cdots, \delta_{ii}, \cdots, \delta_{ni}, -\varphi_i'(t) - (-1)^k r_i'(t) \right), \tag{11}$$

where δ_{mi} is the Kronecker delta symbol.

Considering the sign of the scalar products

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) \quad \text{on } W$$

and

$$P_i^k(x,t) = \left(\nabla B_i^k(x,t), T(x,t)\right) \quad \text{on } W_i^k,\ k = 1,2,\ i = 1,\ldots,n,$$

we shall establish the behaviour of integral curves of $(1),(2)$ and (3) with respect to the sets ω and σ, respectively.

The results of this paper are based on the following *Lemmas 1* and *2* (see [11]) and *Lemma 3* (see [3]). In the following $(n_1,\cdots,n_n)$, denote a permutation of indices $(1,\cdots,n)$.

Lemma 1. *If, for system* (1), *the scalar product*

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) < 0 \quad \text{on } W,$$

then system (1) *has an* $n-$*parameter class of solutions belonging to the set* ω *(graphs of solutions belong to* ω*) for all* $t \in I$.

According to this *Lemma*, for any point $P_0 = \left(x^0, t_0\right) \in \omega$, the integral curve passing through P_0 belongs to ω *for all* $t \geq t_0$.

Lemma 2. *If, for system* (1), *the scalar product*

$$P(x,t) = \left(\frac{1}{2}\nabla B(x,t), T(x,t)\right) > 0 \quad \text{on } W,$$

then system (1) *has at least one solution on* I *whose graph belongs to the set* ω *for all* $t \in I$.

Lemma 3. *If, for the system* (1), *the scalar products*

$$P_i^k = \left(\nabla B_i^k, T\right) < 0 \quad \text{on } W_i^k,\ k = 1,2,\ i = n_1,\cdots,n_p, \tag{12}$$

and

$$P_i^k = \left(\nabla B_i^k, T\right) > 0 \quad \text{on } W_i^k,\ k = 1,2,\ i = n_{p+1},\cdots,n_n, \tag{13}$$

where $p \in \{0,1,\cdots,n\}$, *then system* (1) *has a* $p-$*parameter class of solutions which belongs to the set* σ *(graphs of solutions belong to* σ*) for all* $t \in I$.

The case $p = 0$ means that system (1) has at least one solution belonging to the set σ for all $t \in I$.

2. The n-parameter classes of solutions

First, let us consider the behaviour of integral curves of systems (1), (2) and (3) with respect to the set ω.

Theorem 1. *If*

$$\sum_{i=1}^{n} f_i(x,t)\, x_i^2 < r(t)\, r'(t) \quad \text{or} \tag{14}$$

$$f_i(x,t) < \frac{r'(t)}{r(t)},\ i = 1,\cdots,n, \quad on\ W, \tag{15}$$

then system (1) *has an* $n-$*parameter class of solutions* $x(t)$ *satisfying the condition*

$$\|x(t)\| < r(t), \quad t \in I, \tag{16}$$

i.e. every solution $x(t)$ *of system* (1) *which satisfies the condition*

$$\|x(t_0)\| < r(t_0), \quad t_0 \in I, \tag{17}$$

satisfies (16) *for every* $t \geq t_0$.

Proof. Here we have that the curve Γ is a t-axis ($\varphi = 0$). For the scalar product $P(x,t)$ on W we have

$$P(x,t) = \sum_{i=1}^{n} f_i(x,t)\, x_i^2 - r(t)\, r'(t).$$

According to (14), obviously $P(x,t) < 0$ on W, and according to (15), we have

$$P(x,t) < \frac{r'(t)}{r(t)} \sum_{i=1}^{n} x_i^2 - r(t)\, r'(t) = \frac{r'(t)}{r(t)} r^2(t) - r(t)\, r'(t) = 0 \quad \text{on } W.$$

According to *Lemma 1*, the above estimates confirm the statement of the *Theorem.* □

Using *Theorem 1* we can give special results. For example, the following

Corollary 1. *If*

$$f_i(x,t) < 0,\ i = 1,\cdots,n, \quad on\ W,$$

then system (1) *has an* $n-$*parameter class of solutions* $x(t)$ *satisfying condition* (16), *where function* r *satisfies conditions* (15)*and*

$$r'(t) \leq 0 \quad on\ I. \tag{18}$$

Obviously, in the general case for function r we can take an arbitrary positive constant.

Theorem 2. *Let* Γ *be any integral curve of system* (1), $M \in C(\Omega, \mathbb{R})$ *and*

$$f_i(x,t) \leqslant M(x,t), \quad i = 1,\cdots,n, \quad on\ W. \tag{19}$$

If, on W,

$$|f_i(x,t) - f_i(y,t)| \leqslant L_i \|x - y\|, \quad i = 1,\cdots,n, \quad x, y \in D \tag{20}$$

and

$$\sum_{i=1}^{n} L_i |\varphi_i(t)| < -M(x,t) + \frac{r'(t)}{r(t)} \tag{21}$$

or

$$\sum_{i=1}^{n} |(f_i(x,t) - f_i(\varphi,t))\,\varphi_i(t)| < -M(x,t)\,r(t) + r'(t)\,, \tag{22}$$

then system (1) *has an n-parameter class of solutions $x(t)$ satisfying the condition*

$$\|x(t) - \varphi(t)\| < r(t)\,, \quad t \in I, \tag{23}$$

i.e. every solution $x(t)$ of system (1) *which satisfies the condition*

$$\|x(t_0) - \varphi(t_0)\| < r(t_0)\,, \quad t_0 \in I, \tag{24}$$

satisfies (23) *for every $t \geq t_0$.*

Proof. For the scalar product $P(x,t)$ on W we have

$$\begin{aligned} P(x,t) &= \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]\,x_i - \sum_{i=1}^{n}[x_i - \varphi_i(t)]\,\varphi_i'(t) - r(t)\,r'(t) \\ &= \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]\,\varphi_i(t) \\ &\quad - \sum_{i=1}^{n}[x_i - \varphi_i(t)]\,\varphi_i'(t) - r(t)\,r'(t) \\ &= \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]^2 \\ &\quad + \sum_{i=1}^{n}[f_i(x,t)\,\varphi_i(t) - \varphi_i'(t)]\,[x_i - \varphi_i(t)] - r(t)\,r'(t) \\ &= \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]^2 \\ &\quad + \sum_{i=1}^{n}\{[f_i(x,t) - f_i(\varphi,t) + f_i(\varphi,t)]\,\varphi_i(t) - \varphi_i'(t)\}\,[x_i - \varphi_i(t)] \\ &\quad - r(t)\,r'(t) \\ &= \sum_{i=1}^{n} f_i(x,t)\,[x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n}[f_i(x,t) - f_i(\varphi,t)]\,[x_i - \varphi_i(t)]\,\varphi_i(t) \\ &\quad - r(t)\,r'(t)\,. \end{aligned} \tag{25}$$

Now, we have on W, by (21)

$$\begin{aligned} P(x,t) &\leqslant M(x,t)\,r^2(t) + \sum_{i=1}^{n} L_i\,|\varphi_i(t)|\,r^2(t) - r(t)\,r'(t) \\ &< M(x,t)\,r^2(t) + \left(-M(x,t) + \frac{r'(t)}{r(t)}\right) r^2(t) - r(t)\,r'(t) = 0, \end{aligned}$$

and by (22)

$$P(x,t) \leqslant M(x,t)\, r^2(t) + \sum_{i=1}^{n} \left|\left[f_i(x,t) - f_i(\varphi,t)\right] \varphi_i(t)\right| r(t) - r(t)\, r'(t)$$
$$< M(x,t)\, r^2(t) + \left(-M(x,t)\, r(t) + r'(t)\right) r(t) - r(t)\, r'(t) = 0.$$

The above estimates for $P(x,t)$ on W, according to *Lemma 1*, imply the statement of the *Theorem.* $\square$

Theorem 3. *Let* Γ *be any smooth curve in* Ω *and let* $m, M \in C(\Omega, \mathbb{R})$, $\xi \in C\left(\Omega, \mathbb{R}_0^+\right)$, $\mathbb{R}_0^+ = [0, \infty\rangle$, *such that*

$$\sum_{i=1}^{n} \left|f_i(x,t)\, \varphi_i(t) - \varphi_i'(t)\right| \leqslant \xi(x,t) \quad \text{on } W. \tag{26}$$

(i) If

$$f_i(x,t) \leqslant M(x,t), \quad i = 1, \cdots, n, \quad \text{and} \tag{27}$$
$$\xi(x,t) < -M(x,t)\, r(t) + r'(t) \quad \text{on } W, \tag{28}$$

then system (1) *has an* $n-$*parameter class of solutions* $x(t)$ *satisfying condition* (23).

(ii) If

$$m(x,t) \leqslant f_i(x,t), \quad i = 1, \cdots, n, \quad \text{and} \tag{29}$$
$$\xi(x,t) < m(x,t)\, r(t) - r'(t) \quad \text{on } W, \tag{30}$$

then system (1) *has at least one solution* $x(t)$ *which satisfies condition* (23).

Proof. In view of (25), for the scalar product $P(x,t)$ on W we have:

$$P = \sum_{i=1}^{n} f_i\left(x_i - \varphi_i\right)^2 + \sum_{i=1}^{n} \left(f_i \varphi_i - \varphi_i'\right)\left(x_i - \varphi_i\right) - rr' \quad \text{and}$$

(i) $\quad P \leqslant Mr^2 + \xi r - rr' < 0,$

(ii) $\quad P \geqslant mr^2 - \xi r - rr' > 0.$

According to *Lemmas 1* and *2*, in cases (i) and (ii), respectively, the above estimates for $P(x,t)$ on W imply the statements of the *Theorem.* $\square$

Theorem 4. *Let* $(\varphi(t), t)$, $t \in I$, $\varphi \neq 0$, *be a curve of stationary points of system* (1) $(f_i(\varphi(t), t) = 0,\ t \in I,\ i = 1, \cdots, n)$, *let* (20) *hold and let* $m, M \in C(I, \mathbb{R})$.

(i) If

$$f_i(x,t) \leqslant M(t), \quad i = 1, \cdots, n, \quad (x,t) \in W \quad \text{and}$$
$$r(t) \sum_{i=1}^{n} L_i \left|\varphi_i(t)\right| + \sum_{i=1}^{n} \left|\varphi_i'(t)\right| < -M(t)\, r(t) + r'(t), \quad t \in I,$$

then system (1) *has an* n*-parameter class of solutions* $x(t)$ *satisfying condition* (23).

(ii) If

$$m(t) \le f_i(x,t), \quad i = 1, \cdots, n, \quad (x,t) \in W \quad \textit{and}$$

$$r(t)\sum_{i=1}^{n} L_i |\varphi_i(t)| + \sum_{i=1}^{n} |\varphi_i'(t)| < m(t) r(t) - r'(t), \quad t \in I,$$

then system (1) *has at least one solution* $x(t)$ *which satisfies condition* (23).

Proof. From (25) we have on W

$$\begin{aligned}
P(x,t) &= \sum_{i=1}^{n} f_i(x,t)[x_i - \varphi_i(t)]^2 \\
&\quad + \sum_{i=1}^{n} \{[f_i(x,t) - f_i(\varphi,t)]\varphi_i(t) - \varphi_i'(t)\}[x_i - \varphi_i(t)] - r(t) r'(t) \\
&= \sum_{i=1}^{n} f_i(x,t)[x_i - \varphi_i(t)]^2 + \sum_{i=1}^{n} [f_i(x,t) - f_i(\varphi,t)][x_i - \varphi_i(t)]\varphi_i(t) \\
&\quad - \sum_{i=1}^{n} \varphi_i'(t)[x_i - \varphi_i(t)] - r(t) r'(t).
\end{aligned}$$

The conditions of the Theorem imply the estimates in case (i) and (ii), respectively:

$$P(x,t) \leqslant M(t) r^2(t) + \sum_{i=1}^{n} L_i |\varphi_i(t)| r^2(t) + \sum_{i=1}^{n} |\varphi_i'(t)| r(t) - r(t) r'(t) < 0,$$
$$P(x,t) \geqslant m(t) r^2(t) - \sum_{i=1}^{n} L_i |\varphi_i(t)| r^2(t) - \sum_{i=1}^{n} |\varphi_i'(t)| r(t) - r(t) r'(t) > 0.$$

The above estimates for $P(x,t)$ on W, according to *Lemma 1* in case (i) and according to *Lemma 2* in case (ii), imply the statements of the *Theorem*. □

Now let us consider system (2).

Theorem 5. *Let* $m, M \in C(\Omega, R)$ *and*

$$\varphi_i(t) = C_i \exp\left[\int p_i(t)\,dt\right], \quad i = 1, \cdots, n, \quad C_i \in \mathbb{R}. \tag{31}$$

(i) If

$$p_i(t) + h_i(x,t) \leqslant M(x,t), \quad i = 1, \cdots, n \quad \textit{and}$$
$$\sum_{i=1}^{n} |h_i(x,t)\varphi_i(t)| \leqslant -M(x,t) r(t) + r'(t) \quad \textit{on } W,$$

then system (2) *has an* $n-$*parameter class of solutions* $x(t)$ *satisfying condition* (23).

(ii) If

$$m(x,t) \leqslant p_i(t) + h_i(x,t), \quad i = 1, 2, \cdots, n \quad \textit{and}$$

$$\sum_{i=1}^{n} |h_i (x,t) \varphi_i (t)| \leqslant m (x,t) r (t) + r' (t) \quad \text{on } W,$$

then system (2) *has at least one solution* $x(t)$ *which satisfies condition* (23).

Proof. Firstly, we can note that $p_i (t) \varphi_i (t) - \varphi_i' (t) = 0,\ i = 1, \cdots , n,\ t \in I$. Now, according to (25) and conditions of this theorem, we have on W:

$$\begin{aligned} P &= \sum_{i=1}^{n} (p_i + h_i) (x_i - \varphi_i)^2 + \sum_{i=1}^{n} [(p_i + h_i) \varphi_i - \varphi_i'] (x_i - \varphi_i) - rr' \\ &= \sum_{i=1}^{n} (p_i + h_i) (x_i - \varphi_i)^2 + \sum_{i=1}^{n} h_i \varphi_i (x_i - \varphi_i) - rr' \quad \text{and} \end{aligned}$$

(*i*)

$$\begin{aligned} P &\le r^2 \sum_{i=1}^{n} (p_i + h_i) + r \sum_{i=1}^{n} |h_i \varphi_i| - rr' \\ &\le Mr^2 + r \sum_{i=1}^{n} |h_i \varphi_i| - rr' < 0, \end{aligned} \tag{32}$$

(*ii*)

$$P \ge mr^2 - r \sum_{i=1}^{n} |h_i \varphi_i| - rr' > 0. \tag{33}$$

According to *Lemma 1*, estimate (32) implies that system (2) has the $n-$parameter class of solutions $x(t)$ belonging to the corresponding set ω for all $t \in I$, and estimate (33) implies that system (2) has at least one solution $x(t)$ which satisfies that condition. This confirms the statements of the Theorem. □

Now let us consider system (3) in neighbourhood of (x^0, t), $t \in I$, where $x^0 = (x_1^0, \cdots , x_n^0) \in \mathbb{R}^n$ and

$$q_i + g_i (x^0) = 0,\ i = 1, \cdots , n. \tag{34}$$

Theorem 6. *Let* (34) *and*

$$\begin{aligned} |g_i (x) - g_i (y)| \leqslant L_i \|x - y\|, \quad i = 1, \cdots , n, \quad x, y \in D \\ q_i + g_i (x) \le Q \in \mathbb{R},\ i = 1, \cdots , n, \quad \text{on } D. \end{aligned} \tag{35}$$

If

$$\sum_{i=1}^{n} L_i |x_i^0| < -Q + \frac{r' (t)}{r (t)}, \quad t \in I, \tag{36}$$

then system (3) *has an n-parameter class of solutions* $x(t)$ *satisfying the condition*

$$\|x (t) - x^0\| < r (t), \quad t \in I. \tag{37}$$

Proof. For the scalar product $P(x,t)$ on W, using (25) with $f_i(x,t) = q_i + g_i(x)$, $\varphi_i = x_i^0$, $i = 1, \cdots, n$, we have

$$\begin{aligned} P(x,t) &= \sum_{i=1}^{n} (q_i + g_i(x)) \left(x_i - x_i^0\right)^2 + \sum_{i=1}^{n} (q_i + g_i(x)) \, x_i^0 \left(x_i - x_i^0\right) - r(t) \, r'(t) \\ &= \sum_{i=1}^{n} (q_i + g_i(x)) \left(x_i - x_i^0\right)^2 + \sum_{i=1}^{n} \left(g_i(x) - g_i\left(x^0\right)\right) x_i^0 \left(x_i - x_i^0\right) - r(t) \, r'(t) \\ &\leqslant Qr^2(t) + \sum_{i=1}^{n} L_i \left|x_i^0\right| r^2(t) - r(t) \, r'(t) < 0. \end{aligned}$$

Hence, according to *Lemma 1*, the statement of the *Theorem* is valid. □

Corollary 2. *If in* Theorem 6 *condition* (36) *is replaced by*

$$\sum_{i=1}^{n} L_i \left|x_i^0\right| + Q < 0,$$

the statement of Theorem 6 *holds with the function*

$$r(t) = \alpha e^{-\beta t}, \quad \alpha, \beta \in \mathbb{R}^+ \quad \textit{and}$$

$$0 < \beta \leq s = -\left(\sum_{i=1}^{n} L_i \left|x_i^0\right| + Q \right).$$

For $\beta = s$ condition (37) *should be replaced by*

$$\left\| x(t) - x^0 \right\| \leq r(t), \quad t \in I.$$

In case $x^0 = 0$ for system (3) we can use *Theorem 1* and *Corollary 1.* Moreover, we can note that *Theorem 6* and *Corollary 2* are valid without assumption (35).

3. The p-parameter classes of solutions

Here we consider the behaviour of integral curves of systems (1), (2) and (3) with respect to the set σ.

Theorem 7. *If, on W_i^k, $k = 1, 2$,*

$$f_i(x,t) < \frac{r_i'(t)}{r_i(t)}, \quad i = n_1, \cdots, n_p, \tag{38}$$

$$f_i(x,t) > \frac{r_i'(t)}{r_i(t)}, \quad i = n_{p+1}, \cdots, n_n, \tag{39}$$

then system (1) *has a p-parameter class of solutions $x(t)$ satisfying condition*

$$|x_i(t)| < r_i(t), \quad i = 1, \cdots, n, \quad t \in I. \tag{40}$$

Proof. Here is $\varphi = 0$. For the scalar products $P_i^k(x,t)$ on W_i^k, $k = 1,2$, $i = 1, \cdots, n$, we have

$$\begin{aligned} P_i^k(x,t) &= (-1)^k f_i(x,t)\, x_i - r_i'(t) \\ &= f_i(x,t)\, r_i(t) - r_i'(t) \end{aligned}$$

and

$$\begin{aligned} P_i^k(x,t) &< 0 \quad \text{for } i = n_1, \cdots, n_p, \\ P_i^k(x,t) &> 0 \quad \text{for } i = n_{p+1}, \cdots, n_n. \end{aligned}$$

According to *Lemma 3*, the above estimates imply the statement of the *Theorem.* □

Using *Theorem 7* we can give special results. For example,

Corollary 3. *If, on* W_i^k, $k = 1,2$,

$$\begin{aligned} f_i(x,t) &< 0, \quad i = n_1, \cdots, n_p, \\ f_i(x,t) &> 0, \quad i = n_{p+1}, \cdots, n_n, \end{aligned}$$

then system (1) *has a p-parameter class of solutions* $x(t)$ *which satisfy condition*

$$|x_i(t)| < r(t), \quad i = 1, \cdots, n, \quad t \in I,$$

where function r *satisfies conditions* (18) *and, on* W_i^k, $k = 1,2$,

$$f_i(x,t) - \frac{r'(t)}{r(t)} < 0, \quad i = n_1, \cdots, n_p.$$

For function r we can take, for example, some positive constant.

Theorem 8. *If, on* W_i^k, $k = 1,2$,

$$|f_i(x,t)\, x_i - \varphi_i'(t)| < r_i'(t) \quad \textit{or} \tag{41}$$

$$|f_i(x,t)\, \varphi_i(t) - \varphi_i'(t)| < -f_i(x,t)\, r_i(t) + r_i'(t) \tag{42}$$

for $i = n_1, \cdots, n_p$, *and*

$$|f_i(x,t)\, x_i - \varphi_i'(t)| < -r_i'(t) \quad \textit{or} \tag{43}$$

$$|f_i(x,t)\, \varphi_i(t) - \varphi_i'(t)| < f_i(x,t)\, r_i(t) - r_i'(t) \tag{44}$$

for $i = n_{p+1}, \cdots, n_n$, *then system* (1) *has a p-parameter class of solutions* $x(t)$ *satisfying the condition*

$$|x_i(t) - \varphi_i(t)| < r_i(t), \quad i = 1, \cdots, n, \quad t \in I. \tag{45}$$

Proof. Here for $P_i^k(x,t)$ on W_i^k, $k = 1,2$, $i = 1, \cdots, n$, we have

$$\begin{aligned} P_i^k(x,t) &= (-1)^k f_i(x,t)\, x_i - (-1)^k \varphi_i'(t) - r_i'(t) \\ &= (-1)^k [f_i(x,t)\, x_i - \varphi_i'(t)] - r_i'(t) \end{aligned} \tag{46}$$

or

$$\begin{aligned} P_i^k(x,t) &= (-1)^k [x_i - \varphi_i(t)] f_i(x,t) + (-1)^k [f_i(x,t)\varphi_i(t) - \varphi_i'(t)] - r_i'(t) \\ &= f_i(x,t) r_i(t) + (-1)^k [f_i(x,t)\varphi_i(t) - \varphi_i'(t)] - r_i'(t). \end{aligned} \tag{47}$$

Now (41) with (46) or (42) with (47) imply, respectively,

$$\begin{aligned} P_i^k &\leqslant |f_i x_i - \varphi_i'| - r_i' < 0, \\ P_i^k &\leqslant f_i r_i + |f_i \varphi_i - \varphi_i'| - r_i' < 0 \end{aligned}$$

on $W_i^k,\ k = 1, 2$, for $i = n_1, \cdots, n_p$. The conditions (43) with (46) or (44) with (47) imply, respectively,

$$\begin{aligned} P_i^k &\geqslant -|f_i x_i - \varphi_i'| - r_i' > 0, \\ P_i^k &\geqslant f_i r_i - |f_i \varphi_i - \varphi_i'| - r_i' > 0 \end{aligned}$$

on $W_i^k,\ k = 1, 2$, for $i = n_{p+1}, \cdots, n_n$. These estimates, according to the *Lemma 3*, confirm the statement of this *Theorem*. □

Theorem 9. *Let* Γ *be any integral curve of system* (1), *let* (20) *hold and let*

$$\rho(t) = \sqrt{r_1^2(t) + \cdots + r_n^2(t)}\ . \tag{48}$$

If, on $W_i^k,\ k = 1, 2$,

$$L_i |\varphi_i(t)| \rho(t) < -f_i(x,t) r_i(t) + r_i'(t)$$

for $i = n_1, \cdots, n_p$, *and*

$$L_i |\varphi_i(t)| \rho(t) < f_i(x,t) r_i(t) - r_i'(t)$$

for $i = n_{p+1}, \cdots, n_n$, *then system* (1) *has a p-parameter class of solutions* $x(t)$ *satisfying condition* (45).

Proof. In view of (47) for $P_i^k(x,t)$ on $W_i^k,\ k = 1, 2$, we have

$$\begin{aligned} P_i^k(x,t) &= f_i(x,t) r_i(t) + (-1)^k [(f_i(x,t) - f_i(\varphi,t) + f_i(\varphi,t))\varphi_i(t) - \varphi_i'(t)] - r_i'(t) \\ &= f_i(x,t) r_i(t) + (-1)^k [f_i(x,t) - f_i(\varphi,t)]\varphi_i(t) - r_i'(t). \end{aligned}$$

Now, it is enough to note that, on $W_i^k,\ k = 1, 2$,

$$\begin{aligned} P_i^k(x,t) &\leqslant f_i(x,t) r_i(t) + |f_i(x,t) - f_i(\varphi,t)|\,|\varphi_i(t)| - r_i'(t) \\ &\leqslant f_i(x,t) r_i(t) + L_i \|x - \varphi\|\,|\varphi_i(t)| - r_i'(t) \\ &= f_i(x,t) r_i(t) + L_i\rho(t)\,|\varphi_i(t)| - r_i'(t) < 0 \end{aligned}$$

for $i = n_1, \cdots, n_p$, and

$$P_i^k(x,t) \geqslant f_i(x,t) r_i(t) - L_i\rho(t)\,|\varphi_i(t)| - r_i'(t) > 0$$

for $i = n_{p+1}, \cdots, n_n$. □

Theorem 10. *Let* Γ *be a curve of stationary points of system* (1) *different from the* $t-$ *axis, i.e.* $f_i(\varphi(t), t) = 0$, $i = 1, \cdots, n$, $t \in I$, $\varphi \neq 0$, *and let* (20) *and* (48) *be valid. If, on* W_i^k, $k = 1, 2$,

$$\begin{aligned} L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| &< -f_i(x,t) r_i(t) + r_i'(t), \quad i = n_1, \cdots, n_p, \\ L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| &< f_i(x,t) r_i(t) - r_i'(t), \quad i = n_{p+1}, \cdots, n_n, \end{aligned}$$

then system (1) *has a p-parameter class of solutions* $x(t)$ *satisfying condition* (45).

Proof. In view of (47) we have, on W_i^k, $k = 1, 2$,

$$\begin{aligned} P_i^k(x,t) &= f_i(x,t) r_i(t) + (-1)^k [(f_i(x,t) - f_i(\varphi, t)) \varphi_i(t) - \varphi_i'(t)] - r_i'(t), \\ P_i^k(x,t) &\leqslant f_i(x,t) r_i(t) + L_i \|x - \varphi\| |\varphi_i(t)| + |\varphi_i'(t)| - r_i'(t) \\ &= f_i(x,t) r_i(t) + L_i |\varphi_i(t)| \rho(t) + |\varphi_i'(t)| - r_i'(t) < 0, \quad i = n_1, \cdots, n_p, \\ P_i^k(x,t) &\geqslant f_i(x,t) r_i(t) - L_i |\varphi_i(t)| \rho(t) - |\varphi_i'(t)| - r_i'(t) > 0, \quad i = n_{p+1}, \cdots, n_n. \end{aligned}$$

These estimates confirm the statement of this *Theorem.* □

Let us now consider the behaviour of integral curves of system (2) with respect to the set σ, where φ is defined by (31).

Theorem 11. *If, on* W_i^k, $k = 1, 2$,

$$|h_i(x,t) \varphi_i(t)| < -(p_i(t) + h_i(x,t)) r_i(t) + r_i'(t)$$

for $i = n_1, \cdots, n_p$, *and*

$$|h_i(x,t) \varphi_i(t)| < (p_i(t) + h_i(x,t)) r_i(t) - r_i'(t)$$

for $i = n_{p+1}, \cdots, n_n$, *then system* (2) *has a p-parameter class of solutions* $x(t)$ *satisfying condition* (45), *where* φ *is defined by* (31).

Proof. According to (47), we have, on W_i^k, $k = 1, 2$,

$$\begin{aligned} P_i^k &= (p_i + h_i) r_i + (-1)^k [(p_i + h_i) \varphi_i - \varphi_i'] - r_i' \\ &= (p_i + h_i) r_i + (-1)^k h_i \varphi_i - r_i'. \end{aligned}$$

Now, it is enough to note that, on W_i^k, $k = 1, 2$,

$$P_i^k \leqslant (p_i + h_i) r_i + |h_i \varphi_i| - r_i' < 0$$

for $i = n_1, \cdots, n_p$, and

$$P_i^k \geqslant (p_i + h_i) r_i - |h_i \varphi_i| - r_i' > 0$$

for $i = n_{p+1}, \cdots, n_n$. □

Now let us consider the behaviour of integral curves of system (3) in neighbourhood of (x^0, t), $t \in I$, where $x^0 \in \mathbb{R}^n$.

Theorem 12. *Let* (34) *and* (35) *be valid and let function* $\rho(t)$ *be defined by* (48). *If, on* W_i^k, $k = 1, 2$,

$$L_i |x_i^0| \rho(t) < -(q_i + g_i(x)) r_i(t) + r_i'(t) \tag{49}$$

for $i = n_1, \cdots, n_p$, *and*

$$L_i \left|x_i^0\right| \rho(t) < (q_i + g_i(x)) r_i(t) - r_i'(t) \tag{50}$$

for $i = n_{p+1}, \cdots, n_n$, *then system* (3) *has a p-parameter class of solutions* $x(t)$ *satisfying the condition*

$$\left|x_i(t) - x_i^0\right| < r_i(t), \quad i = 1, \cdots, n, \quad t \in I.$$

Proof. Using (47) we have, on $W_i^k,\ k = 1, 2,$

$$\begin{aligned} P_i^k(x,t) &= (q_i + g_i(x)) r_i(t) + (-1)^k (q_i + g_i(x)) x_i^0 - r_i'(t) \\ &= (q_i + g_i(x)) r_i(t) + (-1)^k \left[q_i + g_i\left(x^0\right) + g_i(x) - g_i\left(x^0\right)\right] x_i^0 - r_i'(t) \\ &= (q_i + g_i(x)) r_i(t) + (-1)^k \left[g_i(x) - g_i\left(x^0\right)\right] x_i^0 - r_i'(t) \end{aligned} \tag{51}$$

and

$$P_i^k(x,t) \leqslant (q_i + g_i(x)) r_i(t) + L_i \left|x_i^0\right| \rho(t) - r_i'(t) < 0$$

for $i = n_1, \cdots, n_p$, and

$$P_i^k(x,t) \geqslant (q_i + g_i(x)) r_i(t) - L_i \left|x_i^0\right| \rho(t) - r_i'(t) > 0$$

for $i = n_{p+1}, \cdots, n_n$. These estimates confirm this *Theorem.* □

We can note that in case $x^0 = 0$ *Theorem 12* holds without assumption (35).

4. Examples

Let us consider two known examples.

Example 1. *The Lotka-Volterra model ([1])*

$$\begin{aligned} \dot{x_1} &= x_1 - x_1 x_2, \\ \dot{x_2} &= -x_2 + x_1 x_2. \end{aligned} \tag{52}$$

Corollary 4. *Let functions* $r_1, r_2 \in C^1(I, \mathbb{R}^+)$ *satisfy the conditions*

$$r_1(t) < 1 + \frac{r_2'(t)}{r_2(t)}, \quad r_2(t) < 1 - \frac{r_1'(t)}{r_1(t)}, \quad t \in I.$$

System (52) *has a one-parameter class of solutions* $(x_1(t), x_2(t))$ *satisfying the condition*

$$|x_1(t)| < r_1(t), \quad |x_2(t)| < r_2(t), \quad t \in I.$$

This *Corollary* follows from *Theorem 7.* Conditions (38) and (39) are valid for $i = 2$ and $i = 1$, respectively. Here we have on W_i^k $(i, k = 1, 2)$:

$$\begin{aligned} P_2^k(x_1, x_2, t) &= (x_1 - 1) r_2(t) - r_2'(t) \le (r_1(t) - 1) r_2(t) - r_2'(t) < 0, \\ P_1^k(x_1, x_2, t) &= (1 - x_2) r_1(t) - r_1'(t) \ge (1 - r_2(t)) r_1(t) - r_1'(t) > 0. \end{aligned}$$

For functions r_i we can take, for example,

$$r_1(t) = r_2(t) = \alpha e^{-\beta t}, \quad \alpha, \beta \in \mathbb{R}^+, \ \alpha + \beta \le 1, \quad t > 0.$$

Example 2. *The Volterra model* (4), *applying appropriate substitution* ([4]), *can be written in the form*

$$\begin{aligned} \dot{x_1} &= (1 - x_1 - a x_2) x_1, \\ \dot{x_2} &= (-b + a x_1) x_2, \quad a, b \in \mathbb{R}^+. \end{aligned} \tag{53}$$

Corollary 5. *Let* $\alpha, \beta \in \mathbb{R}^+$.

(i) System (53) *has a one-parameter class of solutions* $x(t)$ *which satisfy the condition*

$$|x_1(t)| < \alpha e^{-\beta t}, \quad |x_2(t)| < \alpha e^{-\beta t} \quad \mathit{for}\ t > 0,$$

where

$$1 + \beta \ge \alpha(1 + a), \quad b \ge \alpha a + \beta.$$

(ii) System (53) *has a one-parameter class of solutions* $x(t)$ *which satisfy the condition*

$$|x_1(t) - 1| < \alpha e^{-\beta t}, \quad |x_2(t)| < \alpha e^{-\beta t} \quad \mathit{for}\ t > 0,$$

where

$$\beta \ge (1 + a)(1 + \alpha), \quad b \ge a(1 + \alpha) + \beta.$$

The statements of this *Corollary* follow from *Theorem 12.* Conditions (49) and (50) hold for $i = 2$ and $i = 1$, respectively, in both cases (i) and (ii). In this example we consider $x^0 = (0, 0)$ in case (i) and $x^0 = (1, 0)$ in case (ii). According to (51), here we have on W_i^k $(i, k = 1, 2)$:

(i)

$$\begin{aligned} P_2^k(x_1, x_2, t) &= (-b + a x_1) \alpha e^{-\beta t} + \beta \alpha e^{-\beta t} < (-b + a\alpha + \beta) \alpha e^{-\beta t} \le 0, \\ P_1^k(x_1, x_2, t) &= (1 - x_1 - a x_2) \alpha e^{-\beta t} + \beta \alpha e^{-\beta t} > (1 - \alpha - a\alpha + \beta) \alpha e^{-\beta t} \ge 0; \end{aligned}$$

(ii)

$$\begin{aligned} P_2^k(x_1, x_2, t) &= (-b + a x_1) \alpha e^{-\beta t} + \beta \alpha e^{-\beta t} < (-b + a(1 + \alpha) + \beta) \alpha e^{-\beta t} \le 0, \\ P_1^k(x_1, x_2, t) &= (1 - x_1 - a x_2) \alpha e^{-\beta t} + (-1)^k (1 - x_1 - a x_2) + \beta \alpha e^{-\beta t} \\ &\ge \left(-\alpha e^{-\beta t} - a\alpha e^{-\beta t}\right) \alpha e^{-\beta t} - \alpha e^{-\beta t} - a\alpha e^{-\beta t} + \beta \alpha e^{-\beta t} \\ &> (-\alpha - a\alpha - 1 - a + \beta) \alpha e^{-\beta t} \ge 0. \end{aligned}$$

Remark. We can note that the obtained results also contain answers to questions on approximation and stability or instability of solutions $x(t)$ whose existence is established. The errors of the approximation and the functions of stability or instability (including autostability and stability along the coordinates) are defined by functions $r(t)$ and $r_i(t)$, $i = 1, \cdots, n$ (see [9], [10], [11]).

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