On the number of hamiltonian groups*

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Abstract. *Finite hamiltonian groups are counted. The sequence of numbers of all groups of order n whose all subgroups are normal and the sequence of numbers of all groups of order less than or equal to n whose all subgroups are normal are presented.*

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1. Introduction

Subgroups of abelian groups are abelian and hence self-conjugate or *normal*. A nonabelian group all of whose subgroups are normal is called *hamiltonian* [1, 14]. Let $\mathcal{A}$ denote the class of abelian groups and let $\mathcal{H}$ denote the class of hamiltonian groups. In topological graph theory [2, 15], hamiltonian groups have been studied in the past [5, 7, 6]. For several classes of hamiltonian groups the genus is known exactly. For abelian and hamiltonian groups, there are structural theorems available. Let us mention that here we use a different structure theorem. For instance, the cyclic group $\mathbb{Z}_{15}$ can be written as $\mathbb{Z}_3 \times \mathbb{Z}_5$. Since it can be generated by a single generator, the former form is preferred in the topological graph theory over the latter. In this paper we determine the number $h(n)$ of hamiltonian groups of order n and the number $b(n)$ of all groups of order n with the property that all their subgroups are normal. We also determine the number $v(n)$ of all hamiltonian groups of order $\leq n$ and the number $w(n)$ of all groups of order $\leq n$ with the property that all their subgroups are normal.

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2. Results

Before we study hamiltonian groups we will recall the structure of finite abelian groups [13]. Let $\pi(m)$ denote a partition of a natural number m, where

$$\pi(m) := \{m_1, m_2, \ldots, m_s\},$$

such that $m = \sum_{k=1}^{s} m_k$ and $m_i \geq m_j$ for all $1 \leq i < j \leq s$. For $c \in \mathbb{N}$ let $c^{\pi(m)} := \{c^{m_1}, c^{m_2}, \ldots, c^{m_s}\}$ and let $A(n_1, n_2, \ldots, n_r)$ denote the direct product of cyclic groups

$$A(n_1, n_2, \ldots, n_r) := \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r}.$$

Let G be a finite abelian group of order n. Let us write down the prime decomposition of n as

$$n = \prod_{k=1}^{\ell} p_k^{\alpha_k}.$$

It is well-known that G is isomorphic to the group

$$G \approx A\left(p_1^{\pi(\alpha_1)}, p_2^{\pi(\alpha_2)}, \ldots, p_\ell^{\pi(\alpha_\ell)}\right).$$

Let $a(n)$ denote the number of abelian groups of order n and let $P(n)$ denote the number of partitions of the integer n. The previous discussion gives a proof to the following result.

Proposition 1. *The number $a(n)$ of abelian groups of order n is given by $\prod_{i=1}^{\ell} P(\alpha_i)$ where $n = \prod_{k=1}^{\ell} p_k^{\alpha_k}$ is the prime decomposition of n.*
The initial 200 values of the sequence $a(n)$ are given in *Table 1*.

n	1				5					10					15					20
0	1	1	1	2	1	1	1	3	2	1	1	2	1	1	1	5	1	2	1	2
20	1	1	1	3	2	1	3	2	1	1	1	7	1	1	1	4	1	1	1	3
40	1	1	1	2	2	1	1	5	2	2	1	2	1	3	1	3	1	1	1	2
60	1	1	2	11	1	1	1	2	1	1	1	6	1	1	2	2	1	1	1	5
80	5	1	1	2	1	1	1	3	1	2	1	2	1	1	1	7	1	2	2	4
100	1	1	1	3	1	1	1	6	1	1	1	5	1	1	1	2	2	1	1	3
120	2	1	1	2	3	2	1	15	1	1	1	2	1	1	3	3	1	1	1	2
140	1	1	1	10	1	1	2	2	1	2	1	3	2	1	1	2	1	1	1	7
160	1	5	1	2	1	1	1	3	2	1	2	2	1	1	2	5	1	1	1	4
180	1	1	1	3	1	1	1	2	3	1	1	11	1	1	1	4	1	2	1	6

Table 1. *The initial values of $a(n)$, $n = 1, 2, \ldots, 200$, ([8], A000688).*

Note that the sequence $\{a(n)\}_{n \in \mathbb{N}}$ cannot contain multiples of primes in the sequence $s := \{13, 17, 19, 23, 29, 31, 37, \ldots\}$ since $P(n) \neq k \cdot s_i, \forall n, i, k \in \mathbb{N}$ (see [9]). The number $a(n)$ depends only on the prime signature of n. For example, both $24 = 2^3 \cdot 3^1$ and $375 = 3^1 \cdot 5^3$ have the prime signature $(3, 1)$, therefore $a(375) = a(24)$.

A similar structural theorem holds for hamiltonian groups. A hamiltonian group H is isomorphic to a direct product of the quaternion group Q of order 8, an elementary abelian group E of exponent 2 and an abelian group A of odd order

$$H \approx Q \times E \times A \approx Q \times \mathbb{Z}_{2^k} \times A,$$

where $|Q| = 8 = 2^3$, $|E| = 2^k$ and $|A| \not\equiv 0 \pmod 2$. Therefore $|H| = 2^{3+k}|A|$. Let n be an arbitrary natural number. We can write n uniquely in the form $n = 2^e \cdot o$

where $e = e(n) \geq 0$ and $o = o(n)$ is an odd number. These results give the number of hamiltonian groups of order n.

Proposition 2. *Let $n = 2^e \cdot o$, where $e = e(n) \geq 0$ and $o = o(n)$ is an odd number. The number $h(n)$ of hamiltonian groups of order n is given by*

$$h(n) = \begin{cases} 0, & e(n) < 3; \\ a\left(o(n)\right), & \text{otherwise.} \end{cases}$$

The initial 200 values of the sequence $h(n)$ are given in *Table 2*.

n	1				5					10					15					20
0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
20	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
40	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
60	0	0	0	1	0	0	0	0	0	0	0	2	0	0	0	0	0	0	0	1
80	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
100	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
120	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
140	0	0	0	2	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
160	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
180	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	2

Table 2. *The initial values of $h(n)$, $n = 1, 2, \ldots, 200$.*

Combining abelian and hamiltonian groups of order n we may now give the number $b(n) := a(n) + h(n)$ of all groups of order n all of whose subgroups are normal. The initial 300 values of the sequence $b(n)$ are given in *Table 3*.

n	1				5					10					15					20
0	1	1	1	2	1	1	1	4	2	1	1	2	1	1	1	6	1	2	1	2
20	1	1	1	4	2	1	3	2	1	1	1	8	1	1	1	4	1	1	1	4
40	1	1	1	2	2	1	1	6	2	2	1	2	1	3	1	4	1	1	1	2
60	1	1	2	12	1	1	1	2	1	1	1	8	1	1	2	2	1	1	1	6
80	5	1	1	2	1	1	1	4	1	2	1	2	1	1	1	8	1	2	2	4
100	1	1	1	4	1	1	1	6	1	1	1	6	1	1	1	2	2	1	1	4
120	2	1	1	2	3	2	1	16	1	1	1	2	1	1	3	4	1	1	1	2
140	1	1	1	12	1	1	2	2	1	2	1	4	2	1	1	2	1	1	1	8
160	1	5	1	2	1	1	1	4	2	1	2	2	1	1	2	6	1	1	1	4
180	1	1	1	4	1	1	1	2	3	1	1	12	1	1	1	4	1	2	1	8
200	1	1	1	2	1	1	2	6	1	1	1	2	1	1	1	12	1	1	1	2
220	1	1	1	8	4	1	1	2	1	1	1	4	1	2	1	2	1	1	1	6
240	1	2	7	2	2	1	1	4	1	3	1	4	1	1	1	23	1	1	1	2
260	2	1	1	4	1	1	1	2	1	3	1	6	1	1	2	2	1	1	2	4
280	1	1	1	2	1	1	1	16	2	1	1	2	1	2	1	4	3	1	1	4

Table 3. *The initial values of $b(n)$, $n = 1, 2, \ldots, 300$.*

The number $u(n)$ of all abelian groups of order $\leq n$ is presented in [11]. The initial 100 values of the sequence $u(n)$ are given in *Table 4*.

n	1				5					10
0	1	2	3	5	6	7	8	11	13	14
10	15	17	18	19	20	25	26	28	29	31
20	32	33	34	37	39	40	43	45	46	47
30	48	55	56	57	58	62	63	64	65	68
40	69	70	71	73	75	76	77	82	84	86
50	87	89	90	93	94	97	98	99	100	102
60	103	104	106	117	118	119	120	122	123	124
70	125	131	132	133	135	137	138	139	140	145
80	150	151	152	154	155	156	157	160	161	163
90	164	166	167	168	169	176	177	179	181	185

Table 4. *The initial values of $u(n)$, $n = 1, 2, \ldots, 100$, ([11], A063966).*

Let $v(n)$ be the number of all hamiltonian groups of order $\leq n$ and let $w(n)$ be the number of all groups of order $\leq n$ all of whose subgroups are normal. The initial 200 values of the sequences $v(n)$ and $w(n)$ are given in *Table 5* and *Table 6*, respectively.

n	1				5					10
0	0	0	0	0	0	0	0	1	1	1
10	1	1	1	1	1	2	2	2	2	2
20	2	2	2	3	3	3	3	3	3	3
30	3	4	4	4	4	4	4	4	4	5
40	5	5	5	5	5	5	5	6	6	6
50	6	6	6	6	6	7	7	7	7	7
60	7	7	7	8	8	8	8	8	8	8
70	8	10	10	10	10	10	10	10	10	11
80	11	11	11	11	11	11	11	12	12	12
90	12	12	12	12	12	13	13	13	13	13
100	13	13	13	14	14	14	14	14	14	14
110	14	15	15	15	15	15	15	15	15	16
120	16	16	16	16	16	16	16	17	17	17
130	17	17	17	17	17	18	18	18	18	18
140	18	18	18	20	20	20	20	20	20	20
150	20	21	21	21	21	21	21	21	21	22
160	22	22	22	22	22	22	22	23	23	23
170	23	23	23	23	23	24	24	24	24	24
180	24	24	24	25	25	25	25	25	25	25
190	25	26	26	26	26	26	26	26	26	28

Table 5. *The initial values of* $v(n)$, $n = 1, 2, \ldots, 200$.

n	1				5					10
0	1	2	3	5	6	7	8	12	14	15
10	16	18	19	20	21	27	28	30	31	33
20	34	35	36	40	42	43	46	48	49	50
30	51	59	60	61	62	66	67	68	69	73
40	74	75	76	78	80	81	82	88	90	92
50	93	95	96	99	100	104	105	106	107	109
60	110	111	113	125	126	127	128	130	131	132
70	133	141	142	143	145	147	148	149	150	156
80	161	162	163	165	166	167	168	172	173	175
90	176	178	179	180	181	189	190	192	194	198
100	199	200	201	205	206	207	208	214	215	216
110	217	223	224	225	226	228	230	231	232	236
120	238	239	240	242	245	247	248	264	265	266
130	267	269	270	271	274	278	279	280	281	283
140	284	285	286	298	299	300	302	304	305	307
150	308	312	314	315	316	318	319	320	321	329
160	330	335	336	338	339	340	341	345	347	348
170	350	352	353	354	356	362	363	364	365	369
180	370	371	372	376	377	378	379	381	384	385
190	386	398	399	400	401	405	406	408	409	417

Table 6. *The initial values of* $w(n)$, $n = 1, 2, \ldots, 200$.

If we look at the sequences $\{a(n)\}_{n\in\mathbb{N}}$ and $\{h(n)\}_{n\in\mathbb{N}}$ from the inverse perspective, we can define two more sequences. Let $S_a(n)$ denote the smallest number $k \in \mathbb{N}$, for which there exist exactly n nonisomorphic abelian groups of order k ([10]). The first 60 elements of the sequence $\{S_a(n)\}_{n\in\mathbb{N}}$ are given in *Table 7*. Here 0 denotes the case, where $S_a(n)$ does not exist (n is not a product of partition numbers). These indices n are exactly multiples of primes in the sequence s ([9]).

n	1				5					10
0	1	4	8	36	16	72	32	900	216	144
10	64	1800	0	288	128	44100	0	5400	0	3600
20	864	256	0	88200	1296	0	27000	7200	0	512
30	0	5336100	1728	0	2592	264600	0	0	0	176400
40	0	1024	0	2304	3456	0	0	10672200	7776	32400
50	0	0	0	1323000	5184	2048	0	0	0	4608

Table 7. *The initial values of* $S_a(n)$, $n = 1, 2, \dots, 60$, *([10], A046056).*

Let $S_h(n)$ denote the smallest number $k \in \mathbb{N}$, for which there exist exactly n nonisomorphic hamiltonian groups of order k. The first 30 elements of the sequence $\{S_h(n)\}_{n\in\mathbb{N}}$ are given in *Table 8*, where again 0 denotes the case, where n is not a product of partition numbers and $S_h(n)$ does not exist.

n	1				5					10
0	8	72	216	1800	648	5400	1944	88200	27000	16200
10	5832	264600	0	48600	17496	10672200	0	1323000	0	793800
20	243000	52488	0	32016600	405000	0	9261000	2381400	0	157464

Table 8. *The initial values of* $S_h(n)$, $n = 1, 2, \dots, 30$.

Let us finish with two open problems. Think of computing the genus of each of the groups $\Gamma \in \mathcal{A} \cup \mathcal{H}$, counted by $b(n)$. Since $\mathbb{Z}_n \in \mathcal{A} \cup \mathcal{H}$, the minimal genus is 0. A natural question is therefore to determine

$$g(n) := \max\{\gamma(\Gamma) \mid \Gamma \in \mathcal{A} \cup \mathcal{H},\ |\Gamma| = n\}.$$

The sequence $\{g(n)\}_{n\in\mathbb{N}} = (0, 0, 0, 0, 0, 0, 0, 1, \dots)$.

Another interesting problem is a generalization of the considered problem, namely, the problem of determining the number of groups, whose every subgroup is 2-subnormal ([4, 12]). A subgroup H of group G is said to be *2-subnormal* in G if there is a series

$$H = H_2 \triangleleft H_1 \triangleleft H_0 = G$$

of subgroups in G (see [3]). Such a subgroup is also said to be *of defect* 2. Similarly, subgroups H of defect 1 in G are precisely normal subgroups of G.

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