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$\alpha\beta\gamma\sigma$ – technology in the triangle geometry

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Abstract. *The barycentric coordinates of the most important points and circles and the equations of the most important lines, conics and cubics of the geometry of triangle ABC are expressed by means of numbers $\alpha = \cot A$, $\beta = \cot B$, $\gamma = \cot C$ and $\sigma = \alpha + \beta + \gamma$.*

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It is shown in [4] and [5] that the formulae for metrical relations of points, lines and circles connected with the given triangle ABC with sidelengths a, b, c and angles A, B, C can be written in a more contracted form if the barycentric coordinates and numbers

$$\alpha = \cot A, \qquad \beta = \cot B, \qquad \gamma = \cot C \tag{1}$$

are used.

We are also introducing the abbreviation

$$\sigma = \alpha + \beta + \gamma. \tag{2}$$

Because of the identity

$$-\alpha = -\cot A = \cot(\pi - A) = \cot(B + C) = \frac{\cot B \cot C - 1}{\cot B + \cot C} = \frac{\beta\gamma - 1}{\beta + \gamma}$$

the identity

$$\beta\gamma + \gamma\alpha + \alpha\beta = 1, \tag{3}$$

is valid, which can also be used in the forms

$$\alpha(\beta + \gamma) = 1 - \beta\gamma, \quad \beta(\gamma + \alpha) = 1 - \gamma\alpha, \quad \gamma(\alpha + \beta) = 1 - \alpha\beta, \tag{4}$$

$$(\gamma + \alpha)(\alpha + \beta) = 1 + \alpha^2, \quad (\alpha + \beta)(\beta + \gamma) = 1 + \beta^2, \quad (\beta + \gamma)(\gamma + \alpha) = 1 + \gamma^2. \tag{5}$$

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Thus, it follows,

$$\begin{aligned}\sigma - \alpha\beta\gamma = \beta + \gamma + \alpha(1 - \beta\gamma) &= \beta + \gamma + \alpha^2(\beta + \gamma) = (\beta + \gamma)(1 + \alpha^2)\\ &= (\beta + \gamma)(\gamma + \alpha)(\alpha + \beta),\end{aligned}$$

namely, we also get the identity

$$(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) = \sigma - \alpha\beta\gamma. \tag{6}$$

Identity (4) can also be written in the forms

$$\alpha\sigma - \alpha^2 = 1 - \beta\gamma, \quad \beta\sigma - \beta^2 = 1 - \gamma\alpha, \quad \gamma\sigma - \gamma^2 = 1 - \alpha\beta. \tag{7}$$

Because of the identity $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\beta\gamma + \gamma\alpha + \alpha\beta)$ it follows

$$\alpha^2 + \beta^2 + \gamma^2 = \sigma^2 - 2, \tag{8}$$

and then, by means of (2), (3) and (8), we can derive formulae, which express different symmetric functions of α, β, γ by numbers σ and $\alpha\beta\gamma$. For example, we get

$$\begin{aligned}
\beta\gamma(\beta + \gamma) + \gamma\alpha(\gamma + \alpha) + \alpha\beta(\alpha + \beta) &= \alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)\\
&= \alpha(1 - \beta\gamma) + \beta(1 - \gamma\alpha) + \gamma(1 - \alpha\beta)\\
&= \sigma - 3\alpha\beta\gamma,\\
\alpha(\beta + \gamma)^2 + \beta(\gamma + \alpha)^2 + \gamma(\alpha + \beta)^2 &= \alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta) + 6\alpha\beta\gamma\\
&= \sigma - 3\alpha\beta\gamma + 6\alpha\beta\gamma = \sigma + 3\alpha\beta\gamma,\\
\alpha(\beta - \gamma)^2 + \beta(\gamma - \alpha)^2 + \gamma(\alpha - \beta)^2 &= \alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta) - 6\alpha\beta\gamma\\
&= \sigma - 3\alpha\beta\gamma - 6\alpha\beta\gamma = \sigma - 9\alpha\beta\gamma,\\
\alpha^3 + \beta^3 + \gamma^3 &= (\alpha^2+\beta^2+\gamma^2)(\alpha+\beta+\gamma) - \alpha^2(\beta+\gamma) - \beta^2(\gamma+\alpha) - \gamma^2(\alpha+\beta)\\
&= (\sigma^2 - 2)\sigma - (\sigma - 3\alpha\beta\gamma) = \sigma^3 - 3\sigma + 3\alpha\beta\gamma,\\
\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2 &= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) = 1 - 2\alpha\beta\gamma\sigma,\\
\alpha^4 + \beta^4 + \gamma^4 &= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)\\
&= (\sigma^2 - 2)^2 - 2(1 - 2\alpha\beta\gamma\sigma) = \sigma^4 - 4\sigma^2 + 2 + 4\alpha\beta\gamma\sigma,\\
\beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3 &= (\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2)(\beta\gamma + \gamma\alpha + \alpha\beta) - \alpha\beta\gamma[\alpha(\beta^2 + \gamma^2)\\
&\quad +\beta(\gamma^2 + \alpha^2) + \gamma(\alpha^2 + \beta^2)]\\
&= 1 - 2\alpha\beta\gamma\sigma - \alpha\beta\gamma(\sigma - 3\alpha\beta\gamma) = 1 - 3\alpha\beta\gamma\sigma + 3\alpha^2\beta^2\gamma^2.
\end{aligned}$$

Let $\triangle$ and R be the area and the radius of the circumscribed circle of triangle ABC, respectively. Because of, for example

$$b^2 + c^2 - a^2 = 2bc\cos A = 2bc\sin A \cdot \cot\, A = 4\triangle\alpha$$

we have identities

$$b^2 + c^2 - a^2 = 4\triangle\alpha, \quad c^2 + a^2 - b^2 = 4\triangle\beta, \quad a^2 + b^2 - c^2 = 4\triangle\gamma, \tag{9}$$

by addition of which we get the following identities

$$a^2 = 2\triangle(\beta+\gamma), \quad b^2 = 2\triangle(\gamma+\alpha), \quad c^2 = 2\triangle(\alpha+\beta), \tag{10}$$

$$a^2 + b^2 + c^2 = 4\triangle\sigma, \tag{11}$$

and by subtraction there follows

$$b^2 - c^2 = 2\triangle(\gamma-\beta), \quad c^2 - a^2 = 2\triangle(\alpha-\gamma), \quad a^2 - b^2 = 2\triangle(\beta-\alpha). \tag{12}$$

From the formulae $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$ we get

$$abc = 2R\sin A \cdot bc = 2R \cdot 2\triangle,$$

i.e.

$$abc = 4\triangle R, \tag{13}$$

and then there follows $8R^3 \sin A \sin B \sin C = 4\triangle R$, i.e.

$$\triangle = 2R^2 \sin A \sin B \sin C. \tag{14}$$

Because of (10) and (13) we get

$$(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = \frac{a^2b^2c^2}{8\triangle^3} = \frac{16\triangle^2R^2}{8\triangle^3} = \frac{2R^2}{\triangle},$$

namely, because of (6) and (14) there follows

$$\sigma - \alpha\beta\gamma = (\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) = \frac{2R^2}{\triangle} = \frac{1}{\sin A \sin B \sin C}. \tag{15}$$

Thus, we further obtain

$$\alpha\beta\gamma = \frac{\cos A \cos B \cos C}{\sin A \sin B \sin C} = \frac{2R^2}{\triangle}\cos A \cos B \cos C = (\sigma - \alpha\beta\gamma)\cos A \cos B \cos C,$$

so identities

$$\cos A \cos B \cos C = \frac{\triangle}{2R^2}\alpha\beta\gamma = \frac{\alpha\beta\gamma}{\sigma - \alpha\beta\gamma} \tag{16}$$

are valid.

Identities

$$\sin^2 A = \frac{1}{1+\alpha^2}, \quad \sin^2 B = \frac{1}{1+\beta^2}, \quad \sin^2 C = \frac{1}{1+\gamma^2},$$
$$\cos^2 A = \frac{\alpha^2}{1+\alpha^2}, \quad \cos^2 B = \frac{\beta^2}{1+\beta^2}, \quad \cos^2 C = \frac{\gamma^2}{1+\gamma^2}$$

are also valid. If h_a, h_b, h_c are the lengths of the altitudes of triangle ABC, then from (10) and, e.g. formula $2\triangle = ah_a$, we get

$$h_a^2 = \frac{4\triangle^2}{a^2} = \frac{4\triangle^2}{2\triangle(\beta+\gamma)} = \frac{2\triangle}{\beta+\gamma},$$

so we have identities

$$h_a^2 = \frac{2\triangle}{\beta+\gamma}, \quad h_b^2 = \frac{2\triangle}{\gamma+\alpha}, \quad h_c^2 = \frac{2\triangle}{\alpha+\beta}.$$

From numbers α, β, γ in (1) there are always at least two positive ones, and because of (10) and (11) numbers $\beta+\gamma$, $\gamma+\alpha$, $\alpha+\beta$ and σ are always positive, and then the number $\sigma-\alpha\beta\gamma$ from (6) is also positive. If numbers α, β, γ are positive, then, because of the inequality for the arithmetic and geometric mean, we get

$$\sigma-\alpha\beta\gamma = (\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) \geq 2\sqrt{\beta\gamma}\cdot 2\sqrt{\gamma\alpha}\cdot 2\sqrt{\alpha\beta} = 8\alpha\beta\gamma$$

with the equality if and only if $\alpha=\beta=\gamma$. If one of the numbers α, β, γ is negative, then $-8\alpha\beta\gamma>0$, so because of $\sigma-\alpha\beta\gamma>0$ there follows $\sigma-9\alpha\beta\gamma>0$. Because of that the number $\sigma-9\alpha\beta\gamma$ is always non–negative and equal to zero if and only if the triangle ABC is equilateral.

The angle ω such that $\cot\omega=\sigma$ is called a *Brocard angle* of the triangle ABC. Because of (11) for that angle we get

$$\cot\omega = \sigma = \alpha+\beta+\gamma = \frac{1}{4\triangle}(a^2+b^2+c^2)$$

(see [1]). From the identity

$$(\beta-\gamma)^2+(\gamma-\alpha)^2+(\alpha-\beta)^2 = 2(\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta) = 2(\sigma^2-3)$$

follows the inequality $\sigma^2\geq 3$, i.e. $\cot\omega\geq\sqrt{3}$ or $\omega\leq\frac{\pi}{6}$, with the equalities if and only if triangle ABC is equilateral.

Steiner angles (the first and the second) of triangle ABC are the angles ω_1 and ω_2 such that numbers $\sigma_i=\cot\omega_i$ $(i=1,2)$ are solutions by τ of the equation $\tau^2-2\sigma\tau+3=0$. These identities

$$\sigma^2-2\sigma\sigma_i+3=0 \quad (i=1,2),$$

$$\sigma_1=\sigma+\sqrt{\sigma^2-3}, \quad \sigma_2=\sigma-\sqrt{\sigma^2-3}, \tag{17}$$

$$\sigma_1+\sigma_2=2\sigma, \quad \sigma_1\sigma_2=3 \tag{18}$$

are valid for them. Because of $\sigma^2\geq 3$, i.e. $0\leq\sqrt{\sigma^2-3}\leq\sigma$, numbers σ_1 and σ_2 from (17) are positive and the angles ω_1 and ω_2 are acute. Because of $\sigma_1>\sigma>\sigma_2$ follows $0<\omega_1<\omega<\omega_2<\frac{\pi}{2}$, and from equalities

$$\cot(\omega_1+\omega_2) = \frac{\cot\omega_1\cot\omega_2-1}{\cot\omega_1+\cot\omega_2} = \frac{\sigma_1\sigma_2-1}{\sigma_1+\sigma_2} = \frac{3-1}{2\sigma} = \frac{1}{\sigma} = \mathrm{tg}\omega = \cot\left(\frac{\pi}{2}-\omega\right),$$

which we obtained by means of (18), there follows $\omega_1+\omega_2=\frac{\pi}{2}-\omega$, so *Theorem 1* is valid.

Theorem 1. *The sum of Brocard angle and Steiner angles of the triangle is equal to the right angle (see [4]).*

Because of formulae (9), (10) and (11) all those points, lines and circles whose barycentric coordinates are functions of a^2, b^2 and c^2 can be expressed by means of numbers α, β, γ, σ. In this way, we can express a great part of the geometry of the triangle. We are going to give the barycentric coordinates of the most important points, lines and circles without proof, as well as the equations of the most important conics and cubics which can be expressed in this way. If T is the point of the form

$$T = (f(\alpha,\beta,\gamma) : f(\beta,\gamma,\alpha) : f(\gamma,\alpha,\beta)),$$

then we shall write it in a shorter form $T = (f(\alpha,\beta,\gamma))$, and we shall do the same with lines and circles. For each point we shall give the sum of its coordinates $\Sigma = f(\alpha,\beta,\gamma)+f(\beta,\gamma,\alpha)+f(\gamma,\alpha,\beta)$. For more interesting lines we shall give their point at infinity and the point at infinity of the lines perpendicular to them, and for the more interesting circles we shall give the center and the radius.

The most important points, lines and circles of the geometry of triangle ABC expressed by means of numbers α, β, γ and σ, are listed in the following tables.

point	the first coordinate and mark	the sum of coordinates
centroid	$G = (1) = X_2$	3
orthocenter	$H = (\beta\gamma) = X_4$	1
circumcenter	$O = (1-\beta\gamma) = X_3$	2
Euler center	$O_9 = (1+\beta\gamma) = X_5$	4
symmedian center	$K = (\beta+\gamma) = X_6$	2σ
Crelle–Brocard points	$\Omega = (1+\beta^2)$	σ^2+1
	$\Omega' = (1+\gamma^2)$	σ^2+1
Steiner point	$S = ((\gamma-\alpha)(\alpha-\beta)) = X_{99}$	$3-\sigma^2$
Tarry point	$T = ((1-\beta)(1-\gamma\sigma)) = X_{98}$	$3-\sigma^2$
isogonic centers	$V_1 = ((1+\sqrt{3}\beta)(1+\sqrt{3}\gamma)) = X_{13}$	$2\sqrt{3}(\sqrt{3}+\sigma)$
	$V_1 = ((1-\sqrt{3}\beta)(1-\sqrt{3}\gamma)) = X_{14}$	$2\sqrt{3}(\sqrt{3}-\sigma)$
isodynamic centers	$W_1 = ((\beta+\gamma)(1+\sqrt{3}\alpha)) = X_{15}$	$2(\sigma+\sqrt{3})$
	$W_2 = ((\beta+\gamma)(1-\sqrt{3}\alpha)) = X_{16}$	$2(\sigma-\sqrt{3})$
" Napoleon " points	$N_1 = ((\beta+\sqrt{3})(\gamma+\sqrt{3})) = X_{17}$	$2(5+\sqrt{3}\sigma)$
	$N_2 = ((\beta-\sqrt{3})(\gamma-\sqrt{3})) = X_{18}$	$2(5-\sqrt{3}\sigma)$
de Longchamps point	$L = (1-2\beta\gamma) = X_{20}$	1
infinity point of Euler line	$(1-3\beta\gamma) = X_{30}$	0
centroid of the orthic triangle	$((\beta+\gamma)(1+\beta\gamma)) = X_{51}$	$3(\sigma-\alpha\beta\gamma)$
orthocenter of the orthic triangle	$((\beta+\gamma)(1+\beta\gamma)(1-\alpha^2)) = X_{52}$	$2(\sigma-\alpha\beta\gamma)$
symmedian center of the orthic triangle	$(\beta\gamma(1+\beta\gamma)) = X_{53}$	$2(1-\alpha\beta\gamma\sigma)$
Euler center of the orthic triangle	$((\beta+\gamma)(1+\beta\gamma)(3-\alpha^2))$	$8(\sigma-\alpha\beta\gamma)$
centroid of the tangential triangle	$((\beta+\gamma)(1-2\beta\gamma))$	$6\alpha\beta\gamma$
orthocenter of the tangential triangle	$(\alpha(\beta+\gamma)(\sigma-\alpha\beta\gamma-2\alpha))$	$4\alpha\beta\gamma$
circumcenter of the tangential triangle	$((\beta+\gamma)(\alpha\sigma-2\beta\gamma+\alpha^2\beta\gamma)) = X_{26}$	$8\alpha\beta\gamma$
Euler center of the tangential triangle	$(3-5\beta\gamma-\alpha^2-\alpha^2\beta\gamma)$	$6-\sigma^2-\alpha\beta\gamma\sigma$
Feurbach point of the tangential triangle	$((\beta+\gamma)(\gamma-\alpha)(\alpha-\beta))$	$-(\sigma+3\alpha\beta\gamma)$

Other points X_i from [2]:

point	the first coordinate	the sum of coordinates
X_{22}	$((\beta+\gamma)(\alpha\sigma-\beta\gamma))$	$\sigma+3\alpha\beta\gamma$
X_{23}	$((\beta+\gamma)(\alpha\sigma-3\beta\gamma))$	$9\alpha\beta\gamma-\sigma$
X_{24}	$(\beta\gamma(\sigma+\alpha\beta\gamma-2\alpha))$	$\sigma-5\alpha\beta\gamma$
X_{25}	$(\beta\gamma(\beta+\gamma))$	$\sigma-3\alpha\beta\gamma$
X_{32}	$((\beta+\gamma)^2)$	$2(\sigma^2-1)$
X_{39}	$(\sigma^2-\alpha^2)$	$2(\sigma^2+1)$
X_{49}	$(\alpha(\beta+\gamma)^2(1-5\alpha^2))$	$4(7\alpha\beta\gamma-\sigma)$
X_{50}	$((\beta+\gamma)^2(1-3\alpha^2))$	$2(4+\sigma^2+4\alpha\beta\gamma\sigma)$
X_{54}	$((\beta+\gamma)(1+\gamma\alpha)(1+\alpha\beta))$	$3\sigma+5\alpha\beta\gamma$
X_{61}	$((\beta+\gamma)(\alpha+\sqrt{3}))$	$2(1+\sqrt{3}\sigma)$
X_{62}	$((\beta+\gamma)(\alpha-\sqrt{3}))$	$2(1-\sqrt{3}\sigma)$
X_{64}	$((\beta+\gamma)(1-2\gamma\alpha)(1-2\alpha\beta))$	$2\alpha\beta\gamma$
X_{66}	$((\beta\sigma-\gamma\alpha)(\gamma\sigma-\alpha\beta))$	$4\alpha\beta\gamma\sigma$
X_{67}	$((\beta\sigma-3\gamma\alpha)(\gamma\sigma-3\alpha\beta))$	$2\sigma(9\alpha\beta\gamma-\sigma)$
X_{68}	$(\alpha(1-\beta^2)(1-\gamma^2))$	$4\alpha\beta\gamma$
X_{69}	(α)	σ
X_{70}	$((\beta\sigma-2\gamma\alpha+\alpha\beta^2\gamma)(\gamma\sigma-2\alpha\beta+\alpha\beta\gamma^2))$	$-\sigma^2+10\alpha\beta\gamma\sigma+7\alpha^2\beta^2\gamma^2$
X_{74}	$((\beta+\gamma)(1-3\gamma\alpha)(1-3\alpha\beta)$	$9\alpha\beta\gamma-\sigma$
X_{76}	$(1+\alpha^2)$	σ^2+1
X_{83}	$((\sigma+\beta)(\sigma+\gamma))$	$5\sigma^2+1$
X_{93}	$(\beta\gamma(1+\alpha^2)(1-5\beta^2)(1-5\gamma^2))$	$36-5\sigma^2-74\alpha\beta\gamma\sigma+165\alpha^2\beta^2\gamma^2$
X_{94}	$((1+\alpha^2)(1-3\beta^2)(1-3\gamma^2))$	$16-5\sigma^2-6\alpha\beta\gamma\sigma+27\alpha^2\beta^2\gamma^2$
X_{95}	$((1+\gamma\alpha)(1+\alpha\beta))$	$5+\alpha\beta\gamma\sigma$
X_{96}	$((1+\gamma\alpha)(1+\alpha\beta)(1-\beta^2)(1-\gamma^2))$	$12-\sigma^2+\alpha^2\beta^2\gamma^2$
X_{97}	$((1-\beta\gamma)(1+\gamma\alpha)(1+\alpha\beta))$	$2-\alpha\beta\gamma\sigma-3\alpha^2\beta^2\gamma^2$

line	the first coordinate	point at infinity	point at infinity of the perpendicular lines
Euler line	$(\alpha(\beta-\gamma))$	$(1-3\beta\gamma)=X_{30}$	$(\beta-\gamma)$
Brocard diameter	$((\beta-\gamma)(1+\alpha^2))$	$((\beta+\gamma)(1-\alpha\sigma))$	$(\beta^2-\gamma^2)$
ortic line	(α)	$(\beta-\gamma)$	$(1-3\beta\gamma)=X_{30}$
Lemoine line	$(1+\alpha^2)$	$(\beta^2-\gamma^2)$	$((\beta+\gamma)(1-\alpha\sigma))$
Longchamps line	$(\beta+\gamma)$	$(\beta-\gamma)$	$(1-3\beta\gamma)=X_{30}$
Steiner axes:	$((\beta-\gamma)(\sigma_1-3\alpha))$ $((\beta-\gamma)(\sigma_2-3\alpha))$	$((\sigma_1-3\beta)(\sigma_1-3\gamma))$ $((\sigma_2-3\beta)(\sigma_2-3\gamma))$	$((\sigma_2-3\beta)(\sigma_2-3\gamma))$ $((\sigma_1-3\beta)(\sigma_1-3\gamma))$
line through G, $K, X_{69}, X_{81}, X_{86}$	$(\beta-\gamma)$	$(\sigma-3\alpha)$	$((\beta-\gamma)(2\sigma-3\alpha))$
line through G, X_{54}, X_{68}, X_{96}	$((1+\beta\gamma)(1-\alpha^2)(\beta-\gamma))$	$(3\sigma+\alpha\beta\gamma-6\alpha$ $-3\beta\gamma\sigma-3\alpha\beta^2\gamma^2)$	$((\beta-\gamma)(3\sigma-\alpha\beta\gamma-6\alpha))$
line through O, S, T, X_{76}	$((1-\alpha\sigma)(\beta^2-\gamma^2))$	$(\sigma^2-2-3\alpha^2+\alpha\sigma-\beta\gamma\sigma^2)$	$((\beta-\gamma)(1+3\alpha\sigma-\beta\gamma\sigma^2))$
line through O_9, V_1, N_2, X_{62}	$((\alpha-\sqrt{3})(1+\alpha\sqrt{3})(\beta-\gamma))$	$(3\sigma-6\alpha-3\beta\gamma\sigma$ $+3\sqrt{3}\beta\gamma-\sqrt{3})$	$((\beta-\gamma)(3\sigma-6\alpha+\sqrt{3}))$
line through O_9, V_2, N_1, X_{61}	$((\alpha+\sqrt{3})(1-\alpha\sqrt{3})(\beta-\gamma))$	$(3\sigma-6\alpha-3\beta\gamma\sigma$ $-3\sqrt{3}\beta\gamma+\sqrt{3})$	$((\beta-\gamma)(3\sigma-6\alpha-\sqrt{3}))$

From lines from [3] here are given those which are going through at least four points of the form X_i, $i\in\{1,2,\dots,101\}$. Then the Euler line goes through points $G,O,H,O_9,L,X_{21},X_{22},X_{23}$, $X_{24},X_{25},X_{26},X_{27},X_{28}$, X_{29},X_{30}, and the Brocard diameter through points O,K,W_1,W_2, $X_{32},X_{39},X_{50},X_{52},X_{58}$, X_{61},X_{62}.

circle	first coordinate	center	radius
circumscribed	0	$O = (1 - \beta\gamma)$	R
polar	α	$H = (\beta\gamma)$	$\sqrt{-2\triangle\alpha\beta\gamma}$
Euler	$\frac{\alpha}{2}$	$O_9 = (1 + \beta\gamma)$	$\frac{R}{2}$
ortocentroidal	$\frac{2\alpha}{3}$	$(1 + 3\beta\gamma)$	$\frac{1}{2}\lvert GH\rvert$
Longchamps	$\beta + \gamma$	$L = (1 - 2\beta\gamma)$	$\sqrt{-4\triangle\alpha\beta\gamma}$
Brocard	$\frac{1}{2\sigma}(1 + \alpha^2)$	$((\beta + \gamma)(1 + \alpha\sigma))$	$\frac{R}{2\sigma}\sqrt{\sigma^2 - 3}$
1. Lemoine	$\frac{1}{4\sigma^2}(1 + \alpha^2)(\sigma + \alpha)$	$((\beta + \gamma)(1 + \alpha\sigma))$	$\frac{R}{2\sigma}\sqrt{\sigma^2 + 1}$
2. Lemoine	$\frac{\alpha}{\sigma^2}(1 + \alpha^2)$	$K = (\beta + \gamma)$	$\frac{R}{\sigma}$
Taylors	$\frac{\alpha^2}{\sigma - \alpha\beta\gamma}$	$((\beta + \gamma)(1 - \alpha^2\beta\gamma))$	$\frac{R}{\sigma - \alpha\beta\gamma}\sqrt{1 + \alpha^2\beta^2\gamma^2}$
Tuckers circle with parameter κ	$\frac{1 + \alpha^2 - \kappa}{\sigma - \alpha\beta\gamma}\kappa$	$((\beta + \gamma)(\alpha\sigma - \alpha^2\beta\gamma + \kappa - \alpha\sigma\kappa))$	$\frac{R}{\sigma - \alpha\beta\gamma}\sqrt{(\sigma - \alpha\beta\gamma - \sigma\kappa)^2 + \kappa^2}$
Schouters circle with parameter κ	$\frac{1 + \alpha^2}{\sigma + \kappa}$	$((\beta + \gamma)(1 + \alpha\kappa)))$	$\frac{R}{\sigma + \kappa}\sqrt{\kappa^2 - 3}$

Some other triangles are added to triangle ABC. Let us give some most important ones.

The complementary triangle has vertices $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 0)$ and sides $(-1 : 1 : 1)$, $(1 : -1 : 1)$, $(1 : 1 : -1)$.

The anticomplementary triangle has vertices $(-1 : 1 : 1)$, $(1 : -1 : 1)$, $(1 : 1 : -1)$ and sides $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : 1 : 0)$.

The orthic triangle has vertices $(0 : \gamma : \beta)$, $(\gamma : 0 : \alpha)$, $(\beta : \alpha : 0)$ and sides $(-\alpha : \beta : \gamma)$, $(\alpha : -\beta : \gamma)$, $(\alpha : \beta : -\gamma)$.

The tangential triangle has vertices $(-(\beta + \gamma) : (\gamma + \alpha) : (\alpha + \beta))$, $((\beta + \gamma) : -(\gamma + \alpha) : (\alpha + \beta))$, $((\beta + \gamma) : (\gamma + \alpha) : -(\alpha + \beta))$ and sides $(0 : (\alpha + \beta) : (\gamma + \alpha))$, $((\alpha + \beta) : 0 : (\beta + \gamma))$, $((\gamma + \alpha) : (\beta + \gamma) : 0)$.

The first Brocard triangle has vertices $((\beta + \gamma) : (\alpha + \beta) : (\gamma + \alpha))$, $((\alpha + \beta) : (\gamma + \alpha) : (\beta + \gamma))$, $((\gamma + \alpha) : (\beta + \gamma) : (\alpha + \beta))$ and sides $((1 - \sigma^2 + 2\alpha\sigma) : (1 - \sigma^2 + 2\gamma\sigma) : (1 - \sigma^2 + 2\beta\sigma))$, $((1 - \sigma^2 + 2\gamma\sigma) : (1 - \sigma^2 + 2\beta\sigma) : (1 - \sigma^2 + 2\alpha\sigma))$, $((1 - \sigma^2 + 2\beta\sigma) : (1 - \sigma^2 + 2\alpha\sigma) : (1 - \sigma^2 + 2\gamma\sigma))$. That triangle and the triangle ABC have the center of homology $(1 + \alpha^2)$ and the axis of homology $((1 - \sigma^2 + 2\beta\sigma)(1 - \sigma^2 + 2\gamma\sigma))$.

The second Brocard triangle has vertices $(2\alpha : (\gamma + \alpha) : (\alpha + \beta))$, $((\beta + \gamma) : 2\beta : (\alpha + \beta))$, $((\beta + \gamma) : (\gamma + \alpha) : 2\gamma)$ and sides $((4\beta\gamma - 1 - \alpha^2) : (\beta + \gamma)(\sigma - 3\gamma) : (\beta + \gamma)(\sigma - 3\beta))$, $((\gamma + \alpha)(\sigma - 3\gamma) : (4\gamma\alpha - 1 - \beta^2) : (\gamma + \alpha)(\sigma - 3\alpha))$, $((\alpha + \beta)(\sigma - 3\beta) : (\alpha + \beta)(\sigma - 3\alpha) : (4\alpha\beta - 1 - \gamma^2))$. That triangle and the triangle ABC have the center of homology $(\beta + \gamma)$ and the axis of homology $((\sigma - 3\beta)(\sigma - 3\gamma))$.

Besides lines and circles given in the table, there are also the triplets of analogous lines and circles. We are giving the most important ones.

Altitudes $(0 : -\beta : \gamma)$, $(\alpha : 0 : -\gamma)$, $(-\alpha : \beta : 0)$ of triangle ABC and bisectors $((\beta - \gamma) : -(\beta + \gamma) : (\beta + \gamma))$, $((\gamma + \alpha) : (\gamma - \alpha) : -(\gamma + \alpha))$, $(-(\alpha + \beta) : (\alpha + \beta) : (\alpha - \beta))$ of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ have got points at infinity $(-(\beta + \gamma) : \gamma : \beta)$, $(\gamma : -(\gamma + \alpha) : \alpha)$, $(\beta : \alpha : -(\alpha + \beta))$. Medians $(0 : 1 : -1)$, $(-1 : 0 : 1)$, $(1 : -1 : 0)$ have points at infinity $(-2:1:1)$, $(1:-2:1)$, $(1:1:-2)$.

Apollonius circles

$$\left(0, \frac{1+\beta^2}{\beta-\gamma}, \frac{1+\gamma^2}{\gamma-\beta}\right), \left(\frac{1+\alpha^2}{\alpha-\gamma}, 0, \frac{1+\gamma^2}{\gamma-\alpha}\right), \left(\frac{1+\alpha^2}{\alpha-\beta}, \frac{1+\beta^2}{\beta-\alpha}, 0\right)$$

have centers $(0:-(\gamma+\alpha):(\alpha+\beta))$, $((\beta+\gamma):0:-(\alpha+\beta))$,$(-(\beta+\gamma):(\gamma+\alpha):0)$ and radii

$$\frac{2R}{|\beta-\gamma|}, \quad \frac{2R}{|\gamma-\alpha|}, \quad \frac{2R}{|\alpha-\beta|}.$$

Neuberg circles $(0,\beta+\gamma,\beta+\gamma)$, $(\gamma+\alpha,0,\gamma+\alpha)$, $(\alpha+\beta,\alpha+\beta,0)$ have centers $((\beta+\gamma)\sigma:(1-\gamma\sigma):(1-\beta\sigma))$, $((1-\gamma\sigma):(\gamma+\alpha)\sigma:(1-\alpha\sigma))$, $((1-\beta\alpha):(1-\alpha\sigma):(\alpha+\beta)\sigma)$ and radii $\frac{a}{2}\sqrt{\sigma^2-3}$, $\frac{b}{2}\sqrt{\sigma^2-3}$, $\frac{c}{2}\sqrt{\sigma^2-3}$.

M'Cay's circles $(\frac{2\alpha}{3}, \frac{1}{3}(\beta+\gamma), \frac{1}{3}(\beta+\gamma))$, $(\frac{1}{3}(\gamma+\alpha), \frac{2\beta}{3}, \frac{1}{3}(\gamma+\alpha))$, $(\frac{1}{3}(\alpha+\beta), \frac{1}{3}(\alpha+\beta), \frac{2\gamma}{3})$ have centers $((\beta+\gamma)\sigma:(3-\gamma\sigma):(3-\beta\sigma))$, $((3-\gamma\sigma):(\gamma+\alpha)\sigma:(3-\alpha\sigma))$, $((3-\beta\sigma):(3-\alpha\sigma):(\alpha+\beta)\sigma)$ and radii $\frac{a}{6}\sqrt{\sigma^2-3}$, $\frac{b}{6}\sqrt{\sigma^2-3}$, $\frac{c}{6}\sqrt{\sigma^2-3}$.

The six so-called adjoint circles are also interesting $(0,0,\beta+\gamma)$, $(\gamma+\alpha,0,0)$, $(0,\alpha+\beta,0)$, $(0,\beta+\gamma,0)$, $(0,0,\gamma+\alpha)$, $(\alpha+\beta,0,0)$, which successively go through points A,B,C, A,B,C and at points B,C,A, C,A,B successively touch lines BC, CA, AB, BC, CA, AB. They have successively centers $((1+\beta^2):(1+\alpha\beta):-\beta(\alpha+\beta))$, $(-\gamma(\beta+\gamma):(1+\gamma^2):(1+\beta\gamma))$, $((1+\gamma\alpha):-\alpha(\gamma+\alpha):(1+\alpha^2))$, $((1+\gamma^2):-\gamma(\gamma+\alpha):(1+\gamma\alpha))$, $((1+\alpha\beta):(1+\alpha^2):-\alpha(\alpha+\beta))$, $(-\beta(\beta+\gamma):(1+\beta\gamma):(1+\beta^2))$ and radii $R\frac{c}{b}$, $R\frac{a}{c}$, $R\frac{b}{a}$, $R\frac{b}{c}$, $R\frac{c}{a}$, $R\frac{a}{b}$.

The circumscribed conics of triangle ABC have the equation of the form $vyz+pzx+qxy=0$, where $v,p,q\in\mathcal{R}\setminus\{0\}$, and the inscribed conics in the line coordinate have the equation of the form $OYZ+PZX+QXY=0$, where $O,P,Q\in\mathcal{R}\setminus\{0\}$. Let us list the most important circumscribed and inscribed conics which can be expressed by α,β,γ and σ.

circumscribed Steiner ellipse	$yz+zx+xy=0$,
inscribed Steiner ellipse	$YZ+ZX+XY=0$,
Kiepert hyperbola	$(\beta-\gamma)yz+(\gamma-\alpha)zx+(\alpha-\beta)xy=0$,
Kiepert parabola	$(\beta-\gamma)YZ+(\gamma-\alpha)ZX+(\alpha-\beta)XY=0$,
Jeřabek hyperbola	$\alpha(\beta^2-\gamma^2)yz+\beta(\gamma^2-\alpha^2)zx+\gamma(\alpha^2-\beta^2)xy=0$,
Brocard ellipse	$(1+\alpha^2)YZ+(1+\beta^2)ZX+(1+\gamma^2)XY=0$,
K–conic	$\alpha YZ+\beta ZX+\gamma XY=0$,

Three Artzt's parabolas of the first kind have the equations $X^2-4YZ=0$, $Y^2-4ZX=0$, $Z^2-4XY=0$, and the Artzt's parabolas of the second kind have the equations

$$(\beta-\gamma)X^2+(\gamma+\alpha)Y^2-(\alpha+\beta)Z^2-2\alpha ZX+2\alpha XY=0,$$
$$-(\beta+\gamma)X^2+(\gamma-\alpha)Y^2+(\alpha+\beta)Z^2+2\beta YZ-2\beta XY=0,$$
$$(\beta+\gamma)X^2-(\gamma+\alpha)Y^2+(\alpha-\beta)Z^2-2\gamma YZ+2\gamma ZX=0.$$

Three Majcen's hyperbolas have the equations $(\beta-\gamma)yz-(\gamma+\alpha)zx+(\alpha+\beta)xy=0$, $(\beta+\gamma)yz+(\gamma-\alpha)zx-(\alpha+\beta)xy=0$, $-(\beta+\gamma)yz+(\gamma+\alpha)zx+(\alpha-\beta)xy=0$.

Lucas cubic of triangle ABC have the equation $\alpha x(y^2-z^2)+\beta y(z^2-x^2)+\gamma z(x^2-y^2)=0$, and Thomson, Darboux, Mac Cay's and Neuberg cubics, and the so-called ortho cubic have the equations of the form

$$dx[(\alpha+\beta)y^2-(\gamma+\alpha)z^2]+ey[(\beta+\gamma)z^2-(\alpha+\beta)x^2]+fz[(\gamma+\alpha)x^2-(\beta+\gamma)y^2]=0,$$

where the point $(d:e:f)$ is successively: centroid $G=(1)$, Longchamps point $L=(1-2\beta\gamma)$, circumcenter $O=(1-\beta\gamma)$, the point at infinity $(1-3\beta\gamma)$ of the Euler line and orthocenter $H=(\beta\gamma)$. Lemoine cubic have the equation

$$\alpha(\beta+\gamma)^2yz(y-z)+\beta(\gamma+\alpha)^2zx+\gamma(\alpha+\beta)^2xy-(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)xyz=0.$$

Steiner deltoid of triangle ABC (the envelope of Wallace lines) has the equation in line coordinates $\alpha X(Y-Z)^2+\beta Y(Z-X)^2+\gamma Z(X-Y)^2=0$.

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