On the trivial units in finite commutative group rings

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Abstract. *Let G be a finite abelian group and F a finite field. A criterion is found for all units in the group ring FG to be trivial. This attainment is also extended to the general case for arbitrary abelian groups and fields.*

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1. Introduction

Traditionally, throughout the paper, suppose FG is the group ring of an abelian group G over a field F with a group of invertible elements (also called *units*) $U(RG)$ and its subgroup $V(RG)$ of normalized units (that are units of augmentation 1). For such a group G, the letter $G_t = \coprod_{\forall p} G_p$ denotes the torsion part of G with p-primary components G_p taken over each prime number p. For such a field F, the symbol F^* is reserved for its multiplicative group ($F^* = F \setminus \{0\}$), $F(\eta_n)$ for its cyclotomic extension by adding the primitive n-th root of unity η_n and $(F(\eta_n) : F)$ is the degree of this extension over F. As usual, $|M|$ will designate the cardinality of an arbitrary set M. All other undefined or unexplained notions and notation are standard and follow essentially book [4].

A problem of central interest in the group ring theory is the so-called *trivial units question*, which reads:

Problem. *Under what circumstances on F and G, the units in FG are of the form fg, where $f \in F^*$ and $g \in G$?*

This is equivalent to the following two equalities: $V(FG) = G$ and so $U(FG) = V(FG) \times F^* = G \times F^*$.

One of the major studies in that theme, which is the first pioneer's one, is due to Higman [6] (see [8, Lemma 1] too). It reads as follows:

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Theorem 1 [Higman]. *Suppose G is a torsion-free abelian group and F is a field. Then $V(FG) = G$.*

Moreover, in [6] it was established that if G is a finite abelian group and F is the ring of algebraic numbers in an algebraic number field (of $char(F) = 0$), then $V(FG)$ possesses only trivial torsion units, that is, $V(FG)_t = G$.

An important generalization to this result for indecomposable rings of zero characteristic and, more concrete, a global exploration of torsion units in group rings were realized by May [8]. There it was obtained, under certain restrictions on a commutative unitary ring R and an abelian group G, that the identity $V(RG)_t = G_t$ holds.

We also note that another important explorations in the non-commutative aspect of the trivial units in group rings recently appeared in [2], [3], [5] and [9]. These interesting papers investigated similar problems by finding new suitable methods for determining trivial units of group rings in some special situations.

We will bound our attention only on general units, which are not necessarily torsion elements, by finding an appropriate necessary and sufficient condition in terms of powers of F and G. That criterion, listed in the sequel, exhausts the above query in full generality.

Before stating and proving the major criterion, which motivated us to write this paper, we recall explicitly the description of $V(FG)$ up to isomorphism in the case of finite groups and arbitrary fields.

Proposition 1 [1]. *Suppose G is a finite abelian group and F is a field whose characteristic does not divide the orders of the torsion elements in G. Then*

$$V(FG) \cong \coprod_{2 \leq d/|G|} \times_{l_d} F(\eta_d)^*,$$

where $l_d = |\{g \in G | o(g) = d\}|/(F(\eta_d) : F)$ and $\sum_{d/|G|} l_d.(F(\eta_d) : F) = |G|$.

The connection between this isomorphic classification and the present manuscript is that the foregoing isomorphism formula ensures the power estimation of $V(FG)$ that needs to be compared with that of G. The novelty in our approach is comparison of the corresponding cardinalities by the usage of the aforementioned isomorphism relationship.

2. Main result

The next affirmation is our crucial tool. It asserts as follows:

Theorem 2. *Suppose G is a finite abelian group and F is a field. Then $V(FG) = G \iff G = 1$, or $|F| = 2$ and $|G| = 2$, or $|F| = 2$ and $|G| = 3$, or $|F| = 3$ and $|G| = 2$.*

Proof. First of all, we shall show that F must be of finite power. To this end, assume in a way of contradiction that $|F| \geq \aleph_0$. If $char(F)/|G|$, we derive that there exists $1 \neq g \in G$ with $o(g) = char(F)$. By considering the family of elements $x_r = 1 + r(g - 1)$, where $\{0, 1\} \neq r \in R$, we easily check that x_r is a torsion element of order $o(x_r) = o(g)$ and thus $x_r \in V(FG) \setminus G$ gives that $|V(FG)| \geq |F|$. Therefore, as a result, $V(FG) \neq G$. If now FG is semi-simple, that is $char(F)$ does not divide $|G|$, we apply [1] to get that $V(FG)$ is isomorphic to a various number of nontrivial copies of cyclotomic extensions of F, so it is infinite as well and thereby $V(FG) \neq G$ once again. These two derivatives substantiate our claim on the cardinality of F, and even that FG would always be semi-simple provided $|F| \neq |G|$, hence there exist non-trivial idempotents in FG; when $|F| = |G|$ it is simply checked via the method alluded to the above that $V(FG) = G$ only when $|F| = |G| = 2$.

After this, in accordance with [1], we write down the above cited isomorphism formula for $V(FG)$, namely:

$$V(FG) \cong \coprod_{2 \leq d/|G|} \times_{l_d} F(\eta_d)^*,$$

where $l_d = |\{g \in G | o(g) = d\}| / (F(\eta_d) : F)$ and $\sum_{d/|G|} l_d.(F(\eta_d) : F) = |G|$.

Since both $V(FG)$ and G are now finite and $G \subseteq V(FG)$, to obtain the desired equality $V(FG) = G$ it is strong enough to show that $|V(FG)| = |G|$.

For this purpose, we elementarily observe that $|G| = q$ where q is a prime number. Indeed, if there is $1 \neq C < G$ we may select $g \in G \setminus C$ and an idempotent $e^2 = e \in FC \setminus \{0, 1\}$. Thus, by constructing the element $x_g = 1 + e(g - 1)$, we routine verify that $x_g \in V(FG) \setminus G$ because $x_g^{-1} = g^{-1} + (1 - e)(1 - g^{-1})$ and $eg \neq e$; if $g^m = 1$ for some $m \in \mathbb{N}$ we find that $x_g^m = 1$. Thus the fact occurs that G does not have proper subgroups.

For facilitating of the expression, let $char(F) = p \neq 0$. Certainly, p is a prime integer.

We further distinguish two basic cases about η_q, namely:

Case 1. $\eta_q \in F$. Therefore, the previous isomorphism relation takes the form

$$V(FG) \cong \times_{q-1} F^*,$$

since $l_q = q - 1$, which fact follows according to $|\{g \in G | o(g) = q\}| = q - 1$ and $(F(\eta_q) : F) = 1$.

Consequently,

$$|V(FG)| = |G| \iff |(F^*)^{q-1}| = q.$$

Henceforth, $(p^n - 1)^{q-1} = q$ for some $n \in \mathbb{N}$ and we observe that this diophantine equation possesses solely the solution $p = 3$, $n = 1$ and $q = 2$. That is why $|F| = 3$ and $|G| = 2$, whence the wanted last condition from the theorem is true.

Case 2. $\eta_q \notin F$. Furthermore, the preceding isomorphism relationship takes the form

$$V(FG) \cong F(\eta_q)^*,$$

since $l_q = 1$, which fact follows owing to $|\{g \in G | o(g) = q\}| = q - 1 = (F(\eta_q) : F)$.

Consequently,

$$|V(FG)| = |G| \iff |F(\eta_q)^*| = q.$$

Thereby, because the polynomial $1 + x + x^2 + \cdots + x^{q-1}$ is of minimal degree $q - 1$ so that η_q is its solution and is also irreducible over F, appealing to ([7], p. 187, Proposition 3) we infer that $|F(\eta_q)| = |F|^{(F(\eta_q):F)} = |F|^{q-1} = p^{n(q-1)}$ and thus $p^{n(q-1)} - 1 = q$ for some $n \in \mathbb{N}$. But this diophantine equation has unique solution in the current situation like this $p = 2$, $n = 1$ and $q = 3$, since the other possibility $p = 3$, $n = 1$ and $q = 2$ must be excluded by observing that $\eta_q = \eta_2 = \pm 1 \in F$. That is why $|F| = 2$ and $|G| = 3$, hence the desired condition is sustained.

It is also worthwhile noticing that because, referring to ([4], p. 368, Theorem 127.2 - Skolem) or to ([7], p. 210, Theorem 11), F^* is a cyclic group of order $p^n - 1$, we derive that $p^n - 1 \leq q$.

The proof is completed. □

The following main statement completely settles the foregoing investigated Problem over a field.

General Criterion. *Let G be an abelian group and let F be a field. Then $V(FG) = G \iff$ (1) $G = 1$; (2) $G \neq 1$, $G_t = 1$; (3) $|F| = 2, |G| = 2$; (4) $|F| = 2, |G| = 3$; (5) $|F| = 3, |G| = 2$.*

Proof. Foremost, given that $|G| \geq \aleph_0$ with $G_t \neq 1$. Consequently, as we have already obtained, there is a finite subgroup $1 \neq C < G$ with the properties: $e^2 = e \in FC \setminus \{0, 1\}$ and $g \in G \setminus C$, whence $ge \neq e$. Furthermore, we consider the element $1 \neq x_g = 1 + e(g - 1)$. Since $x_g^{-1} = g^{-1} + (1 - e)(1 - g^{-1})$ exists, we obviously deduce that $x_g \in V(FG) \setminus G$. So, $V(FG) \neq G$ and this discrepancy with our assumption leads us to $|G| < \aleph_0$ whenever $G_t \neq 1$. Henceforth, the equivalence follows immediately by the quoted result of Higman combined with the previous Theorem. The proof is finished. □

Example 1. *To be more complete, we show that non-trivial units exist in certain concrete situations.*

1) *Let we assume that $|F| = 2$ and $|G| = 5$. So $F = \{0, 1\}$ and $G = \langle a | a^5 = 1 \rangle$. Construct $u = 1 + a + a^2 \in FG$. Clearly, $u.(a + a^2 + a^4) = 1$ and $1 \neq u \in V(FG) \setminus G$, so we are done.*

2) *Let we assume that $|F| = 4$ and $|G| = 3$. Then $F = \{0, 1, r, r^2 | r^3 = 1\}$ with $char(R) = 2$ and $G = \langle g \rangle$ with $o(g) = 3$ that is $G = \{1, g, g^2 | g^3 = 1\}$. Then F contains all primitive 3-th roots of unity, i.e. $\eta_3 \in F$. Select $v = 1 + rg + rg^2$. Evidently, $v^3 = 1$ since $1 + r + r^2 = 0$ and so v is a torsion unit with $1 \neq v \in V(FG) \setminus G$. More precisely, $V(FG)$ possesses nine elements of order 3, six of which are non-trivial units, unlike G which has only three elements.*

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