

A fixed fuzzy point for fuzzy mapping in complete metric spaces*

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Abstract. *In this paper, we prove a fixed fuzzy point theorem for fuzzy mappings over a complete metric space.*

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1. Introduction

After the introduction of the concept of a fuzzy set by Zadeh [11], several researches were conducted on the generalizations of the concept of a fuzzy set. The idea of an intuitionistic fuzzy set is due to Atanassov [1], [2], [3] and Çoker [5] has defined the concept of fuzzy topological spaces induced by Chang [4]. Heilpern [7], introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [8]. Estruch and Vidal [6] give a fixed point theorem for fuzzy contraction mappings over a complete metric spaces which is a generalization of the given Heilpern's fixed point theorem. In this paper we give a fixed point theorem for a fuzzy contraction mapping over a complete metric space which is a generalization of the fixed point theorem given by Estruch and Vidal. They give a fixed point theorem under the condition $D_\alpha(F(x), F(y)) \leq qd(x, y)$ for each $x, y \in X$. We give a fixed point theorem under the condition $D_\alpha(F(x), F(y)) \leq K(M(x, y))$ and some examples, where D_α, F, q, X, K and M are following defined. One of these example has been an extended version of their main theorem.

2. Preliminaries

Let X be a nonempty set and $I = [0, 1]$. A fuzzy set of X is an element of I^X . For $A, B \in I^X$ we denote $A \subseteq B$ if and only if $A(x) \leq B(x)$ for each $x \in X$.

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Definition 1 [3]. An intuitionistic fuzzy set (*i*-fuzzy set) A of X is an object having the form $A = \langle A^1, A^2 \rangle$ where $A^1, A^2 \in I^X$ and $A^1(x) + A^2(x) \leq 1$ for each $x \in X$.

We denote by $IFS(X)$ the family of all *i*-fuzzy sets of X .

Remark 1. If $A \in I^X$, then A is identified with the *i*-fuzzy set $\langle A, 1 - A \rangle$ denoted by $[A]$.

For $x \in X$ we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of X . For $\alpha \in (0, 1]$ the fuzzy point [9] x_α of X is the fuzzy set of X given by $x_\alpha(x) = \alpha$ and $\alpha \neq x$. Now we give the following definition.

Definition 2 [6]. Let x_α be a fuzzy point of X . We will say that $\langle x_\alpha, 1 - x_\alpha \rangle$ is an *i*-fuzzy point of X and it will be denoted by $[x_\alpha]$. In particular $[x] = \langle \{x\}, 1 - \{x\} \rangle$ will be called an *i*-point of X .

Definition 3 [3]. Let $A, B \in IFS(X)$. Then $A \subset B$ if and only if $A^1 \subset B^1$ and $B^2 \subset A^2$.

Remark 2. Notice $[x_\alpha] \subset A$ if and only if $x_\alpha \subset A^1$.

Let (X, d) be a metric linear space (i. e., a complex or real vector space). The α - level set of A , denoted by A_α , is defined by

$$A_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1],$$

$$A_0 = \overline{\{x : A(x) > 0\}}$$

where \overline{B} denotes the closure of the (non fuzzy)set B . Heilpern [7] called a fuzzy mapping a mapping from the set of X into a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ if and only if A_α is compact and convex in X for each $\alpha \in [0, 1]$ and $\sup\{Ax : x \in X\} = 1$. In this context we give the following definitions.

Definition 4 [7]. Let $A, B \in W(X)$, $\alpha \in [0, 1]$. Define

$$p_\alpha(A, B) = \inf \{d(x, y) : x \in A_\alpha, y \in B_\alpha\},$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

$$D(A, B) = \sup_\alpha D_\alpha(A, B),$$

where H is the Hausdorff distance.

For $x \in X$ we write $p_\alpha(x, B)$ instead of $p_\alpha(\{x\}, B)$.

Definition 5 [6]. Let X be a metric space and $\alpha \in [0, 1]$. Consider the following family $W_\alpha(X)$:

$$W_\alpha(X) = \{A \in I^X : A_\alpha \text{ is nonempty, compact and convex}\}$$

Now we define the family $\Phi_\alpha(X)$ of *i*-fuzzy sets of X as follows:

$$\Phi_\alpha(X) = \{A \in IFS(X) : A^1 \in W_\alpha(X)\}.$$

Clearly, for $\alpha \in I$, $W(X) \subset \Phi_\alpha(X)$ in the sense of Remark 1.

We need the following lemmas. Let (X, d) be metric space.

Lemma 1. [7] Let $x \in X$ and $A \in W(X)$. Then $x_\alpha \subset A$ if $p_\alpha(x, A) = 0$.

Lemma 2. [7] $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$ for $x, y \in X$, $A \in W(X)$.

Lemma 3. [7] If $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$, for each $A, B \in W(X)$.

3. Fixed fuzzy point theorem

In mathematical programming, problems are expressed as optimizing some goal function given certain constraints and there are real-life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that brings us to the optimum of all objective functions. A possible method of resolution that is quite useful is the one using Fuzzy Sets. The idea is to relax the pretenses of optimization by means of a subjective gradation which can be modelled into fuzzy membership functions μ_i . If $\cap \mu_i$ the objective will be to search x such that $\max F = F(x)$. If $\max F = 1$, then there exists x such that $F(x) = 1$, but if $\max F = \alpha$, $\alpha \in (0, 1)$, the solution of the multiobjective optimization is a fuzzy point x_α and $F(x) = \dot{\alpha}$.

In a more general sense than the one given by Heilpern, a mapping $F : X \rightarrow I^X$ is a fuzzy mapping over X ([10]) and $(F(x))(x)$ is the fixed degree of x for F . In this context we give the following definition.

Definition 6 [6]. Let x_α be a fuzzy point of X . We will say that x_α is a fixed fuzzy point of the fuzzy mapping F over X if $x_\alpha \subset F(x)$ (i. e., the fixed degree of x is at least α). In particular, and according to [7], if $\{x\} \subset F(x)$, we say that x is a fixed point of F .

Theorem 1. Let $\alpha \in (0, 1]$ and (X, d) be a complete metric space. Let F be a continuous fuzzy mapping from X into $W_\alpha(X)$ satisfying the following condition:

There exists $K : [0, \infty) \rightarrow [0, \infty)$, $K(0) = 0$, $K(t) < t$ for all $t \in (0, \infty)$ and K is non-decreasing such that

$$D_\alpha(F(x), F(y)) \leq K(M(x, y)) \tag{1}$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), p_\alpha(x, F(x)), p_\alpha(y, F(y)), p_\alpha(x, F(y)), p_\alpha(y, F(x))\}.$$

Then there exists $x \in X$ such that x_α is a fixed fuzzy point of F if and only if there exists $x_0, x_1 \in X$ such that $x_1 \in (F x_0)_\alpha$ with $\sum_{n=1}^{\infty} K^n(d(x_0, x_1)) < \infty$. In particular, if $\alpha = 1$, then x is a fixed point of F .

Proof. If there exists $x \in X$ such that x_α is a fixed fuzzy point of F , then $x_\alpha \subset F(x)$ and $d(x, x) = 0$, $0 = K(0) = K^2(0) = \dots = K^n(0) = \dots$ and $\sum_{n=1}^{\infty} K^n(d(x, x)) = 0$. Let $x_0 \in X$ and suppose that there exists $x_1 \in (F(x_0))_\alpha$ such that $\sum_{n=1}^{\infty} K^n(d(x_0, x_1)) < \infty$. Since $(F(x_1))_\alpha$ is a nonempty compact subset of X , then there exists $x_2 \in (F(x_1))_\alpha$ such that

$$d(x_1, x_2) = p_\alpha(x_1, F(x_1)) \leq D_\alpha(F(x_0), F(x_1))$$

by Lemma 3. By induction we construct a sequence $\{x_n\}$ in X such that $x_n \in (F(x_{n-1}))_\alpha$ and $d(x_n, x_{n+1}) \leq D_\alpha(F(x_n), F(x_{n-1}))$.

Noting that K is nondecreasing and using inequality (1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(F(x_n), F(x_{n-1})) \\ &\leq K(\max\{d(x_{n-1}, x_n), p_\alpha(x_{n-1}, F(x_{n-1})), p_\alpha(x_n, F(x_n)), \\ &\quad p_\alpha(x_{n-1}, F(x_n)), p_\alpha(x_n, F(x_{n-1}))\}) \end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq K(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \quad (2)$$

Suppose $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$ for some n . Then from (2) and $K(t) < t$ for all $t \in (0, \infty)$, we have

$$d(x_n, x_{n+1}) \leq K(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$$

which is a contradiction. Therefore, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq D_\alpha(F(x_{n-1}), F(x_n)) \\ &\leq K(d(x_{n-1}, x_n)) \\ &= K(p_\alpha(x_{n-1}, F(x_{n-1}))) \\ &\leq K(D_\alpha(F(x_{n-1}), F(x_{n-2}))) \\ &\leq K(K(d(x_{n-1}, x_{n-2}))) \\ &\quad \vdots \\ &\leq K^n(d(x_0, x_1)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + \dots + d(x_{n+m-1}, x_{n+m}) \\ &\leq K^n(d(x_0, x_1)) + \dots + K^{n+m-1}(d(x_0, x_1)) \\ &= \sum_{k=n}^{n+m-1} K^k(d(x_0, x_1)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} K^n(d(x_0, x_1)) < \infty$, it follows that there exists r such that $d(x_n, x_{n+m}) < r$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in X . Suppose $\{x_n\}$ converges to $x \in X$. Now by *Lemmas 1* and *2* we have

$$\begin{aligned} p_\alpha(x, F(x)) &\leq d(x, x_n) + p_\alpha(x_n, F(x)) \\ &\leq d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \\ &\leq d(x, x_n) + K(d(x_{n-1}, x)). \end{aligned}$$

consequently, $p_\alpha(x, F(x)) = 0$ and by *Lemma 1*, $x_\alpha \subset F(x)$. \square

Remark 3. K needs only be defined on the range d . If one replaces the metric with an equivalent metric with $d(x, y) < 1$, then clearly Theorem 4 holds for K :

$[0, 1) \rightarrow [0, \infty)$. To apply Theorem 4, one needs a nondecreasing function K and x in X with $\sum_{n=1}^{\infty} K^n(p_\alpha(x, F(x))) < \infty$.

The following examples satisfy these conditions and therefore illustrate the generality of Theorem 4. Let (X, d) be a complete metric space.

Example 1. Suppose $0 < q < 1$. Let $K(t) = qt$ for $t \geq 0$. Then $D_\alpha(F(x), F(y)) \leq K(M(x, y)) = qM(x, y)$ and $K^n(p_\alpha(x, F(x))) = q^n p_\alpha(x, F(x))$ for any x in X . It is known that there exist $x \in X$ such that $x_\alpha \subset F(x)$.

Remark 4. Example 1 is a generalization of Theorem 3.2. of [6].

Remark 5. Theorem 3.1. of [7]. A fixed point of the fuzzy mapping $F : X \rightarrow W(X)$ exists whenever $D(F(x), F(y)) \leq qd(x, y)$ for each $x, y \in X$, being $q \in (0, 1)$. Now, in Example 1, it is clear that the condition $D(F(x), F(y)) \leq qd(x, y)$ can be weakened to $D_1(F(x), F(y)) \leq qd(x, y)$.

Example 2. Suppose that F satisfies $D_\alpha(F(x), F(y)) \leq \phi(M(x, y))d(x, y)$ for all x, y in X , where $\phi : [0, \infty) \rightarrow [0, 1)$ and ϕ is nondecreasing. Then $K(t) = t\phi(t)$ is nondecreasing and $K : [0, \infty) \rightarrow [0, \infty)$. It follows by induction that $K^n(t) \leq t[\phi(t)]^n$, since $\phi(t) < 1$ and $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Example 3. Consider $K(M(x, y)) = M(x, y)\phi(M(x, y))$ for all x, y in X , where $\phi : [0, \infty) \rightarrow [0, \infty)$, $\phi(t) \leq t$ for $t \leq 1$. If $t < 1$, it follows that $K^n(t) \leq t[\phi(t)]^n$. If K is nondecreasing, then Theorem 4 can be applied.

Example 4. $K(M(x, y)) = M(x, y)\phi(M(x, y))$ for all x, y in X , where $\phi : [0, \infty) \rightarrow [0, \infty)$, $\phi(qM(x, y)) \leq q\phi(M(x, y))$ for $q \in (0, 1)$. If $\phi(t) < 1$, then $K^n(t) \leq K(t)[\phi(t)]^n$ for all $n \geq 2$.

Example 5. Assume that K is nondecreasing, K is convex on $[0, 1)$ and $K(M(x, y)) < M(x, y)$ for all x, y in X . If $t < 1$, $K(t) < t$ for all $0 < t < 1$, then $K(t) = qt$ for some $0 < q < 1$. It can be shown that $K^n(t) \leq q^n t$ for all n and $\sum_{n=1}^{\infty} K^n(t) < \infty$.

Example 6. Let $X = [0, 1]$ and let $d : X \times X \rightarrow IR^+$ be the Euclidean metric. Let $\alpha \in (0, \frac{1}{2})$ and suppose $F : X \rightarrow I^X$ defined by

$$F(0)(x) = \begin{cases} 1 & , x = 0 \\ \alpha, & x \in (0, \frac{1}{2}] \\ \frac{\alpha}{2}, & x \in (\frac{1}{2}, 1] \end{cases}$$

$$F(1)(x) = \begin{cases} 1 & , x = 0 \\ 2\alpha, & x \in (0, \frac{1}{2}] \\ \frac{\alpha}{2}, & x \in (\frac{1}{2}, 1] \end{cases}$$

and for $z \in (0, 1)$

$$F(z)(x) = \begin{cases} 1 & , x = 0 \\ \alpha, & x \in (0, \frac{1}{2}] \\ 0, & x \in (\frac{1}{2}, 1] \end{cases}.$$

Then $F(0)_1 = F(z)_1 = F(1)_1 = \{0\}$, $F(0)_\alpha = F(z)_\alpha = F(1)_\alpha = [0, \frac{1}{2}]$, and

$F(0)_{\frac{\alpha}{2}} = F(1)_{\frac{\alpha}{2}} = [0, 1]$, $F(z)_{\frac{\alpha}{2}} = [0, \frac{1}{2}]$. Consequently

$$\begin{aligned} D_1(F(x), F(y)) &= H(F(x)_1, F(y)_1) = 0, \quad \forall x, y \in X, \\ D_\alpha(F(x), F(y)) &= H(F(x)_\alpha, F(y)_\alpha) = 0, \quad \forall x, y \in X, \\ D_{\frac{\alpha}{2}}(F(x), F(y)) &= H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}}) = 0, \quad \forall x, y \in \{0, 1\} \text{ and } \forall x, y \in (0, 1), \\ D_{\frac{\alpha}{2}}(F(x), F(y)) &= H(F(x)_{\frac{\alpha}{2}}, F(y)_{\frac{\alpha}{2}}) = \frac{1}{2}, \quad \forall x \in \{0, 1\} \text{ and } \forall y \in (0, 1). \end{aligned}$$

Let $\phi : [0, \infty) \rightarrow [0, \infty)$, $\phi(t) = \frac{t^2}{1+t}$. Thus $\phi(t) = t \frac{t}{1+t} < t$. Let $K(t) = \frac{t^3}{1+t} = t\phi(t)$. If $t \leq 1$, it follows that $K^n(t) \leq t[\phi(t)]^n = \frac{t^{2n+1}}{(1+t)^n}$ and $\sum_{n=1}^{\infty} K^n(t) < \infty$. Therefore $D_\alpha(F(x), F(y)) = 0 \leq K(d(x, y))$ for each $x, y \in X$. By Theorem 4, there exists a fixed fuzzy point (a fixed point) of the fuzzy mapping F . We can see by the definition of F that 0 is a fixed point. Nevertheless, Heilpern's theorem is not useful in this example because $D_{\frac{\alpha}{2}}(F(0), F(\frac{1}{3})) = H(F(0)_{\frac{\alpha}{2}}, F(\frac{1}{3})_{\frac{\alpha}{2}}) = \frac{1}{2} > \frac{1}{3} = d(0, \frac{1}{3})$ and then $D(F(0), F(\frac{1}{3})) = \sup_k H(F(0)_k, F(\frac{1}{3})_k) > d(0, \frac{1}{3})$.

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