Fixed points of asymptotically regular mappings

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Abstract. *Two general fixed point theorems for asymptotically regular self-mappings on a metric space X which satisfy the contractive condition (1) below are proved. Our results extend and generalize results of Sharma and Yuel [4] and Guay and Singh [3].*

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1. Introduction

Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions, which imply the existence of a fixed point. Almost all of conditions imply the asymptotic regularity of the mappings under consideration. So the investigation of the asymptotically regular maps plays an important role in fixed point theory.

Sharma and Yuel [4] and Guay and Singh [3] were among the first who used the concept of asymptotic regularity to prove fixed point theorems for a wider class of mappings than a class of mappings introduced and studied by Ćirić [2].

The purpose of this short paper is to study a wide class of asymptotically regular mappings which possess fixed points in complete metric spaces. Our results generalize and unify the results of Sharma and [4] and Guay and Singh [3].

2. Main Results

Browder and Petryshyn [1] defined the following notion.

Definition 1. *A selfmapping T on a metric space (X, d) is said to be* asymptotically regular at a point x in X, *if*

$$d(T^n x, T^n Tx) \to 0 \quad as \quad n \to \infty, \tag{1}$$

where $T^n x$ denotes the n-th iterate of T at x.

Let R^+ be the set of nonnegative reals and let $F_i : R^+ \to R^+$ be functions such that $F_i(0) = 0$ and F_i is continuous at 0 $(i = 1, 2)$.

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Our main result is the following theorem.

Theorem 1. *Let (X, d) be a complete metric space and T a selfmapping on X satisfying the following condition:*

$$d(Tx,Ty) \le a_1F_1[\min\{d(x,Tx),d(y,Ty)\}] + a_2F_2[d(x,Tx)\cdot d(y,Ty)] \\ +a_3d(x,y) + a_4[d(x,Tx)+d(y,Ty)] + a_5[d(x,Ty)+d(y,Tx)] \quad (2)$$

for all x, y in X, where $a_i = a_i(x,y)$ $(i = 1,2,3,4,5)$ are nonnegative functions such that for arbitrarily fixed $K > 0$ and $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$:

$$a_1(x,y), a_2(x,y) \le K, \quad (3)$$

$$a_4(x,y) + a_5(x,y) \le \lambda_1, \quad (4)$$

$$a_3(x,y) + 2a_5(x,y) \le \lambda_2. \quad (5)$$

If T is asymptotically regular at some x_0 in X, then T has a unique fixed point in X and at this point T is continuous.

Proof. We show that $\{x_n\}$ is a Cauchy sequence, where $x_n = T^n x_0$. Denote

$$d_n = d(x_n, x_{n+1}). \quad (6)$$

Using the triangle inequality, from (2) we have

$$d(x_n,x_m) \le d_n + d(Tx_n,Tx_m) + d_m \le d_n + d_m \\ +a_1F_1[\min\{d_n,d_m\}] + a_2F_2(d_n\cdot d_m) \\ +a_3d(x_n,x_m) + a_4(d_n+d_m) + a_5[d(x_n,Tx_m)+d(x_m,Tx_n)],$$

where $a_i = a_i(x_n,x_m)$. Using again the triangle inequality, we get

$$d(x_n,x_m) \le (a_3+2a_5)d(x_n,x_m) + (1+a_4+a_5)(d_n+d_m) \\ +a_1F_1[\min\{d_n,d_m\}] + a_2F_2(d_n\cdot d_m).$$

Hence, because of (3), (4) and (5), we obtain

$$(1-\lambda_2)d(x_n,x_m) \le (1+\lambda_1)(d_n+d_m) + KF_1[\min\{d_n,d_m\}] + KF_2(d_n\cdot d_m). \quad (7)$$

Since T is asymptotically regular and F_1 and F_2 are continuous at zero, taking the limit as m tends to infinity we obtain

$$(1-\lambda_2)\lim_{n>m\to\infty} d(x_n,x_m) \le 0, \quad (8)$$

which implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there is some u in X such that

$$\lim x_n = u. \quad (9)$$

Now we show that u is a unique fixed point of T. Suppose that $d(u,Tu) > 0$. From (2) we have

$$d(u,Tu) \le d(u,Tx_n) + d(Tx_n,Tu) \le d(u,x_{n+1}) \\ +a_1F_1[\min\{d_n,d(u,Tu)\}] + a_2F_2[d_n\cdot d(u,Tu)] \\ +a_3d(x_n,u) + a_4[d_n+d(u,Tu)] + a_5[d(x_n,Tu)+d(u,x_{n+1})],$$

where $a_i = a_i(x_n, u)$. Using the triangle inequality we get

$$\begin{aligned} d(u,Tu) &\le (1+a_5)d(u,x_{n+1}) + a_1F_1[\min\{d_n, d(u,Tu)\}] \\ &\quad + a_2F_2[d_n \cdot d(u,Tu)] + (a_3+a_5)d(x_n,u) + a_4d_n + (a_4+a_5)d(u,Tu). \end{aligned}$$

Therefore, from (3), (4) and (5),

$$\begin{aligned} d(u,Tu) &\le \lambda_1 d(u,Tu) + (1+\lambda_2)d(u,x_{n+1}) + \lambda_2 d(u,x_n) \\ &\quad + K\cdot F_1[\min\{d_n, d(u,Tu)\}] + KF_2[d_n\cdot d(u,Tu)]. \end{aligned}$$

Taking the limit we get $d(u,Tu) \le \lambda_1 d(u,Tu) < d(u,Tu)$, a contradiction. Therefore, $d(u,Tu) = 0$; hence $Tu = u$.

To prove the uniqueness of u, let us suppose that u and v are two fixed points of T. From (2), with $a_i = a_i(u,v)$,

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\le a_1F_1(0) + a_2F_2(0) + a_3d(u,v) + a_4\cdot 0 + 2a_5d(u,v) \\ &= (a_3+2a_5)d(u,v). \end{aligned}$$

Hence, because of (5),

$$(1-\lambda_2)d(u,v) \le 0, \tag{10}$$

which implies $v = u$.

To prove that T is continuous at u, suppose that $x_n \to u = Tu$. Then from (2),

$$\begin{aligned} d(Tx_n,u) = d(Tx_n,Tu) &\le a_1\cdot F_1(0) + a_2F_2(0) \\ &\quad + a_3d(x_n,u) + a_4d(x_n,Tx_n) \\ &\quad + a_5[d(x_n,u) + d(Tx_n,u)] \\ &\le (a_3+a_4+a_5)d(x_n,u) + (a_4+a_5)d(u,Tx_n), \end{aligned}$$

where $a_i = a_i(x_n,u)$. Hence, using (4) and (5),

$$(1-\lambda_1)d(Tx_n,u) \le (\lambda_1+\lambda_2)d(x_n,u). \tag{11}$$

Letting n go to infinity, we obtain

$$(1-\lambda_1)\lim_{n\to\infty} d(Tx_n,u) \le 0, \tag{12}$$

which implies that $\lim_{n\to\infty} Tx_n = u$. □

Remark 1. *The contractive condition considered by Sharma and Yuel [4] is defined as follows:*

$$d(Tx,Ty) \le \alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y), \tag{13}$$

where α,β are nonnegative reals, satisfying

$$\alpha < 1, \qquad \beta < 1. \tag{14}$$

By symmetry of d, it is clear that (13) implies

$$d(Tx,Ty) \le \alpha \frac{\min\{d(x,Tx), d(y,Ty)\} + d(x,Tx)\cdot d(y,Ty)}{1+d(x,y)} + \beta d(x,y). \tag{15}$$

Our condition (2) becomes (15) if

$$a_1(x,y) = a_2(x,y) = \frac{1}{1+d(x,y)}, F_1(t) = F_1(t) = \alpha \cdot t,$$
$$a_3(x,y) = \beta, \quad a_4(x,y) = a_5(x,y) = 0,$$

and clearly (3) and (5) becomes $a_1 = a_2 \le 1$ and $\beta < 1$, respectively.

Remark 2. *The contractive condition, introduced and considered by Guay and Singh [3], is defined as follows:*

$$d(Tx,Ty) \le pd(x,y) + q[d(x,Tx) + d(y,Ty)] + r[d(x,Tx) + d(y,Ty)], \tag{16}$$

where p, q and r are fixed real numbers such that $q + r < 1$, $p + 2r < 1$. It is clear that our condition (2) becomes (16), if $a_1 = 0$, $a_2 = 0$ and $a_3 = p$, $a_4 = q$ and $a_5 = r$.

Remark 3. *The example of Sharma and Yuel [4] shows that the assumption of asymptotically regularity in the above theorems cannot be dropped.*

The following theorem generalizes Theorem 2 of Sharma and Yuel [4].

Theorem 2. *Let (X,d) be a metric space, not necessarily complete, and let T be as in* Theorem 1. *If the sequence of iterates $\{T^n x_0\}$ at some x_0 has a subsequence converging to a point u in X, then u is the unique fixed point of T, and $\{T^n x_0\}$ also converges to u and T is continuous at u.*

Proof. As shown in the proof of *Theorem 1*, $\{T^n x_0\}$ is a Cauchy sequence. Since it contains a subsequence converging to u, $\lim T^n x_0 = u$. The rest of the result follows by the same method of proof as in *Theorem 1.* □

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