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A NOTE ON GENERALIZED DERIVATIONS OF PRIME RINGS

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Abstract. We show that a generalized derivation on a prime ring, that acts as a homomorphism or an anti-homomorphism on a non-zero ideal in the ring, is the zero map or the identity map.

Let R be an associative ring, let d be a derivation on R (i.e. an additive function on R satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$) and let $F : R \to R$ be a generalized derivation associated to d (i.e. an additive function satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$).

We say that R is prime if the relation $aRb = 0$ implies that $a = 0$ or $b = 0$, for all $a, b \in R$. Note that if R is a prime ring and I is a non-zero ideal of R, then the relation $aIb = 0$ implies that $a = 0$ or $b = 0$, for all $a, b \in R$

In [R, Theorem 1.2] the following statement is stated.

Assume that R is 2-torsion free and prime.

(i) *If $d \neq 0$ and F acts as a homomorphism on a non-zero ideal I in R then R is commutative.*

(ii) *If $d \neq 0$ and F acts as an anti-homomorphism on a non-zero ideal I in R then R is commutative.*

It seems that the assumptions in this statement are contradictory. Also, despite an ingenious argument the conclusion is incomplete. Using a similar argument we prove the following:

Theorem 1. *Let R be an associative prime ring, let d be any function on R (not necessary a derivation nor an additive function), let F be any function on R (not necessarily additive) satisfying $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, and let I be a non-zero ideal in R.*

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(a) *Assume that* $F(xy) = F(x)F(y)$ *for all* $x, y \in I$. *Then* $d = 0$, *and* $F = 0$ *or* $F(x) = x$ *for all* $x \in R$.

(b) *Assume that* $F(xy) = F(y)F(x)$ *for all* $x, y \in I$. *Then* $d = 0$, *and* $F = 0$ *or* $F(x) = x$ *for all* $x \in R$ *(in this case* R *should be commutative).*

PROOF. (a) Assume that $F|I$ is a homomorphism of rings. Then calculating $F(xyz)$ in two different ways (as in [R]) we get $(F(x) - x)yd(z) = 0$ for all $x, y, z \in I$. Since R is prime, we conclude that if $d|I \neq 0$ then $F(x) = x$, for all $x \in I$. From this we get $xd(y) = 0$ for all $x, y \in I$. Since R is prime it implies that $d(y) = 0$ for all $y \in I$, a contradiction. Hence, $d|I = 0$.

Now, from $F(x)y = F(x)F(y)$ for all $x, y \in I$, replacing x by zt we get $F(z)t(y - F(y)) = 0$ for all $z, t, y \in I$. This implies $F(z) = 0$ for all $z \in I$ or $F(y) = y$ for all $y \in I$. If $F(z) = 0$ for all $z \in I$, then $0 = F(rz) = F(r)z + rd(z) = F(r)z$ for all $r \in R$ and $z \in I$, hence F is zero on R. If $F(y) = y$ for all $y \in I$ then $ry = F(ry) = F(r)y + rd(y) = F(r)y$ for all $r \in R$ and $y \in I$. Therefore $F(r) = r$ for all $r \in R$.

To prove that d is zero on R we first assume that $F = 0$ (although it is sufficient to assume $F|I = 0$). We get $0 = F(zr) = F(z)r + zd(r) = zd(r)$ for all $z \in I$ and $r \in R$. This implies $d(r) = 0$ for all $r \in R$. Assume, now, that F is the identity (although it is sufficient to assume that $F|I$ is the identity). We get $zr = F(zr) = F(z)r + zd(r) = zr + zd(r)$ for all $z \in I$ and $r \in R$. This implies $d(r) = 0$ for all $r \in R$.

(b) Assume that $F|I$ is an anti-homomorphism. As in [R] we get $[F(z), y]xd(z) = 0$ for all $x, y, z \in I$. Assume that $d(z) \neq 0$ for some $z \in I$. Then $F(z)y = yF(z)$ for all $y \in I$. This implies $F(z)r = rF(z)$ for all $r \in R$ (namely if $ax = xa$ for some $a \in R$ and all $x \in I$, then $(ar - ra)x = a(rx) - r(ax) = rxa - rxa = 0$ for any $r \in R$ and all $x \in I$). Now we have

$$\begin{aligned} F(xy)z + xyd(z) &= F(xyz) = F(z)F(y)F(x) = F(y)F(z)F(x) \\ &= F(y)F(xz) = F(y)(F(x)z + xd(z)) \\ &= F(xy)z + F(y)xd(z), \end{aligned}$$

for all $x, y \in I$, and $z \in I$ such that $d(z) \neq 0$, hence

$$(xy - F(y)x)d(z) = 0 \tag{1}$$

for all $x, y \in I$ and $z \in I$ such that $d(z) \neq 0$. Replacing x by tx, $t \in R$ in (1) we get $txyd(z) = F(y)txd(z)$, while multiplying (1) by t we get $txyd(z) = tF(y)xd(z)$, hence $(F(y)t - tF(y))xd(z) = 0$, for all $x, y \in I$, $z \in I$ such that $d(z) \neq 0$, and $t \in R$. Since R is prime we get $F(y)t = tF(y)$ for all $y \in I$ and $t \in R$. Therefore $F|I$ is a homomorphism. Using (a) we get $d = 0$. This is a contradiction, so that $d|I = 0$.

Now we have $F(x)zy = F(xz)y = F(z)F(x)y = F(z)F(xy) = F(xyz) = F(x)yz$, for all $x, y, z \in I$, i.e. $F(x)t(zy - yz) = 0$ for all $x, z, t, y \in I$.

Therefore $F(x) = 0$ for all $x \in R$ or $zy = yz$ for all $y, z \in I$. The second relation implies that R is commutative and that $F|I$ is a homomorphism. Using (a), we get $F(x) = x$ for all $x \in R$ or $F = 0$ on R. Finally, we get, as in (a) that $d = 0$ on R. ☐

Example 2. Assume that $R = \mathbf{Z}[x] \oplus \mathbf{Z}[x]$. Then R is not prime. Notice that $I := (0) \oplus \mathbf{Z}[x]$ and $J := \mathbf{Z}[x] \oplus (0)$ are ideals in R. Let us define a derivation d on R by $d|I := 0$ and $(d|J)((f(X), 0)) := (f'(X), 0)$. Then F defined by $F|J := d|J$ and $(F|I)(0, g(X)) := (0, g(X))$ is a generalized derivation on R associated to d, that acts as a homomorphism on I.

References

[R] N. Rehman, *On generalized derivations as homomorphisms and anti-homomorphisms*, Glasnik Mat. **39(59)** (2004), 27-30.

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