GLASNIK MATEMATIČKI
Vol. 40(60)(2005), 261 – 301

$\mathcal{S}$-DIAGONALIZABLE OPERATORS IN QUANTUM MECHANICS

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Abstract. In this paper we study a certain class of endomorphisms on the space of tempered distributions. More precisely, the core of the paper deals with endomorphisms, defined on the whole space of tempered distributions, for which there exists an $\mathcal{S}$-basis of the space (see section 5) formed by their eigenvectors. We call these operators $\mathcal{S}$-diagonalizable operators. One of the goals of the paper is the realization that this class of endomorphisms represents in the infinite-dimensional case what in finite-dimension is represented by the diagonalizable matrices.

We concentrate our examination on two aspects: the study of the spectrum of these operators and the foundation of a functional calculus for them. Concerning the first aspect, we do not assume nothing about the spectrum of these operators. The circumstance that the eigenvalue-spectrum of these operators will be continuous (more precisely it will be a connected subset of the complex plane, as it is proved in the present note) is a consequence of our definition. Moreover, the spectral measures will be not used in the construction of the functional calculus. In such a way, the definition of the function of an operator, presented in the paper, differs deeply from the usual one, in which the spectral measures of the operators play a fundamental role (as in the spectral decomposition of an operator). Note that, even in the case in which the eigenvalues-spectrum is a subset of the real line, we show that it is not necessarily coinciding with the whole real line.

1. Introduction

The first part of the paper is devoted to the operators acting on a continuous superposition of vectors as the linear operators act on a finite linear

2000 *Mathematics Subject Classification.* 46F10, 46F99, 47A05, 47N50, 70A05, 70B05, 81P05, 81Q99.

Key words and phrases. Linear operator, tempered distribution, basis, linear superposition, eigenvalue, diagonalizability.

combinations of vectors. The continuous superpositions was defined by the author to give a precise mathematical sense to the *principle of superposition* of quantum mechanics, even in the case in which a state-vector must be expanded as a superposition of a continuous family of state-vectors. Several observables of the Dirac formulation of quantum mechanics enjoy the property of linearity with respect to a continuous superposition, but only in a *formal* sense, because *neither the continuous superpositions nor the continuous-linearity* was defined rigorously by Dirac. This lack was sensed by several mathematicians, that decided to eliminate it with the help of Lebesgue-Stiljes integral or of the Bochner integral. Unfortunately, the obtained results do not replicate completely those of Dirac. We follow another way, *not using integrals, not using measure* but using the *operation of superposition*. Giving a rigorous mathematical model of the formal procedures of Dirac, we show several applications to the foundations of Quantum Mechanics.

2. Motivations and connections with the Rigged Hilbert spaces

The Rigged Hilbert space formulation of quantum mechanics, the formulation developed by Bohm and Gadella in [3], presents rigorous mathematics for Dirac formalism. But the theory presented did not justify all the features of Dirac's formulation of Quantum Mechanics. To understand where the Rigged Hilbert space formulation fails, we relate a brief history.

The Hilbert space formulation of Von Neumann does not cover the following aspects of Dirac calculus:

1) There are states of a physical system that cannot be normalized in the Hilbert sense;

2) here are states not normalizable in the Hilbert sense that can be normalized in the sense of Dirac;

3) There are some continuous families of vector states for which is reasonable to write:

$$\int_{\mathbb{R}} a_x v_x dx;$$

4) The wave-functions representing the vector-states are always smooth, i.e., of class C^∞;

5) It is possible to calculate a kind of scalar product among the non-normalizable states;

6) It is possible to decompose a vector in the following way:

$$u = \int_{\mathbb{R}} \langle u \mid v_x \rangle \, v_x dx;$$

7) In the space of vector-states of a quantum system, there are continuous families of vectors that are "bases" of the space, not in the Hamel sense but in a new sense of Dirac. In the Dirac sense, a continuous family v is a basis if:

every vector of the space is decomposable in the form $\int_{\mathbb{R}} a_x v_x dx$; moreover, the relation $\int_{\mathbb{R}} a_x v_x dx = \overrightarrow{0}$ implies $a_x = 0$, for every x;

1') The observables are defined in the whole space of vector-states;

2') The operation among the observables are always possible, in particular, the commutation relations are identities and not inclusions;

3') The observables could be treated as continuous operators;

4') Some observables admit a continuous system of eigenkets which is a basis in the sense of Dirac;

5') It is possible to decompose an operator as follows

$$A(u) = \int_{\mathbb{R}} a(x) \langle u \mid v_x \rangle v_x dx.$$

6') The observables are linear with respect to the superpositions of a continuous family of states:

$$A(u) = A(\int_{\mathbb{R}} \langle u \mid v_x \rangle v_x dx) = \int_{\mathbb{R}} \langle u \mid v_x \rangle A(v_x) dx.$$

1") Finally, it is possible, superposing certain continuous families of solutions of a linear differential equation, to obtain new states, which are yet solutions of the equation.

The Rigged Hilbert space formulation of quantum mechanics gives a justification to 1), 4), and, only in some particular cases, to 6), 1'), 2'), 3'), 4') and 5').

The $\mathcal{S}$-linear algebra in the space of tempered distributions of David Carfi gives a unitary justification to all the aspects, as we shall show in the paper. Actually, the space of tempered distributions is the third component of the Gelfand' triple $(\mathcal{S}_n, L^2_n, \mathcal{S}'_n)$, and thus, the $\mathcal{S}$-linear algebra is based on a Rigged Hilbert Space, but it is characterized by two fundamental *extra-tools*: *the operations of superposition* (see section 1) and the L^2-*product* introduced in [5]. In this way, we shall use the structure

$$(\mathcal{S}'_n, (\int_{\mathbb{R}^m})_{m\in\mathbb{N}}, (\cdot \mid \cdot)_{L^2}),$$

that is more rich than $(\mathcal{S}_n, L^2_n, \mathcal{S}'_n)$, and through the new tools we found rigorously the Dirac' calculus.

3. Preliminaries and notations on tempered distributions

In this paper we shall use the following notations:

1) n, m, k are natural numbers, $\mathbb{N}(\leq k) = \{i\}_{i=1}^k$;
2) μ_n is the Lebesgue measure on $\mathbb{R}^n$; $\mathbb{I}_{(\mathbb{R},\mathbb{C})}$ is the immersion of $\mathbb{R}$ in $\mathbb{C}$ and, if X is a non-empty set, $\mathbb{I}_X$ is the identity map on X;

3) if X and Y are two topological vector spaces on $\mathbb{K}$, $\mathrm{Hom}(X,Y)$ is the set of all the linear operators from X to Y, $\mathcal{L}(X,Y)$ is the set of all the linear and continuous operators from X to Y, $X' = \mathrm{Hom}\,(X,\mathbb{K})$ is the algebraic dual of X and $X^* = \mathcal{L}\,(X,\mathbb{K})$ is the topological dual of X;
4) $\mathcal{S}_n = \mathcal{S}_n(\mathbb{K}) := \mathcal{S}(\mathbb{R}^n,\mathbb{K})$ is the $(n,\mathbb{K})$-Schwartz space, that is to say the set of all the smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ in $\mathbb{K}$ rapidly decreasing at infinity (the functions and all its derivatives tend to 0 at $\mp\infty$ faster than the reciprocal of any polynomial):
$$\mathcal{S}(\mathbb{R}^n,\mathbb{K}) = \left\{ f \in C^\infty(\mathbb{R}^n,\mathbb{K}) : \forall\alpha,\beta \in \mathbb{N}_0^n \lim_{|x|\to\infty} \left|x^\beta D^\alpha f(x)\right| = 0 \right\}.$$
5) $\mathcal{S}_{(n)}$ is the *standard Schwartz topology* on $\mathcal{S}_n$. It is a topology induced by a metric. In fact, $\mathcal{S}_n$ is closed under differentiation and multiplication by polynomials, and defining, for each non-negative integer k, the seminorm p_k on $\mathcal{S}_n$ by
$$p_k\,(f) = \sup_{x\in\mathbb{R}^n} \max_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ 0\le|\alpha|,|\beta|\le k}} \left|x^\beta D^\alpha f(x)\right|,$$
the topology $\mathcal{S}_{(n)}$ is induced by the family $(p_k)_{k\in\mathbb{N}_0}$. Each p_k is a norm on $\mathcal{S}_n$, and $p_k\,(f) \le p_{k+1}\,(f)$ for all $f \in \mathcal{S}_n$. The pair $\left(\mathcal{S}_n,(p_k)_{k\in\mathbb{N}_0}\right)$ is a complete countably-normed space and so a Frèchet space (see also [8] and [1]);
6) $\mathcal{S}'_n := \mathcal{S}'(\mathbb{R}^n,\mathbb{K})$ is the space of tempered distributions from $\mathbb{R}^n$ to $\mathbb{K}$, that is, the topological dual of the topological vector space $(\mathcal{S}_n,\mathcal{S}_{(n)})$ i.e., $\mathcal{S}'_n{=}(\mathcal{S}_n,\mathcal{S}_{(n)})^*$;
7) if $x \in \mathbb{R}^n$, δ_x is the distribution of Dirac on $\mathcal{S}_n$ centered at x, i.e., the functional: $\delta_x : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi\,(x)\,;$
8) if $f \in \mathcal{O}_M(\mathbb{R}^n,\mathbb{K}) = \{g \in C^\infty(\mathbb{R}^n,\mathbb{K}) : \forall\phi \in \mathcal{S}_n(\mathbb{K}), \phi g \in \mathcal{S}_n(\mathbb{K})\}$, then the functional $[f] = [f]_n : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \int_{\mathbb{R}^n} f\phi d\mu_n$ is a tempered distribution, called the regular distribution generated by f (see [1] page 110);
9) Let $a,b \in \mathbb{R}^{\neq} = \mathbb{R}\backslash\{0\}$, $\underset{(a,b)}{\mathcal{S}}$ is the (a,b) -Fourier-Schwartz transformation, i.e., the operator $\underset{(a,b)}{\mathcal{S}} : \mathcal{S}_n{\to}\mathcal{S}_n$, such that, for all $f \in \mathcal{S}_n$ and $\xi \in \mathbb{R}^n$, one has
$$\underset{(a,b)}{\mathcal{S}}\,(f)(\xi) = \left(\frac{1}{a}\right)^n \int_{\mathbb{R}^n} f e^{-ib(\cdot|\xi)} d\mu_n = \left[\left(\frac{1}{a}\right)^n e^{-ib(\cdot|\xi)}\right](f),$$
where $(\cdot\mid\cdot)$ is the standard scalar product on $\mathbb{R}^n$. Moreover, we recall that $\underset{(a,b)}{\mathcal{S}}$ is a homeomorphism with respect to the standard topology

$\mathcal{S}_{(n)}$ and, concerning its inverse, for every $x \in \mathbb{R}^n$ and $g \in \mathcal{S}_n$, one has

$$\underset{(a,b)}{\mathcal{S}}^{-}(g)(x) = \left(\frac{|b|\,a}{2\pi}\right)^n \int_{\mathbb{R}^n} g e^{ib(x|\cdot)} d\mu_n = \underset{(2\pi/(|b|a),-b)}{\mathcal{S}}(g)(x);$$

10) Let $a, b \in \mathbb{R}^{\neq} = \mathbb{R}\backslash\{0\}$, $\underset{(a,b)}{\mathcal{F}}$ is the (a,b)-Fourier transformation on the space of tempered distributions, i.e., the operator $\underset{(a,b)}{\mathcal{F}} : \mathcal{S}'_n \to \mathcal{S}'_n$, such that, for all $u \in \mathcal{S}'_n$ and for every $\phi \in \mathcal{S}_n$, one has

$$\underset{(a,b)}{\mathcal{F}}(u)(\phi) = u\left(\underset{(a,b)}{\mathcal{S}}(\phi)\right),$$

i.e., the transpose of $\underset{(a,b)}{\mathcal{S}}$:

$$\underset{(a,b)}{\mathcal{F}} = {}^t\left(\underset{(a,b)}{\mathcal{S}}\right).$$

Moreover, we recall that $\underset{(a,b)}{\mathcal{F}}$ is a homeomorphism in the weak* topology $\sigma_n^* = \sigma\left(\mathcal{S}'_n, \mathcal{S}_n\right)$ (even more it is a topological isomorphism). Moreover, one has

$$\underset{(a,b)}{\mathcal{F}}^{-} = \underset{(2\pi/(|b|a),-b)}{\mathcal{F}}.$$

Two fundamental properties that we shall use are the following ones: for all $\alpha \in \mathbb{N}_0^n$,

$$\underset{(a,b)}{\mathcal{F}}\left(u^{(\alpha)}\right) = (bi)^\alpha \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha \underset{(a,b)}{\mathcal{F}}(u);$$

and

$$\underset{(a,b)}{\mathcal{F}}\left(\left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha u\right) = \left(\frac{i}{b}\right)^\alpha \left(\underset{(a,b)}{\mathcal{F}}(u)\right)^{(\alpha)},$$

where, $\mathbb{I}_{\mathbb{R}^n}$ is (as we said) the identity operator on $\mathbb{R}^n$, and $\left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha$ the α-th power of the identity in multi-indexed notation.

4. Some concepts from $\mathcal{S}$-linear algebra

Let I be a non-empty set. We denote by $s\left(I, \mathcal{S}'_n\right)$ the space of all the families in $\mathcal{S}'_n$ indexed by I, i.e., the set of all the surjective maps from I onto a subset of $\mathcal{S}'_n$. Moreover, if v is one of these families, for each $p \in I$, the distribution $v(p)$ is denoted by v_p, and v also by $\left(v_p\right)_{p\in I}$. The set $s\left(I, \mathcal{S}'_n\right)$ is a vector space with respect to the following standard operations: the addition

$$+ : s\left(I, \mathcal{S}'_n\right)^2 \to s\left(I, \mathcal{S}'_n\right) : (v, w) \mapsto v + w,$$

where $v + w = \left(v_p + w_p\right)_{p\in I}$, i.e., $(v+w)_p = v_p + w_p$; the multiplication by scalars

$$\cdot : \mathbb{K} \times s\left(I, \mathcal{S}'_n\right) \to s\left(I, \mathcal{S}'_n\right) : (\lambda, v) \mapsto \lambda v$$

where $\lambda v = (\lambda v_p)_{p\in I}$, i.e., $(\lambda v)(p) = (\lambda v)_p = \lambda v_p$. Moreover, we shall use the following definitions of David Carfi:

DEFINITION 4.1 (family of tempered distributions of class $\mathcal{S}$). *Let $v \in s(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. The family v is called family of class $\mathcal{S}$ or $\mathcal{S}$-family if, for each $\phi \in \mathcal{S}_n$, the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$, defined by $v(\phi)(p) = v_p(\phi)$, for each $p \in \mathbb{R}^m$, belongs to the space $\mathcal{S}_m$. The set of all these families is denoted by $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$.*

EXAMPLE 4.1 (a family of class $\mathcal{S}$). The Dirac family in $\mathcal{S}'(\mathbb{R}^n, \mathbb{K})$, i.e., the family $\delta = (\delta_y)_{y\in\mathbb{R}^n}$, is of class $\mathcal{S}$. In fact, for each $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ and for each p in $\mathbb{R}^n$, one has $\delta(\phi)(p) = \delta_p(\phi) = \phi(p)$, thus one has $\delta(\phi) = \phi$, so the image of ϕ under the family δ is ϕ itself which lies in $\mathcal{S}_n$.

EXAMPLE 4.2 (a family that is not of class $\mathcal{S}$). The constant family defined by $v_y = v(y) = \delta_0$, for each $y \in \mathbb{R}$, is not of class $\mathcal{S}$. In fact, let $\phi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ be such that $\phi(0) \neq 0$, for every $y \in \mathbb{R}$, one has $v(\phi)(y) = v_y(\phi) = \delta_0(\phi) = \phi(0)$, i.e., $v(\phi)$ is a non-null constant real functional on $\mathbb{R}$. So, ϕ doesn't decrease and then doesn't lie in $\mathcal{S}(\mathbb{R}, \mathbb{K})$.

DEFINITION 4.2 (operator generated by an $\mathcal{S}$-family). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of class $\mathcal{S}$. The operator generated by the family v (or associated with v is the operator*

$$\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi).$$

EXAMPLE 4.3 (on the Dirac family). The operator on $\mathcal{S}_n$ generated by the Dirac family, i.e., by the family $\delta = (\delta_y)_{y\in\mathbb{R}^n}$, is the identity operator on $\mathcal{S}_n$. In fact, for each $y \in \mathbb{R}^n$, one has $\widehat{\delta}(\phi)(y) = \delta_y(\phi) = \phi(y) = \mathbb{I}_{\mathcal{S}_n}(\phi)(y)$.

The set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is a subspace of the vector space $(s(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ and for each $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ the operator $\widehat{v}$ is linear and the map

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$$

is an injective linear operator.

THEOREM 4.1 (basic lemma for the superpositions of an $\mathcal{S}$-*family*). *Let $a \in \mathcal{S}'_m$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-family. Then, the composition $u = a \circ \widehat{v}$, i.e., the functional $u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\widehat{v}(\phi))$, is a tempered distribution.*

PROOF. Let $a \in \mathcal{S}'_m$. Because the subspace $\mathrm{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ is sequentially dense in $\mathcal{S}'_m$ (see [2] page 205), there is a sequence of distributions $(\alpha_k)_{k\in\mathbb{N}}$ in $\mathrm{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ such that

$$^{\sigma^*_m}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since $\alpha_k \in \operatorname{span}(\{\delta_y\}_{y\in\mathbb{R}^m})$ there exist a finite set $\{y_i\}_{i=1}^h$ in $\mathbb{R}^m$ and a finite set $\{\lambda_i\}_{i=1}^h$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^{h} \lambda_i \delta_{y_i},$$

and consequently

$$\alpha_k \circ \widehat{v} = \sum_{i=1}^{h} \lambda_i v_{y_i}.$$

Hence, for every $k \in \mathbb{N}$, the composition $\alpha_k \circ \widehat{v}$ belongs to $\mathcal{S}'_n$.

Let τ be the topology of the pointwise convergence in $\operatorname{Hom}(\mathcal{S}_n, \mathbb{K})$, one has

$${}^{\tau}\lim_{k\to+\infty} \alpha_k \circ \widehat{v} = a \circ \widehat{v}.$$

In fact, for every ϕ in $\mathcal{S}_n$, we obtain

$$\lim_{k\to+\infty} (\alpha_k \circ \widehat{v})(\phi) = \lim_{k\to+\infty} \alpha_k (\widehat{v}(\phi)) = a(\widehat{v}(\phi)),$$

so we have that

$$(\alpha_k \circ \widehat{v})_{k\in\mathbb{N}} \xrightarrow{\tau} a \circ \widehat{v}.$$

At this point, being $\{\alpha_k \circ \widehat{v}\}_{k\in\mathbb{N}} \subset \mathcal{S}'_n$, then, by the completeness theorem of $\mathcal{S}'_n$ (see [7] page 602), one has $a \circ \widehat{v} \in \mathcal{S}'_n$. □

Definition 4.3 (linear superpositions of an *$\mathcal{S}$-family*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $a \in \mathcal{S}'_m$. The distribution*

$$a \circ \widehat{v} = {}^t(\widehat{v})(a)$$

is called the $\mathcal{S}$-linear superposition of v with respect to (the system of coefficients) a and we denote it by

$$\int_{\mathbb{R}^m} av.$$

Moreover, if $u \in \mathcal{S}'_n$ and there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$, u is said an $\mathcal{S}$-linear superposition of v.

As a particular case, we can consider the linear superposition of v with respect to the distribution $1_{\mathcal{S}'_m} := \left[1_{(\mathbb{R}^m,\mathbb{K})}\right]$. We denote it simply by $\int_{\mathbb{R}^m} v$, and then

$$\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} 1_{\mathcal{S}'_m} v,$$

recall that $\left[1_{(\mathbb{R}^m,\mathbb{K})}\right]$ is the distribution generated by the $\mathbb{K}$ -constant functional on $\mathbb{R}^m$ of value 1.

Example 4.4 (the Dirac family). Let δ be the Dirac family. Then, for each tempered distribution $u \in \mathcal{S}'_n$, one has $\int_{\mathbb{R}^n} u\delta = u \circ \widehat{\delta} = u \circ \mathbb{I}_{\mathcal{S}_n} = u$, thus each tempered distribution is an $\mathcal{S}$ -superposition of the Dirac family.

We propose now a property of the superpositions analogous to the following fundamental property of a complex vector space: $\sum \delta_{(i,\cdot)} v = \sum_{j=1}^{k} \delta_{ij} v_j = v_i$, where $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is the Kronecker's delta and v is a finite family of vectors.

THEOREM 4.2 (selection property of the delta distributions). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, for each $p \in \mathbb{R}^m$, one has $\int_{\mathbb{R}^m} \delta_p v = v_p$.*

PROOF. For every $\phi \in \mathcal{S}_n$, one has $\delta_p(\widehat{v}(\phi)) = \delta_p(v(\phi)) = v(\phi)(p) = v_p(\phi)$, and consequently $\int_{\mathbb{R}^m} \delta_p v = \delta_p \circ \widehat{v} = v_p$. ∎

In the sense of the above theorem, the Dirac family is a continuous version of the Kronecker delta.

DEFINITION 4.4 (the (a,b)-*Fourier family*). *Let $a, b \in \mathbb{R}^{\neq}$. The (a,b)-Fourier family in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ is the following family of regular tempered distributions*

$$\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p \in \mathbb{R}^n}.$$

PROPOSITION 4.1 (on the operator associated with the Fourier families). *Let $a, b \in \mathbb{R}^{\neq}$ and φ be the (a,b)-Fourier family. Then φ is of class $\mathcal{S}$, and more precisely, one has $v(\phi) = \underset{(a,b)}{\mathcal{S}}(\phi)$, for each test function $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$, and thus φ generates the (a,b)-Fourier Schwartz transformation, i.e., $\widehat{v} = \underset{(a,b)}{\mathcal{S}}$.*

PROOF. For each test function $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ and for each p in $\mathbb{R}^n$, one has that

$$\begin{aligned} \varphi(\phi)(p) &= \varphi_p(\phi) = \left[(1/a)^n e^{-ib(p|\cdot)}\right](\phi) = \int_{\mathbb{R}^n} (1/a)^n e^{-ib(p|\cdot)} \phi \, d\mu_n \\ &= \underset{(a,b)}{\mathcal{S}}(\phi)(p), \end{aligned}$$

and thus $\varphi(\phi) = \underset{(a,b)}{\mathcal{S}}(\phi)$. Now, because the (a,b) -Fourier-Schwartz transform is into $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$, the map $\varphi(\phi)$ lies in $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$. ∎

EXAMPLE 4.5. We have, for all $\alpha \in \mathbb{N}_0^n$,

$$\underset{(a,b)}{\mathcal{F}}(u^{(\alpha)}) = (bi)^\alpha (\mathbb{I}_{\mathbb{R}^n})^\alpha \underset{(a,b)}{\mathcal{F}}(u);$$

and

$$\underset{(a,b)}{\mathcal{F}}((\mathbb{I}_{\mathbb{R}^n})^\alpha u) = \left(\frac{i}{b}\right)^\alpha \left(\underset{(a,b)}{\mathcal{F}}(u)\right)^{(\alpha)},$$

where, $\mathbb{I}_{\mathbb{R}^n}$ is the identity operator on $\mathbb{R}^n$, and $(\mathbb{I}_{\mathbb{R}^n})^\alpha$ the α-th power of the identity in multi-indexed notation. These two properties can be immediately translate in terms of superposition. Let φ be the (a,b)-Fourier family. We have, for all $\alpha \in \mathbb{N}_0^n$,

$$\int_{\mathbb{R}^n} u^{(\alpha)}\varphi = (bi)^\alpha \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha \int_{\mathbb{R}^n} u\varphi;$$

and

$$\int_{\mathbb{R}^n} \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha u\varphi = \left(\frac{i}{b}\right)^\alpha \left(\int_{\mathbb{R}^n} u\varphi\right)^{(\alpha)}.$$

The next theorem affirms that each tempered distribution is an $\mathcal{S}$ -superposition of the (a,b)-Fourier family, and this without technical assumptions.

THEOREM 4.3 (Fourier expansion theorem). *Let $u \in \mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ be a tempered distribution and φ be the family $\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}$. Then, one has*

$$u = \int_{\mathbb{R}^n} \underset{(a,b)}{\mathcal{F}}{}^{-}(u)\varphi,$$

in other words, u is the superposition of φ under $\underset{(a,b)}{\mathcal{F}}{}^{-}(u)$.

PROOF. For every function $\phi \in \mathcal{S}(\mathbb{R}^n,\mathbb{C})$, we have

$$\begin{aligned} u(\phi) &= u(\underset{(a,b)}{\mathcal{S}}{}^{-}(\underset{(a,b)}{\mathcal{S}}(\phi)) = \underset{(a,b)}{\mathcal{F}}{}^{-}(u)(\widehat{\varphi}(\phi)) \\ &= \left(\int_{\mathbb{R}^n} \underset{(a,b)}{\mathcal{F}}{}^{-}(u)\varphi\right)(\phi). \end{aligned}$$

∎

5. $\mathcal{S}$-BASES AND APPLICATIONS

DEFINITION 5.1 (of *$\mathcal{S}$-linear independence*). *Let $v \in \mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$. v is said $\mathcal{S}$-linearly independent, if $a \in \mathcal{S}'_m$ and $\int_{\mathbb{R}^m} av = 0_{\mathcal{S}'_n}$ implies $a = 0_{\mathcal{S}'_m}$.*

EXAMPLE 5.1. The Dirac family in $\mathcal{S}'_n$ is $\mathcal{S}$-linearly independent. In fact, one has $\int_{\mathbb{R}^n} u\delta = u$, for all $u \in \mathcal{S}'_n$, and then $\int_{\mathbb{R}^n} u\delta = 0_{\mathcal{S}'_n}$ implies $u = 0_{\mathcal{S}'_n}$.

EXAMPLE 5.2 (the Fourier families). The Fourier families are $\mathcal{S}$-linearly independent. In fact, let φ be the (a,b)-Fourier family, and let $\int_{\mathbb{R}^n} u\varphi = 0_{\mathcal{S}'_n(\mathbb{C})}$. For every $\phi \in \mathcal{S}_n(\mathbb{C})$, one has

$$0 = \left(\int_{\mathbb{R}^n} u\varphi\right)(\phi) = u\left(\widehat{\varphi}\left(\phi\right)\right) = u\left(\underset{(a,b)}{\mathcal{S}}(\phi)\right) = \underset{(a,b)}{\mathcal{F}}(u)(\phi),$$

i.e., $\underset{(a,b)}{\mathcal{F}}(u) = 0_{\mathcal{S}'_n(\mathbb{C})}$, and thus $u = 0_{\mathcal{S}'_n(\mathbb{C})}$, being $\underset{(a,b)}{\mathcal{F}}$ injective.

Definition 5.2 (of $\mathcal{S}$-*linear hull*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The $\mathcal{S}$-linear hull of v is the set*

$$\mathcal{S}\,\mathrm{span}\,(v) = \left\{ u \in \mathcal{S}'_n : \exists a \in \mathcal{S}'_m : u = \int_{\mathbb{R}^m} av \right\}.$$

Example 5.3 (on the Dirac and Fourier families). Let δ be the Dirac family, one has $\mathcal{S}\,\mathrm{span}\,(\delta) = \mathcal{S}'_n$. In fact, for all $u \in \mathcal{S}'_n$, one has $u = u \circ \mathbb{I}_{\mathcal{S}_n} = u \circ \widehat{\delta} = \int_{\mathbb{R}^n} u\delta$. Let φ be the family $\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}$, one has $\mathcal{S}\,\mathrm{span}\,(\varphi) = \mathcal{S}'_n$, it's an immediate consequence of the Fourier expansion theorem.

Definition 5.3 (system of $\mathcal{S}$-*generators*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. v is called system of $\mathcal{S}$-generators for $V \subseteq \mathcal{S}'_n$ if and only if*

$$\mathcal{S}\,\mathrm{span}(v) = V.$$

Example 5.4. *The Dirac family and the Fourier families are systems of $\mathcal{S}$-generators for $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$.*

Definition 5.4 (of $\mathcal{S}$-*basis*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $V \subseteq \mathcal{S}'_n$. v is an $\mathcal{S}$-basis of V if it is $\mathcal{S}$-linearly independent, and $\mathcal{S}\,\mathrm{span}(v) = V$.*

Now we can give a more complete version of the Fourier theorem.

Theorem 5.1 (Fourier expansion theorem in geometric form). *The Fourier families in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ are $\mathcal{S}$-bases of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$.*

The Dirac family δ in $\mathcal{S}'_n$ is an $\mathcal{S}$-basis of $\mathcal{S}'_n$. We call δ the canonical $\mathcal{S}$-basis of $\mathcal{S}'_n$ or the Dirac basis of $\mathcal{S}'_n$, and we call the Fourier families the Fourier bases of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$.

Application 5.1. The decomposition $\int_{\mathbb{R}^n} u\delta = u$, justifies completely the following formal expression of the physicists (see [6, page 78])

$$\int_{\mathbb{R}^n} \delta(x-p)\delta(y-x)dx = \delta(y-p).$$

In fact, one has $\int_{\mathbb{R}^n} \delta_p\delta = \delta_p$, so the correct interpretation of the above formal equality, when it derives from the superposition principle of quantum mechanics, is not properly the convolution of two distributions but it's the following one: the vector state δ_p is the linear superposition of the infinite continuous family of vector states $(\delta_y)_{y\in\mathbb{R}^n}$ with respect to the system of coefficients δ_p. Hence, for instance, we can *rigorously* affirm that: the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of "eigenstates" of the position operator $Q : \mathcal{S}'(\mathbb{R}, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}, \mathbb{C}) : u \mapsto \mathbb{I}_{\mathbb{R}}u$.

Moreover, the Fourier expansion theorem justifies completely another formal expression used by physicists (see [6, page 38 formula (10)])

$$\delta(x-p) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{piy} e^{-iyx} dy. \qquad (1)$$

In fact, a classic result gives

$$\underset{(a,1)}{\mathcal{F}}^{-}(\delta_p) = \frac{a}{2\pi}\left[e^{pi(\cdot)}\right],$$

and thus, from the Fourier expansion theorem, set $a = 1$, one has

$$\delta_p = \int_{\mathbb{R}} \frac{1}{2\pi}\left[e^{pi(\cdot)}\right]\left(\left[e^{-i(x|\cdot)}\right]\right)_{x\in\mathbb{R}}.$$

So the correct interpretation of the formal expression (1), when it derives from the superposition principle of quantum mechanics, is the following one: the vector state δ_p is the linear superposition of the infinite continuous family of vector states $\left(\left[e^{-i(p|\cdot)}\right]\right)_{p\in\mathbb{R}}$ with respect to the system of coefficients $(1/2\pi)\left[e^{pi(\cdot)}\right]$. Once more, we can affirm *rigorously* that the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of "eigenstates" of the momentum operator

$$P : \mathcal{S}'(\mathbb{R},\mathbb{C}) \to \mathcal{S}'(\mathbb{R},\mathbb{C}) : u \mapsto -i\hbar u'$$

(see [6]). Another interpretation of the expansion theorem is the following one: at every time $t \in \mathbb{R}$ a wave $u : \mathbb{R} \to \mathcal{S}'(\mathbb{R}^n,\mathbb{C})$ is an $\mathcal{S}$ -superposition of the family of the harmonic waves

$$\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}$$

with respect to the system of coefficients $\underset{(a,b)}{\mathcal{F}}^{-}(u_t)$.

THEOREM 5.2 (on the structure of $\mathcal{S}\,\mathrm{span}$). *Let $u \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$. Then, $\mathcal{S}\,\mathrm{span}\,(u)$ is a subspace of $\mathcal{S}'_n$, it contains all the elements of u and consequently*

$$\mathrm{span}\,(u) \subseteq \mathcal{S}\,\mathrm{span}\,(u).$$

PROOF. Let $\lambda \in \mathbb{K}$ and $v, w \in \mathcal{S}\,\mathrm{span}\,(u)$, then, there exist $a, b \in \mathcal{S}'_m$ such that $v = \int_{\mathbb{R}^m} au$ and $w = \int_{\mathbb{R}^m} bu$. Now, one has $\lambda v + w = \lambda \int_{\mathbb{R}^m} au + \int_{\mathbb{R}^m} bu = \int_{\mathbb{R}^m} (\lambda a + b)\,u$, and then $\lambda v + w \in \mathcal{S}\,\mathrm{span}\,(u)$. Moreover, let δ be the Dirac basis of $\mathcal{S}'_m$, we have $\int_{\mathbb{R}^m} \delta_p u = u_p$ and then $u_p \in \mathcal{S}\,\mathrm{span}\,(u)$. □

THEOREM 5.3 (about the finite linear combinations). *Let $k \in \mathbb{N}$, $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ and $u = \sum_{i=1}^{k} \lambda_i v_{\alpha_i}$, with $\lambda \in \mathbb{C}^k$, and $\alpha \in (\mathbb{R}^m)^k$. Then, there exists a $\Lambda \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} \Lambda v$. So, each linear combination of a finite subfamily of v is an $\mathcal{S}$-superposition of v.*

PROOF. Let $\Lambda = \sum_{i=1}^{k} \lambda_i \delta_{\alpha_i}$, then, one has

$$\begin{aligned} \int_{\mathbb{R}^m} \Lambda v &= \int_{\mathbb{R}^m} \left(\sum_{i=1}^{k} \lambda_i \delta_{\alpha_i} \right) v = \sum_{i=1}^{k} \int_{\mathbb{R}^m} \lambda_i \delta_{\alpha_i} v \\ &= \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \sum_{i=1}^{k} \lambda_i v_{\alpha_i} = u. \end{aligned}$$

□

THEOREM 5.4. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family $\mathcal{S}$-linearly independent. Then, v is linearly independent. Consequently $\mathcal{S}\,\mathrm{span}\,(v)$ is an infinite-dimensional subspace of $\mathcal{S}'_n$.*

PROOF. Let $k \in \mathbb{N}$, $\alpha \in (\mathbb{R}^m)^k$, and let $v_\alpha = (v_{\alpha_i})_{i=1}^{k}$. By contradiction, let v_α be a linearly dependent system of $\mathcal{S}'_n$, then there exists a $\lambda \in \left(\mathbb{C}^k\right)^{\neq} = \mathbb{C}^k \backslash \{0_k\}$ such that $\sum_{i=1}^{k} \lambda_i v_{\alpha_i} = 0_{\mathcal{S}'_n}$. Thus, put $\Lambda = \sum_{i=1}^{k} \lambda_i \delta_{\alpha_i}$, one has

$$\int_{\mathbb{R}^m} \Lambda v = \int_{\mathbb{R}^m} \sum_{i=1}^{k} \lambda_i \delta_{\alpha_i} v = \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \sum_{i=1}^{k} \lambda_i v_{\alpha_i} = 0_{\mathcal{S}'_n}.$$

Now, because it is $\Lambda \neq 0_{\mathcal{S}'_m}$, one has that v is a family $\mathcal{S}$-linearly dependent, against the assumption. □

The following is a meaningful generalization of the Fourier expansion theorem.

THEOREM 5.5 (characterization of an $\mathcal{S}$-*basis*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, v is an $\mathcal{S}$-basis of $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is bijective.*

PROOF. First of all ${}^t(\widehat{v})$ is well defined because v is an $\mathcal{S}$-family. Moreover, it is obvious that v $\mathcal{S}$-generates $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is surjective, and that v is $\mathcal{S}$-linearly independent if and only if ${}^t(\widehat{v})$ is injective. □

Actually, it is possible to prove the following

THEOREM 5.6 (characterization of an $\mathcal{S}$-*basis*). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, v is an $\mathcal{S}$-basis of $\mathcal{S}'_n$ if and only if ${}^t(\widehat{v})$ is a topological isomorphism.*

6. SYSTEMS OF COORDINATES IN AN $\mathcal{S}$-LINEARLY INDEPENDENT FAMILY

It is simple to prove that, if v is an $\mathcal{S}$-linearly independent family and if $u \in \mathcal{S}\,\mathrm{span}\,(v)$, then there exists a unique $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$. So, we can give the following

Definition 6.1 (system of coordinates). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family and $u \in \mathcal{S}\,\mathrm{span}(v)$. The only tempered distribution $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} av$ is denoted by $[u|v]$ and is called the system of coordinates of u in v.*

Definition 6.2 (of coordinate operator in an *$\mathcal{S}$-linearly independent family*). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. The coordinate operator in w is the following operator*

$$[\cdot \mid w] : \mathcal{S}\,\mathrm{span}(w) \to \mathcal{S}'_m : u \mapsto [u \mid w]\,.$$

Example 6.1 (on the Dirac family). Let δ be the Dirac family in $\mathcal{S}'_n$. For all $u \in \mathcal{S}'_n$, we have $[u \mid \delta] = u$, and hence $[\cdot \mid \delta] = (\cdot)_{\mathcal{S}'_n}$.

Example 6.2 (on the (a,b)-*Fourier family*). Let f be the (a,b)-Fourier family in $\mathcal{S}'_n$. For each $u \in \mathcal{S}'_n$ we have $[u \mid f] = \underset{(h,\omega)}{\mathcal{F}^-}(u)$, and hence $[\cdot \mid f] = \underset{(h,\omega)}{\mathcal{F}^-}$.

Theorem 6.1. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. Then,*

$$[\cdot \mid w] \in \mathrm{Hom}(\mathcal{S}\,\mathrm{span}(w), \mathcal{S}'_m).$$

Proof. Let $\lambda \in \mathbb{C}$ and $u, v \in \mathcal{S}\,\mathrm{span}(w)$, then we have

$$u + \lambda v = \int_{\mathbb{R}^m} [u \mid w]\, w + \lambda \int_{\mathbb{R}^m} [v \mid w]\, w = \int_{\mathbb{R}^m} ([u \mid w] + \lambda\, [v \mid w])\, w,$$

and thus, we deduce

$$[u + \lambda v \mid w] = [u \mid w] + \lambda\, [v \mid w]\,.$$

□

Theorem 6.2. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $\lambda \in \mathbb{C}^{\neq}$. Then, the following assertions hold true*

1) *if w is $\mathcal{S}$-linearly independent then the family λw is $\mathcal{S}$-linearly independent;*
2) $\mathcal{S}\,\mathrm{span}(w) = \mathcal{S}\,\mathrm{span}(\lambda w)$*;*
3) *if w is $\mathcal{S}$-linearly independent, for each $u \in \mathcal{S}\,\mathrm{span}(w)$, we have*

$$[u \mid \lambda w] = (1/\lambda)\, [u \mid w] = [(1/\lambda)\, u \mid w].$$

Proof. 1) Let $a \in \mathcal{S}'_m$ be such that $\int_{\mathbb{R}^m} a(\lambda w) = 0_{\mathcal{S}'_n}$, one has

$$0_{\mathcal{S}'_n} = \int_{\mathbb{R}^m} a(\lambda w) = \lambda \int_{\mathbb{R}^m} aw,$$

thus $\int_{\mathbb{R}^m} aw = 0_{\mathcal{S}'_n}/\lambda = 0_{\mathcal{S}'_n}$, now because w is $\mathcal{S}$-linearly independent we desume $a = 0_{\mathcal{S}'_n}$.

2) Let $u \in \mathcal{S}\,\mathrm{span}(w)$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} aw$. Then, we have

$$u = \int_{\mathbb{R}^m} \left(\frac{a}{\lambda}\right)(\lambda w)\,,$$

so $u \in \mathcal{S}\,\mathrm{span}(\lambda w)$, and hence $\mathcal{S}\,\mathrm{span}(w) \subseteq \mathcal{S}\,\mathrm{span}(\lambda w)$. Viceversa, let $u \in \mathcal{S}\,\mathrm{span}(\lambda w)$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} a(\lambda w)$. Now, one has

$$u = \int_{\mathbb{R}^m} (\lambda a)w,$$

and hence $u \in \mathcal{S}\,\mathrm{span}(w)$, hence $\mathcal{S}\,\mathrm{span}(\lambda w) \subseteq \mathcal{S}\,\mathrm{span}(w)$. Concluding $\mathcal{S}\,\mathrm{span}(w) = \mathcal{S}\,\mathrm{span}(\lambda w)$.

3) For any $u \in \mathcal{S}'_n$, one has $u = \int_{\mathbb{R}^m} [u \mid w]\, w$, hence

$$u = \int_{\mathbb{R}^m} \left(\frac{1}{\lambda}[u \mid w]\right)(\lambda w) = \int_{\mathbb{R}^m} [(1/\lambda)\, u \mid w]\, (\lambda w)\,.$$

$\square$

DEFINITION 6.3 (product of two $\mathcal{S}$-*families*). *Let $k \in \mathbb{N}$, $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_m\right)$ and $w \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ be two families of distributions. The product of v by w is the family (in $\mathcal{S}'_n$) defined by $v \cdot w = \left(\int_{\mathbb{R}^m} v_p w\right)_{p \in \mathbb{R}^k}$. It is also denoted by $\int_{\mathbb{R}^m} vw$. Hence, for each $p \in \mathbb{R}^k$, we have $(v \cdot w)_p = \left(\int_{\mathbb{R}^m} vw\right)_p = \int_{\mathbb{R}^m} v_p w$.*

It can be proved that the product of two $\mathcal{S}$-families is an $\mathcal{S}$-family and that

$$(a \cdot b)^{\wedge} = \widehat{a} \circ \widehat{b}.$$

DEFINITION 6.4 (of invertible $\mathcal{S}$-*family*). *Let $a \in \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right)$. a is called invertible if there exists a $b \in \mathcal{S}\left(\mathbb{R}^n, \mathcal{S}'_n\right)$ such that $a \cdot b = b \cdot a = \delta$.*

REMARK 6.1. It's easy to prove that, for each invertible family $a \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$, there exists only a $b \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ such that $a \cdot b = b \cdot a = \delta$. This family is denoted by a^-. Moreover, it can be proved that the operator generated by a is invertible and $(a^-)^{\wedge} = (\widehat{a})^-$. In fact, for each $\phi \in \mathcal{S}_n$, one has

$$\begin{aligned} \left[(a^-)^{\wedge} \circ \widehat{a}\right](\phi)(p) &= (a^-)^{\wedge}(\widehat{a}(\phi))(p) = (a^-)_p(\widehat{a}(\phi)) \\ &= \left(\int_{\mathbb{R}^n} (a^-)_p\, a\right)(\phi) = (a^- \cdot a)_p(\phi) = \delta_p(\phi) = \phi(p)\,; \end{aligned}$$

so $(a^-)^{\wedge} \circ \widehat{a} = (\cdot)_{\mathcal{S}_n}$. Analogously we have $\widehat{a} \circ (a^-)^{\wedge} = (\cdot)_{\mathcal{S}_n}$, and hence $\widehat{a}$ is invertible and $(a^-)^{\wedge} = (\widehat{a})^-$.

THEOREM 6.3. *Let $v, w \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}_n')$ be two $\mathcal{S}$-linearly independent family. Then, $v \cdot w$ is $\mathcal{S}$-linearly independent. Moreover, if w is invertible one has*

$$[u \mid w \cdot v] = \int_{\mathbb{R}^n} [u \mid v]\, w^-.$$

PROOF. Let $a \in \mathcal{S}_n'$ be such that $\int_{\mathbb{R}^n} a(v \cdot w) = 0_{\mathcal{S}_n'}$, one has

$$0_{\mathcal{S}_n'} = \int_{\mathbb{R}^n} a(v \cdot w) = a \circ (v \cdot w)^\wedge = a \circ (\widehat{v} \circ \widehat{w}) = (a \circ \widehat{v}) \circ \widehat{w} = \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} av)w.$$

Since w is $\mathcal{S}$-linearly independent, we have $\int_{\mathbb{R}^n} av = 0_{\mathcal{S}_n'}$. And since v is $\mathcal{S}$ -linearly independent, one has $a = 0_{\mathcal{S}_n'}$. So $v \cdot w$ is $\mathcal{S}$-linearly independent.

If w is invertible then $\widehat{w}$ is invertible and one has

$$\begin{aligned} u &= \int_{\mathbb{R}^n} [u \mid v]\, v = [u \mid v] \circ \widehat{v} = [u \mid v] \circ \widehat{w}^- \circ \widehat{w} \circ \widehat{v} = \\ &= \left([u \mid v] \circ \widehat{w}^-\right) \circ (\widehat{w} \circ \widehat{v}) = ([u \mid v] \circ \widehat{w}^-) \circ (w \cdot v)^\wedge = \\ &= \int_{\mathbb{R}^n} \left([u \mid v] \circ \widehat{w}^-\right)(w \cdot v), \end{aligned}$$

and hence

$$[u \mid w \cdot v] = [u \mid v] \circ \widehat{w}^- = [u \mid v] \circ \left(w^-\right)^\wedge = \int_{\mathbb{R}^n} [u \mid v]\, w^-.$$

□

DEFINITION 6.5 (superposition of a family with respect to a family). *Let $v \in s(\mathbb{R}^k, \mathcal{S}_m')$ and $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}_n')$. The family*

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w\right)_{p \in \mathbb{R}^k},$$

is called the superposition of w with respect to v.

EXAMPLE 6.3. Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}_n')$ be an $\mathcal{S}$-linearly independent family and let $v \in s(\mathbb{R}^k, \mathcal{S}_n')$ be such that $v_p \in \mathcal{S}\,\mathrm{span}\,(w)$, for each $p \in \mathbb{R}^k$. Then, for each $p \in \mathbb{R}^k$, we have $v_p = \int_{\mathbb{R}^m} [v_p \mid w]\, w$, i.e., $v = \int_{\mathbb{R}^m} [v \mid w]\, w$, where $[v \mid w]$ is the family in $\mathcal{S}_m'$ defined by

$$[v \mid w] := ([v_p \mid w])_{p \in \mathbb{R}^k}.$$

In fact

$$\left(\int_{\mathbb{R}^m} [v \mid w]\, w\right)(p) = \int_{\mathbb{R}^m} [v \mid w]_p\, w = \int_{\mathbb{R}^m} [v_p \mid w]\, w = v_p.$$

REMARK 6.2. Obviously, in the conditions of the above definition, if $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}_m')$, we have

$$\int_{\mathbb{R}^m} vw = v \cdot w,$$

and thus $\int_{\mathbb{R}^m} vw \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}_n')$.

THEOREM 6.4 ($\mathcal{S}$-*linearity of the* $\mathcal{S}$-*linear combinations*). *Let* $a \in \mathcal{S}'_k$, $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_m\right)$ *and* $w \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ *be two families of distributions. Then, one has*

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av \right) w = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw \right) = \int_{\mathbb{R}^k} a(v \cdot w).$$

PROOF. For every $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av \right) w \right)(\phi) &= \left(\int_{\mathbb{R}^m} av \right)(\widehat{w}(\phi)) = a(\widehat{v}(\widehat{w}(\phi))) \\ &= a((\widehat{v} \circ \widehat{w})(\phi)) = a\left((v \cdot w)^{\wedge}(\phi)\right) = \\ &= \int_{\mathbb{R}^k} a(v \cdot w)(\phi) = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw \right)(\phi); \end{aligned}$$

note that

$$\begin{aligned} (v \cdot w)^{\wedge}(\phi)(p) &= (v \cdot w)_p(\phi) = \left(\int_{\mathbb{R}^m} v_p w \right)(\phi) = v_p(\widehat{w}(\phi)) = \\ &= \widehat{v}(\widehat{w}(\phi))(p) = (\widehat{v} \circ \widehat{w})(\phi)(p). \end{aligned}$$

□

NOTATION (the set of the $\mathcal{S}$-*bases of a subspace*). Let $X \subseteq \mathcal{S}'_n$ be a subspace. In the following we shall use the notation

$$\mathcal{SB}(\mathbb{R}^m, X) = \{v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : \mathrm{Im}(v) \subseteq X \text{ and } v \text{ is an } \mathcal{S}\text{-basis for } X\}.$$

DEFINITION 6.6 (the family of change for two $\mathcal{S}$ -*bases*). *Let* $v \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ *and* $w \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$. *We say family of change from* v *to* w *the following family*

$$[w \mid v] := ([w_p \mid v])_{p \in \mathbb{R}^n}.$$

THEOREM 6.5 (on the change of basis). *Let* $v, w \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ *be such that*

$$[v \mid w] \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n),$$

and $u \in \mathcal{S}'_n$. *Then,*

$$[u \mid w] = \int_{\mathbb{R}^n} [u \mid v]\,[v \mid w].$$

PROOF. From $v = \int_{\mathbb{R}^n} [v \mid w]\, w$, applying the $\mathcal{S}$-linearity of the $\mathcal{S}$-linear combinations, we have

$$\begin{aligned} u &= \int_{\mathbb{R}^n} [u \mid v]\, v = \int_{\mathbb{R}^n} [u \mid v] \left(\int_{\mathbb{R}^n} [v \mid w]\, w \right) = \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} [u \mid v]\,[v \mid w] \right) w, \end{aligned}$$

and thus by definition of system of coordinates in an $\mathcal{S}$-basis

$$[u \mid w] = \int_{\mathbb{R}^n} [u \mid v] \, [v \mid w] \, .$$

$\square$

Definition 6.7 (superposition of a family with respect to an operator). *Let $X \subseteq \mathcal{S}'_n$ be a subspace of $\mathcal{S}'_n$, $A \in \operatorname{Hom}(X, \mathcal{S}'_m)$ and $v \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}'_n\right)$ be a family of distributions. We define superposition of v with respect to A, the operator*

$$\int_{\mathbb{R}^m} Av : X \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

Theorem 6.6 (the resolution of the identity). *Let $v \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family. Then, $v \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ if and only if one has*

$$(\cdot)_{\mathcal{S}'_n} = \int_{\mathbb{R}^n} [\cdot \mid v]v.$$

Proof. ($\Rightarrow$) If $v \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$, then $\mathcal{S}\operatorname{span}(v) = \mathcal{S}'_n$. Thus, for all $u \in \mathcal{S}'_n$ we have $u = \int_{\mathbb{R}^n} [u \mid v]v$, i.e., by definition, $(\cdot)_{\mathcal{S}'_n} = \int_{\mathbb{R}^n} [\cdot \mid v]v$.

($\Leftarrow$) If $(\cdot)_{\mathcal{S}'_n} = \int_{\mathbb{R}^n} [\cdot \mid v]v$, then, for all $u \in \mathcal{S}'_n$ one has $u = \int_{\mathbb{R}^n} [u \mid v]v$, and hence $\mathcal{S}'_n = \mathcal{S}\operatorname{span}(v)$, and so $v \in \mathcal{SB}(n, \mathcal{S}'_n)$. $\square$

Theorem 6.7 (the general resolution of the identity). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family in a subspace X of $\mathcal{S}'_n$. Then, $v \in \mathcal{SB}(\mathbb{R}^m, X)$ if and only if*

$$(\cdot)_X = \int_{\mathbb{R}^m} [\cdot \mid v]v.$$

Proof. Note that the identity operator on X and the operator $\int_{\mathbb{R}^m} [\cdot \mid v]v$ coincide on the subspace $X \cap \mathcal{S}\operatorname{span}(v)$, in fact, if u is a point of this intersection,

$$u = \int_{\mathbb{R}^m} [u \mid v]v = \left(\int_{\mathbb{R}^m} [\cdot \mid v]v \right)(u) \, .$$

Consequently, the equality $(\cdot)_X = \int_{\mathbb{R}^m} [\cdot \mid v]v$ holds if and only if the two operators have the same domain, that is to say, if and only if $X = \mathcal{S}\operatorname{span}(v)$.

$\square$

7. $\mathcal{S}$-linear operators and the existence of the $\mathcal{S}$-bases

Definition 7.1 (image of a family of distributions). *Let $W \subseteq \mathcal{S}'_n$, $A : W \to \mathcal{S}'_m$ be an operator and $v = (v_p)_{p \in \mathbb{R}^k}$ be a family of tempered distributions in W, i.e., such that $\{v_p\}_{p \in \mathbb{R}^k} \subseteq W$. The image of v under A is the family in $\mathcal{S}'_m$*

$$A(v) = (A(v_p))_{p \in \mathbb{R}^k} \, ,$$

i.e., the family such that, for all $p \in \mathbb{R}^k$, one has $A(v)_p = A(v_p)$.

We can read the above definition saying that "the image of a family of vectors is the family of the images of vectors".

DEFINITION 7.2 (operator of class $\mathcal{S}$). *Let $W \subseteq \mathcal{S}'_n$ and $L : W \to \mathcal{S}'_m$ be an operator. L is an $\mathcal{S}$-operator or operator of class $\mathcal{S}$ if, for each natural k and for each $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, such that $\{v_p\}_{p\in\mathbb{R}^k} \subseteq W$, one has $L(v) \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$.*

We can read the above definition as follows: "L is of class $\mathcal{S}$ if the image of an $\mathcal{S}$-family is an $\mathcal{S}$-family". In the following we put $\sigma_n = \sigma(\mathcal{S}_n, \mathcal{S}'_n)$.

EXAMPLE 7.1 (the transpose). Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a (σ_n, σ_m)-continuous operator. A is transposable (i.e., for every $a \in \mathcal{S}'_m$, $a \circ A$ is in $\mathcal{S}'_n$) and its transpose is

$$^{t}A : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto a \circ A.$$

Let $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$, one has, by definition,

$$^{t}A\,(v)_p = {}^{t}A\,(v_p)\,,$$

and hence one deduces

$$^{t}A\,(v)\,(\phi)\,(p) = {}^{t}A\,(v)_p\,(\phi) = {}^{t}A\,(v_p)\,(\phi) = v_p\,(A\,(\phi)) = v\,(A\,(\phi))\,(p),$$

so, taking into account that v is an $\mathcal{S}$-family, one has ${}^{t}A\,(v)\,(\phi) = \widehat{v}\,(A\,(\phi)) \in \mathcal{S}_k$. Concluding one has ${}^{t}A\,(v) \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}'_n\right)$, and thus the operator ${}^{t}A$, sending $\mathcal{S}$-family in $\mathcal{S}$-family, is an $\mathcal{S}$-operator.

APPLICATION 7.1. Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a differential operator with constant coefficients and v be an $\mathcal{S}$-family in $\mathcal{S}'_n$. Then $L(v)$ is an $\mathcal{S}$-family, in fact L is the transpose of a certain differential operator on $\mathcal{S}_n$. For instance, the family $(\delta_x)_{x\in\mathbb{R}^n}$ is obviously an $\mathcal{S}$-family, and so the families of derivatives $\left(\delta_x^{(i)}\right)_{x\in\mathbb{R}^n}$ are $\mathcal{S}$ -families for every multi-index i.

DEFINITION 7.3 ($\mathcal{S}$-*linear operator*). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $\mathcal{S}$-operator. L is called $\mathcal{S}$-linear operator if, for each natural k, for each $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ and for every $a \in \mathcal{S}'_k$, one has*

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v).$$

The set of all the $\mathcal{S}$-linear operators from $\mathcal{S}'_n$ to $\mathcal{S}'_m$ is denoted by

$$\mathcal{S}\,\mathrm{Hom}(\mathcal{S}'_n, \mathcal{S}'_m).$$

In the following we denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operator among the two topological vector spaces $(\mathcal{S}_n, \mathcal{S}_{(n)})$ and $(\mathcal{S}_m, \mathcal{S}_{(m)})$. Since these two spaces are complete and metrizable, the space $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ coincides with the space $\mathcal{L}\,(\sigma_n, \sigma_m)$, the space of all the linear and (σ_n, σ_m)-continuous operators from $\mathcal{S}_n$ to $\mathcal{S}_m$ (see [8] page 258, Corollary).

On the other hand, $\mathcal{L}\left(\sigma_n, \sigma_m\right)$ is also the set of all the transposable linear operators from $\mathcal{S}_n$ to $\mathcal{S}_m$ (see [8] page 254, §12, Proposition 1).

It's, at this point, obvious that the two vector spaces $\mathcal{S}\left(\mathbb{R}^m, \mathcal{S}_n'\right)$ and $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ are isomorphic, being the map

$$(\cdot)^\wedge : \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}_n'\right) \to \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right) : v \mapsto \widehat{v}$$

an isomorphism, moreover, its inverse is the map

$$(\cdot)^\vee : \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right) \to \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}_n'\right) : A \mapsto A^\vee := \left(\delta_x \circ A\right)_{x \in \mathbb{R}^m}.$$

Now, we can show the intimate essence of the $\mathcal{S}$ -linear operators defined on $\mathcal{S}_n'$.

Definition 7.4 (superposition of a family with respect to a family). *Let $v \in s\left(\mathbb{R}^k, \mathcal{S}_m'\right)$ and $w \in \mathcal{S}\left(\mathbb{R}^m, \mathcal{S}_n'\right)$. The family in $\mathcal{S}_n'$*

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w\right)_{p \in \mathbb{R}^k},$$

is called the superposition of w with respect to v.

If $v \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}_m'\right)$ then $\int_{\mathbb{R}^m} vw \in \mathcal{S}\left(\mathbb{R}^k, \mathcal{S}_n'\right)$ and

$$\left(\int_{\mathbb{R}^m} vw\right)^\wedge = \widehat{v} \circ \widehat{w}.$$

In this case, $\int_{\mathbb{R}^m} vw$ is denoted by vw and it is called the $\mathcal{S}$-product of v by w.

Lemma 7.1 (the image under a transpose operator). *Let $B \in \mathcal{L}\left(\mathcal{S}_n, \mathcal{S}_m\right)$ and $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}_m')$. Then,*

$$^t B(v) = \int_{\mathbb{R}^k} vB^\vee,$$

so in particular, $^t B$ is an $\mathcal{S}$-operator.

Proof. For each $p \in \mathbb{R}^k$, one has

$$\left(\int_{\mathbb{R}^k} vB^\vee\right)_p = \int_{\mathbb{R}} v_p B^\vee = v_p \circ (B^\vee)^\wedge = v_p \circ B = {}^t B\left(v_p\right) = {}^t B(v)(p),$$

and hence

$$\int_{\mathbb{R}^k} vB^\vee = {}^t B(v).$$

□

Theorem 7.1 ($\mathcal{S}$-*linearity of a transpose operator*). *Let* $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ *and* $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$. *Then, for each* $a \in \mathcal{S}'_k$ *one has*

$$^tB\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} a\, ^tB(v).$$

Proof. One has

$$\begin{aligned} ^tB\left(\int_{\mathbb{R}^k} av\right) &= \left(\int_{\mathbb{R}^k} av\right)\circ B = (a\circ\widehat{v})\circ B = a\circ(\widehat{v}\circ B) = \\ &= \int_{\mathbb{R}^k} a(\widehat{v}\circ B)^\vee = \int_{\mathbb{R}^k} a(\int_{\mathbb{R}^k} vB^\vee) = \int_{\mathbb{R}^k} a\, ^tB(v). \end{aligned}$$

□

Application 6.2. As a simple application, we prove the formula: $u' = \int_{\mathbb{R}} u\delta'$, where δ' is the $\mathcal{S}$-family in $\mathcal{S}'_1$ defined by $\delta' = \left(\delta'_p\right)_{p\in\mathbb{R}}$. Let δ be the Dirac family of $\mathcal{S}'_1$, then for each $u \in \mathcal{S}'_1$, one has $u = \int_{\mathbb{R}} u\delta$, and thus

$$u' = D\left(\int_{\mathbb{R}} u\delta\right) = \int_{\mathbb{R}} uD(\delta) = \int_{\mathbb{R}} u\delta'.$$

Theorem 7.2 (characterization of $\mathcal{S}$-*linearity*). *Let* $L : \mathcal{S}'_n \to \mathcal{S}'_m$. *Then,* L *is* $\mathcal{S}$-*linear if and only if there exists a* $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ *such that* $L = {}^t(B)$.

Proof. Sufficiency. Follows from the above theorem.
Necessity. Let δ be the Dirac's family in $\mathcal{S}'_n$, one has

$$L(u) = L\left(\int_{\mathbb{R}^n} u\delta\right) = \int_{\mathbb{R}^n} uL(\delta) = {}^t\left(L(\delta)^\wedge\right)(u),$$

so

$$L = {}^t\left(L(\delta)^\wedge\right).$$

□

The preceding theorem allow us to state and prove some definitive results on the existence of $\mathcal{S}$-bases for a subspace of $\mathcal{S}'_n$.

Theorem 7.3. *Let* V *be a subspace of* $\mathcal{S}'_n$. *Then*

1) V *has a system of* $\mathcal{S}$-*generators if and only if there is an* $\mathcal{S}$-*linear operator* $A : \mathcal{S}'_m \to \mathcal{S}'_n$, *for some* m, *such that* $A(\mathcal{S}'_m) = V$;
2) V *has an* $\mathcal{S}$-*basis if and only if there is an injective* $\mathcal{S}$-*linear operator* $A : \mathcal{S}'_m \to \mathcal{S}'_n$, *for some* m, *such that* $A(\mathcal{S}'_m) = V$.

Proof. 1) It's obvious, because every $\mathcal{S}$-family univocally determines a transposable operator $\hat{v} : \mathcal{S}_n \to \mathcal{S}_m$ that is univocally determined by the $\mathcal{S}$-linear operator ${}^t\hat{v} : \mathcal{S}'_m \to \mathcal{S}'_n$, and viceversa. Moreover, for every $\mathcal{S}$-family one has $\mathcal{S}\,\mathrm{span}\,(v) = {}^t\hat{v}(\mathcal{S}'_m)$.

2) Remember that

$$^t\hat{v}(a) = \int_{\mathbb{R}^m} av.$$

Then the conclusion follows immediately from the definition of $\mathcal{S}$-linear independence. ∎

The preceding can be reread in the following way

Theorem 7.4. *Let V be a subspace of $\mathcal{S}'_n$. Then*

1) *V has a system of $\mathcal{S}$-generators if and only if there is a continuous linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, for some m, such that $^tA(\mathcal{S}'_m) = V$;*
2) *V has an $\mathcal{S}$-basis if and only if there is a continuous linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, for some m, such that* $\operatorname{im} A$ *is dense in $\mathcal{S}_m$ and $^tA(\mathcal{S}'_m) = V$.*

By the preceding and by the Dieudonnè-Schwartz theorem (see later), it follows

Theorem 7.5. *Let V be a weakly* closed subspace of $\mathcal{S}'_n$. Then*

1) *V has a system of $\mathcal{S}$-generators if and only if there is a strict morphism $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some m, such that $A(\mathcal{S}'_m) = V$;*
2) *V has an $\mathcal{S}$-basis if and only if there is an injective strict morphism $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some m, such that $A(\mathcal{S}'_m) = V$.*

For the reader, we recall the two classic results used above.

Theorem 7.6 (Dieudonné-Schwartz). *Let E and F be two Frèchet spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ respectively, E' and F' their topological duals, and let $u : E \to F$ be a linear map. Then the following conditions are equivalent:*

(α) *u is a strict morphism for $\mathcal{T}_E$ and $\mathcal{T}_F$;*
(β) *u is a strict morphism for $\sigma(E, E')$ and $\sigma(F, F')$;*
(γ) *$u(E)$ is closed in F;*
(δ) *^tu is a strict morphism for $\sigma(F', F)$ and $\sigma(E', E)$;*
(ϵ) *$^tu(F')$is closed in E' for $\sigma(E', E)$.*

Corollary 7.1. *Let E and F be two Frèchet spaces, E' and F' their topological duals, and $u : E \to F$ be a linear map.*

(i) *u is an injective strict morphism if and only if $^tu(F') = E'$.*
(ii) *u is a surjective strict morphism if and only if $^tu(F')$ is closed in E' for $\sigma(E', E)$ and ^tu is injective.*
(iii) *u is an isomorphism if and only if ^tu is an isomorphism for the topologies $\sigma(F', F)$ and $\sigma(E', E)$.*

8. Representations in quantum theory: continuous case

In the present section we give a rigorous and greatly enriched version of the representation theory introduced by Dirac in [6] (page 66). A pure state

σ of a quantum system is a mono-dimensional subspace of the space $\mathcal{S}'_n$, each $\psi \in \sigma$ is a vector-state representing σ.

Let $\psi = (\psi_p)_{p\in\mathbb{R}^n}$ be an $\mathcal{S}$ -basis of $\mathcal{S}'_n$ and $A \in \mathcal{S}\,\mathrm{End}\,(\mathcal{S}'_n)$. For every $p \in \mathbb{R}^n$we have

$$A\,(\psi_p) = \int_{\mathbb{R}^n} [A\,(\psi_p) \mid \psi]\,\psi,$$

we call the family

$$(A)_\psi = ([A\,(\psi_p) \mid \psi])_{p\in\mathbb{R}^n}\,,$$

the representation of A in ψ.

Let, now, A, B be two $\mathcal{S}$-linear operators, one has

$$\begin{aligned} A\,(B\,(\psi_p)) &= A\int_{\mathbb{R}^n} [B\,(\psi_p) \mid \psi]\,\psi = \int_{\mathbb{R}^n} (B)_\psi^p\,A\,(\psi) = \\ &= \int_{\mathbb{R}^n} (B)_\psi^p \int_{\mathbb{R}^n} (A)_\psi\,\psi = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} (B)_\psi^p\,(A)_\psi\right)\psi, \end{aligned}$$

so it follows

$$(AB)_\psi = (A)_\psi\,(B)_\psi\,,$$

the family representing the product of two operators is the product of the families representing the two operators.

The family representing an $A \in \mathcal{S}\,\mathrm{End}\,(\mathcal{S}'_n)$ takes the place of the matrix representing a linear operator among two finite dimensional vector spaces. For this reason we shall call the $\mathcal{S}$-families also with the name "$\mathcal{S}$-matrices".

We have, moreover, $u = \int_{\mathbb{R}^n} [u \mid \psi]\,\psi$. We call the non-locally defined family

$$(u)_\psi = [u \mid \psi]\,,$$

the representation of A in ψ.

We have

$$A\,(u) = \int_{\mathbb{R}^n} (u)_\psi\,A\,(\psi) = \int_{\mathbb{R}^n} (u)_\psi \int_{\mathbb{R}^n} (A)_\psi\,\psi = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} (u)_\psi\,(A)_\psi\right)\psi,$$

thus we proved that

$$(A\,(u))_\psi = \int_{\mathbb{R}^n} (u)_\psi\,(A)_\psi\,.$$

If we regard the multiplication by a number c as an operator: $M_c\,(u) = cu$, we have

$$M_c\,(u) = cu = c\int_{\mathbb{R}^n} (u)_\psi\,\psi = \int_{\mathbb{R}^n} (u)_\psi\,(c\psi) = \int_{\mathbb{R}^n} c\,(u)_\psi\,\psi.$$

Then we have

$$M_c\,(\psi_p) = \int_{\mathbb{R}^n} c\,(\psi_p)_\psi\,\psi = \int_{\mathbb{R}^n} c\delta_p\psi,$$

so the $\mathcal{S}$-family representing the operator M_c in the basis ψ is the "diagonal" family $(c\delta_p)_{p\in\mathbb{R}^n}$, i.e., the family $c\delta$.

The correspondence that sends every $\mathcal{S}$-endomorphism to its corresponding $\mathcal{S}$-matrix, say

$$(\cdot)_{\psi} : \mathcal{S}\,\mathrm{End}\,(\mathcal{S}'_n) \to \mathcal{S}\,(\mathbb{R}^n, \mathcal{S}'_n)\,,$$

is bijective.

In fact, $(A)_{\psi} = (B)_{\psi}$ implies

$$(A\,(u))_{\psi} = \int_{\mathbb{R}^n} (u)_{\psi}\,(A)_{\psi} = \int_{\mathbb{R}^n} (u)_{\psi}\,(B)_{\psi} = (B\,(u))_{\psi}\,,$$

and thus $Au = Bu$, i.e., $A = B$, so it is injective.

The mapping $(\cdot)_{\psi}$ is also surjective. In fact, if $v = (v_p)_{p\in\mathbb{R}^n}$ is an $\mathcal{S}$ -family, putting

$$A\,(u) = \int_{\mathbb{R}^n} (u)_{\psi} \left(\int_{\mathbb{R}^n} v\psi \right),$$

one has

$$\begin{aligned} A\,(\psi_p) &= \int_{\mathbb{R}^n} (\psi_p)_{\psi} \left(\int_{\mathbb{R}^n} v\psi \right) = \int_{\mathbb{R}^n} \delta_p \left(\int_{\mathbb{R}^n} v\psi \right) \\ &= \left(\int_{\mathbb{R}^n} v\psi \right)_p = \int_{\mathbb{R}^n} v_p\psi, \end{aligned}$$

and thus $(A)_{\psi} = v$.

Let $(\cdot)^{-}_{\psi}$ the inverse of $(\cdot)_{\psi}$. In the above proof we deduced that

$$(v)^{-}_{\psi}\,(u) = \int_{\mathbb{R}^n} (u)_{\psi} \left(\int_{\mathbb{R}^n} v\psi \right).$$

If we choose the canonical basis δ, we have

$$(v)^{-}_{\delta}\,(u) = \int_{\mathbb{R}^n} (u)_{\delta} \left(\int_{\mathbb{R}^n} v\delta \right) = \int_{\mathbb{R}^n} uv.$$

If we put (as in the finite-dimensional case)

$$vu := (v)^{-}_{\delta}\,(u) = \int_{\mathbb{R}^n} uv,$$

(vu is called the image of the vector u under the matrix v) we obtain

$$(A\,(u))_{\psi} = \int_{\mathbb{R}^n} (u)_{\psi}\,(A)_{\psi} = (A)_{\psi}\,(u)_{\psi}\,.$$

The generalization to the case $A \in \mathcal{S}\,\mathrm{Hom}(\mathcal{S}'_n, \mathcal{S}'_m)$ is, at this point, very natural:

Let ψ an $\mathcal{S}$-basis of $\mathcal{S}'_n$ and φ be an $\mathcal{S}$-basis of $\mathcal{S}'_m$. We define $\mathcal{S}$-matrix associated with A in the pair of basis (ψ, φ) the $\mathcal{S}$-family $(A)_{(\psi,\varphi)}$ defined by

$$(A\,(u))_{\varphi} = \int_{\mathbb{R}^n} (u)_{\psi}\,(A)_{(\psi,\varphi)}\,.$$

Or, with arguments similar to the preceding ones, the $\mathcal{S}$-matrix such that

$$(A(u))_{\varphi} = (A)_{(\psi,\varphi)} (u)_{\psi} .$$

It's simple to prove that ψ is an $\mathcal{S}$-basis of the entire space if and only if ${}^t\widehat{\psi}$ is bijective. In this case one has

$$u_{\psi} = \left({}^t\widehat{\psi}\right)^{-}(u) = {}^t\left(\widehat{\psi}^{-}\right)(u).$$

And moreover, denoted by ψ^{-} the family associated with the operator $\widehat{\psi}^{-}$, the following decomposition holds

$$(A)_{\psi}{}^{p} = \int_{\mathbb{R}^n} A\psi_p\psi^{-}.$$

This relations will be used in the following examples.

Example 8.1 (the representation of the position operator in the momentum basis). Let

$$X : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto (\cdot)u$$

be the position operator and let φ be the $(1, -1/\hbar)$-Fourier family, then one has

$$(X)_{\varphi}{}^{p} = \int_{\mathbb{R}} X\varphi_p\varphi^{-} = \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}}\varphi_p\varphi^{-} = \left(\frac{i}{1/\hbar}\right)^1 \left(\int_{\mathbb{R}} \varphi_p\varphi^{-}\right)' = i\hbar\,(\varphi_p)_{\varphi}{}' = i\hbar\delta'_p.$$

Example 8.2 (the representation of the momentum operator in the momentum basis). Let

$$P : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto -i\hbar u'$$

be the momentum operator of a particle. One has

$$(P)_{\varphi}{}^{p} = \int_{\mathbb{R}} P\varphi_p\varphi^{-} = \int_{\mathbb{R}} p\varphi_p\varphi^{-} = p\,(\varphi_p)_{\varphi} = p\delta_p.$$

and hence $(P)_{\varphi} = \mathbb{I}_{\mathbb{R}}\delta$.

Example 8.3 (the representation of the kinetic energy operator in the momentum basis). Let

$$T : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto \frac{\hbar^2}{2m}u'' = \frac{1}{2m}P^2(u)$$

be the kinetic energy operator of a nonrelativistic particle, one has

$$\begin{aligned}
(T)_{\varphi}^{\ p} &= \int_{\mathbb{R}} T\varphi_p\varphi^- = \\
&= \int_{\mathbb{R}} \frac{1}{2m}P^2\varphi_p\varphi^- = \\
&= \frac{1}{2m}\int_{\mathbb{R}} p^2\varphi_p\varphi^- = \\
&= \frac{1}{2m}p^2\int_{\mathbb{R}} \varphi_p\varphi = \\
&= \frac{1}{2m}p^2\left(\varphi_p\right)_{\varphi} = \\
&= \frac{1}{2m}p^2\delta_p.
\end{aligned}$$

9. The space $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ and the $\mathcal{S}$-families

First of all we recall, for convenience of the reader, some basic notions from theory of distributions.

Definition 9.1. *We denote by $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ the space of all $f \in C^\infty(\mathbb{R}^n, \mathbb{K})$ such that for every $\phi \in \mathcal{S}_n$ one has $\phi f \in \mathcal{S}_n$. The set $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is said to be the space of C^∞ functions from $\mathbb{R}^n$ to $\mathbb{K}$ slowly increasing at infinity.*

Proposition 9.1. *Let $f \in C^\infty(\mathbb{R}^n, \mathbb{K})$. The following are equivalent conditions:*

1. For all $p \in \mathbb{N}_0^n$ there is a polynomial P_p such that $\forall x \in \mathbb{R}^n$, $|\partial^p f(x)| \leq |P_p(x)|$.

2. For all $\phi \in \mathcal{S}_n$ one has $\phi f \in \mathcal{S}_n$.

3. For every $p \in \mathbb{N}_0^n$ and for every $\phi \in \mathcal{S}_n$ the function $(\partial^p f)\,\phi$ is bounded in $\mathbb{R}^n$.

The standard topology of $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is the locally convex topology defined by the family of seminorms

$$\gamma_{\phi,p}(\phi) = \sup_{x\in\mathbb{R}^n} |\phi(x)\partial^p f(x)|$$

where $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ and $p \in \mathbb{N}_0^n$. This topology does not have a countable basis. Also, it can be shown that $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is a complete space. A sequence (or filter) $(f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ if and only if for every $\phi \in \mathcal{S}_n$ and for every $p \in \mathbb{N}_0^n$, the sequence $(\phi\partial^p f_j)_{j\in\mathbb{N}}$ converges to zero uniformly on $\mathbb{R}^n$. Or, equivalently, for every $\phi \in \mathcal{S}_n$, $(\phi f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{S}_n$. A set B is bounded in $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ if and only if for all $p \in \mathbb{N}_0^n$ there is a polynomial P_p such that $\forall x \in \mathbb{R}^n$, $\forall f \in B$, $|\partial^p f(x)| \leq P_p(x)$. Moreover, the bilinear map

$$\Phi : \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \times \mathcal{S}_n \to \mathcal{S}_n : (\phi, f) \mapsto \phi f$$

is separately continuous.

Proposition 9.2. *Let* $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ *and* $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$*. Then, the mapping*

$$fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$$

is a linear and continuous operator.

Proof. First of all we note that fA is well defined in fact $(fA)(\phi) = fA(\phi) \in \mathcal{S}_m$ because $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$ and $A(\phi) \in \mathcal{S}_m$. Moreover, the bilinear application

$$\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{S}_m \to \mathcal{S}_m : \ (f, \psi) \mapsto f\psi$$

is separately continuous and since

$$(fA)(\phi) = fA(\phi) = \Phi(f, A(\phi))$$

i.e.,

$$fA = \Phi(f, A) := \Phi(f, \cdot) \circ A$$

the operator fA is the composition of two linear continuous maps and then is a linear and continuous operator. $\square$

Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. The operator $fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$ is called the product of A by f.

Proposition 9.3. *Let* $A, B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ *and* $f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$*. Then,*

1) $(f+g)A = fA + gA$; $f(A+B) = fA + fB$; $1_{\mathcal{O}_M} A = A$;
2) *the map* $\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : (f, A) \mapsto fA$ *is a bilinear map.*

Proof. It's a straightforward computation. $\square$

The above bilinear application is called multiplication of operators by $\mathcal{O}_M$ functions.

Remark 9.1. It's easy to see that the algebraic structure $(\mathcal{O}_M(\mathbb{R}^n, \mathbb{K}), +, \cdot)$ is a commutative ring with identity, where:

$$\cdot \ : \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \times \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) \to \mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) : (f, g) \mapsto fg$$

(obviously if $f, g \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ one has $fg \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$) and $1_{(\mathcal{O}_M, +, \cdot)} = 1_{(\mathbb{R}^n, \mathbb{K})}$). Moreover, one has that $\mathcal{S}_n$ is an ideal of $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$.

Proposition 9.4. *Let* $\cdot$ *the operation defined in the above theorem. Then, the algebraic structure* $(\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m), +, \cdot)$ *is a left module over the ring* $(\mathcal{O}_M(\mathbb{R}^m, \mathbb{K}), +, \cdot)$.

PROOF. Recall the preceding theorem, we have to prove only the pseudo-associative law, i.e. we have to prove that for every $f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$, for every $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$, one has $(fg)A = f(gA)$. In fact, for each $\phi \in \mathcal{S}_n$, one has

$$[(fg)A](\phi) = (fg)A(\phi) = f(gA(\phi)) = f(gA)(\phi)) = [f(gA)](\phi).$$

□

DEFINITION 9.2 (product of a family by an $\mathcal{O}_M$*function*). *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *and* $f \in C^\infty(\mathbb{R}^m, \mathbb{K})$. *The product of* v *by* f*is the family* $fv = (f(p)v_p)_{p\in\mathbb{R}^m}$.

THEOREM 9.1. *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *and* $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$. *Then, the family* fv *lies in* $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. *Moreover, one has* $(fv)^\wedge = f\widehat{v}$.

PROOF. Let $\phi \in \mathcal{S}_n$, one has

$$(fv)(\phi)(p) = (fv)_p(\phi) = (f(p)v_p)(\phi) = f(p)v_p(\phi) = f(p)\widehat{v}(\phi)(p)$$

and hence $(fv)(\phi) = f\widehat{v}(\phi) \in \mathcal{S}_m$. Thus, one has $fv \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, $\forall \phi \in \mathcal{S}_n$, $(fv)^\wedge(\phi) = f\widehat{v}(\phi)$, i.e. $(fv)^\wedge = f\widehat{v}$, where $f\widehat{v}$, is the product of $\widehat{v}$ by f and $f\widehat{v} \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$. □

THEOREM 9.2. *Let* $f, g \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$, $v, w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. *Then,*

1) $(f + g)v = fv + gv$; $f(v + w) = fv + fw$; $1_{\mathcal{O}_M} v = v$.
2) *The map* $\Phi : \mathcal{O}_M(\mathbb{R}^m, \mathbb{K}) \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : (f, v) \mapsto fv$ *is a bilinear map.*

PROOF. 1) For all $p \in \mathbb{R}^m$, one has

$$\begin{aligned}[(f+g)\, v]\, (p) &= (f+g)(p)v_p = (f(p)+g(p))v_p = f(p)v_p + g(p)v_p \\ &= (fv)_p + (gv)_p,\end{aligned}$$

i.e. $(f + g)v = fv + gv$; For all $p \in \mathbb{R}^m$, one has

$$\begin{aligned}[f\,(v+w)]\,(p) &= f(p)(v+w)_p = f(p)(v_p + w_p) = f(p)v_p + f(p)w_p \\ &= (fv)_p + (fw)_p,\end{aligned}$$

i.e. $f(v + w) = fv + fw$. For all $p \in \mathbb{R}^m$, one has $(1_{(\mathbb{R}^m,\mathbb{K})}\, v)(p) = 1_{(\mathbb{R}^m,\mathbb{K})}\,(p)v_p = v_p$; i.e. $1_{\mathcal{O}_M} v = v$. 2) To the reader. □

The bilinear application of the point 2) of the preceding theorem is called multiplication of families by $\mathcal{O}_M$ functions.

THEOREM 9.3. *Let* $\cdot$ *the operation defined above. Then, the algebraic structure* $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ *is a left module over the ring* $(\mathcal{O}_M(\mathbb{R}^m, \mathbb{K}), +, \cdot)$.

PROOF. It's analogous to the proof of the Proposition 9.4. □

THEOREM 9.4 (of isomorphism). *The application* $(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ *is a module isomorphism.*

Proof. It follows easily from Theorem 9.1. ∎

Now we can improve the Theorem 6.2.

Theorem 9.5. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and $\lambda \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$ a function different form 0 at every point. Then, the following assertions hold true*

1) *if w is $\mathcal{S}$-linearly independent the family λw is $\mathcal{S}$-linearly independent;*
2) *$\mathcal{S}\,\mathrm{span}(w) = \mathcal{S}\,\mathrm{span}(\lambda w)$;*
3) *if w is $\mathcal{S}$-linearly independent, for each $u \in \mathcal{S}\,\mathrm{span}(w)$, one has*

$$[u \mid \lambda w] = (1/\lambda)\,[u \mid w].$$

Proof. 1) Let $a \in \mathcal{S}'_m$ be such that $\int_{\mathbb{R}^m} a(\lambda w) = 0_{\mathcal{S}'_n}$, one has

$$0_{\mathcal{S}'_n} = \int_{\mathbb{R}^m} a(\lambda w) = \int_{\mathbb{R}^m} (\lambda a) w,$$

thus, because w is $\mathcal{S}$-linearly independent one has $\lambda a = 0_{\mathcal{S}'_n}$. Since λ is different form 0 at every point we can conclude $a = 0_{\mathcal{S}'_n}$.

2) Let $u \in \mathcal{S}\,\mathrm{span}(w)$. Then, there exists an $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} aw.$$

Now, one has

$$u = \int_{\mathbb{R}^m} \left(\frac{a}{\lambda}\right)(\lambda w)\,,$$

so $u \in \mathcal{S}\,\mathrm{span}(\lambda w)$, and hence $\mathcal{S}\,\mathrm{span}(w) \subseteq \mathcal{S}\,\mathrm{span}(\lambda w)$. Viceversa, let $u \in \mathcal{S}\,\mathrm{span}(\lambda w)$. Then, there exists an $a \in \mathcal{S}'_m$ such that $u = \int_{\mathbb{R}^m} a(\lambda w)$. Now, one has

$$u = \int_{\mathbb{R}^m} (\lambda a) w,$$

and hence $u \in \mathcal{S}\,\mathrm{span}(w)$, hence $\mathcal{S}\,\mathrm{span}(\lambda w) \subseteq \mathcal{S}\,\mathrm{span}(w)$. Concluding $\mathcal{S}\,\mathrm{span}(w) = \mathcal{S}\,\mathrm{span}(\lambda w)$.

3) For any $u \in \mathcal{S}'_n$, one has $u = \int_{\mathbb{R}^m} [u \mid w]\, w$, hence

$$u = \int_{\mathbb{R}^m} (\frac{1}{\lambda}\,[u \mid w])\,(\lambda w)\,.$$

∎

10. Spectral $\mathcal{S}$-expansion and $\mathcal{S}$-diagonalizable operators

In the following we shall use the notation

$$\mathcal{S}\,\mathrm{End}(\mathcal{S}'_n) = \mathcal{S}\,\mathrm{Hom}(\mathcal{S}'_n, \mathcal{S}'_n).$$

Let D, C be two vector spaces and $A \in \mathrm{Hom}(D, C)$. The set of all the eigenvectors of the operator A is denoted by $\mathrm{EV}(A)$ and is called the family of the eigenvectors of A. The set of all the eigenvalues of the operator A is denoted by ${}^e\sigma(A)$; moreover the eigenspace relative to an eigenvalue $a \in \mathbb{K}$ is denoted by $|a\rangle_A$. Moreover $\mathcal{S}_{ind}(\mathbb{R}^m, \mathcal{S}'_n)$ is the set of the $\mathcal{S}$-linearly independent families in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$.

THEOREM 10.1 (of continuous spectral expansion). *Let $A \in \mathcal{S}\,\mathrm{End}(\mathcal{S}'_n)$, $f \in \mathcal{O}_M(\mathbb{R}^m, \mathbb{K})$ and $v \in \mathcal{S}_{ind}(\mathbb{R}^m, \mathcal{S}'_n)$ such that, for each $p \in \mathbb{R}^m$, one has $A(v_p) = f(p)v_p$, i.e. $A(v) = fv$. Then, for each $u \in \mathcal{S}\,\mathrm{span}(v)$, one has*

$$A(u) = \int_{\mathbb{R}^m} f\,[u \mid v]\, v.$$

PROOF. For each $u \in \mathcal{S}\,\mathrm{span}(v)$, one has

$$\begin{aligned} A(u) &= A\left(\int_{\mathbb{R}^m} [u \mid v]\, v\right) = \int_{\mathbb{R}^m} [u \mid v]\, A(v) = \int_{\mathbb{R}^m} [u \mid v]\,(fv) \\ &= \int_{\mathbb{R}^m} (f\,[u \mid v])\, v. \end{aligned}$$

In fact, the third equality holds because,

$$A(v)_p = A(v_p) = f(p)v_p = (fv)(p),$$

and the fourth because

$$\begin{aligned} \int_{\mathbb{R}^m} [u|v]\,(fv)\,(\phi) &= [u|v]\,((fv)^\wedge\,(\phi)) = [u|v]\,(f\widehat{v}\,(\phi)) = (f\,[u|v])(\widehat{v}\,(\phi)) \\ &= \int_{\mathbb{R}^m} (f\,[u|v])v. \end{aligned}$$

This concludes the proof. ∎

Let $X \subseteq \mathcal{S}'_n$ be a subspace of $\mathcal{S}'_n$, $A \in \mathrm{Hom}(X, \mathcal{S}'_m)$ and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. The superposition of v with respect to A, is the operator

$$\int_{\mathbb{R}^m} Av : X \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

In the condition of the above theorem one has: $A_{|X} = \int_{\mathbb{R}^n} f\,[\cdot \mid v]\, v$, where $X = \mathcal{S}\,\mathrm{span}(v)$.

The above theorem holds in the particular case in which there exists an $\mathcal{S}$-basis of the space $\mathcal{S}'_n$ constituted by eigenvectors of the operator A. In this case we give the following

DEFINITION 10.1 (of $\mathcal{S}$-*diagonalizable operator*). *Let $A \in \mathcal{S}\,\mathrm{End}(\mathcal{S}'_n)$. The operator A is said $\mathcal{S}$-diagonalizable if there exist a function $a \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ and an $\mathcal{S}$-basis $\alpha \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ such that, for every $p \in \mathbb{R}^n$, one has $A(v_p) = a(p)\alpha_p$, i.e., $A(v) = a\alpha$.*

The origin of the preceding definition is natural: concerning the representation of A in the basis of the definition, one has

$$(A)_\alpha = a\delta.$$

In other words:

an $\mathcal{S}$-linear operator is $\mathcal{S}$-diagonalizable if and only if there exists a $\mathcal{S}$-basis of the space $\mathcal{S}'_n$ in which its representation is $\mathcal{S}$-diagonal.

We recall that an $\mathcal{S}$-matrix is said diagonal iff it is of the form $a\delta$ for some $\mathcal{O}_M$-function a.

Concerning the topological structure of the eigenvalues-spectrum of an $\mathcal{S}$-diagonalizable operator we have the following definitive results (I thank very much an anonymous referee, since, to answer his questions, I found and proved them):

Theorem 10.2 (on the topological structure of the eigenvalues-spectrum of an *$\mathcal{S}$-diagonalizable operator*). *Let A be an $\mathcal{S}$-diagonalizable operator. Then, if a is the ordered system of eigenvalues of A associated to an eigenbasis of A for $\mathcal{S}'_n$, we have*

$$\operatorname{im} a = {}^{e}\sigma(A).$$

In particular the eigenvalues-spectrum of A is a connected subset of $\mathbb{C}$.

We need a lemma:

Lemma 10.1. *Let $u \in \mathcal{S}'_n$ be a distribution and f be a smooth function. Assume that*

$$fu = 0_{\mathcal{S}'_n}.$$

Then u vanishes on the complement of the zero-level set of f.

Proof. Consider the set

$$\Omega = \{p \in \mathbb{R}^n : f(p) \neq 0\} = \mathbb{R}^n \setminus f^{\leftarrow}(0),$$

we have to prove that for every test function $\phi \in \mathcal{D}(\Omega)$ is $u(\phi) = 0$. Let $\phi \in \mathcal{D}(\Omega)$, the restriction $f_{|\Omega}$ does not vanish, so the quotient $\phi/f_{|\Omega}$ is defined on Ω, it is smooth and it belongs to $\mathcal{D}(\Omega)$. Now

$$u(\phi) = u\left(f\phi/f_{|\Omega}\right) = fu\left(\phi/f_{|\Omega}\right) = 0,$$

as desired. So the distribution u must be vanish in the open set $\{p \in \mathbb{R}^n : f(p) \neq 0\}$. ∎

Proof of the theorem 10.2. Since A is an $\mathcal{S}$-diagonalizable operator then there exist a function $a \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ and an $\mathcal{S}$-basis $\alpha \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ such that, for every $p \in \mathbb{R}^n$, one has $A(\alpha_p) = a(p)\alpha_p$, i.e., $A(\alpha) = a\alpha$.

We shall prove that the eigenvalues-spectrum of A is $\operatorname{im} a$.

Assume that e is an eigenvalue of A then there exists a non zero vector η such that $A(\eta) = e\eta$. We then have

$$\begin{aligned} A\eta &= A\int_{\mathbb{R}^n} [\eta \mid \alpha]\, \alpha = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha]\, A\alpha = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha]\, a\alpha = \\ &= \int_{\mathbb{R}^n} a\, [\eta \mid \alpha]\, \alpha, \end{aligned}$$

but on the other hand

$$\begin{aligned} A\eta &= e\eta = \\ &= e\int_{\mathbb{R}^n} [\eta \mid \alpha]\, \alpha = \\ &= \int_{\mathbb{R}^n} e\, [\eta \mid \alpha]\, \alpha. \end{aligned}$$

from the $\mathcal{S}$-independence of α we have

$$a\,[\eta \mid \alpha] = e\,[\eta \mid \alpha]\,,$$

then

$$(a - e)\,[\eta \mid \alpha] = 0.$$

so the distribution $[\eta \mid \alpha]$ must be vanish in the open set

$$\Omega_\eta = \{p \in \mathbb{R}^n : a(p) \neq e\} = \mathbb{R}^n \setminus a^{\leftarrow}(e).$$

Assume by contradiction that $e \notin \operatorname{im} a$, then there are no p such that $a(p) = e$, and then

$$\Omega_\eta = \mathbb{R}^n,$$

this implies

$$[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

so we deduce that η is null, and this is an absurd. We then saw that ${}^e\sigma(A) \subseteq \operatorname{im} a$, the converse is true by definition of eigenbasis.

Concluding the eigenvalues-spectrum of A is the image of a:

$${}^e\sigma(A) = \operatorname{im} a = a(\mathbb{R}^n),$$

that is a connected set because a is continuous and $\mathbb{R}^n$ is connected. ∎

Corollary 10.1. *If the eigenvalues-spectrum of an $\mathcal{S}$-diagonalizable operator is real then it is an interval of the real line (eventually degenerate).*

Only a question remains open:

What about the so called residual spectrum and continuous spectrum of an $\mathcal{S}$-diagonalizable operator ?

Recall that

1) the eigenvalues-spectrum is the set of all the complex numbers z such that the z-characteristic operator of a, that is the operator $C_z = A - zI$, is not injective;
2) the continuous spectrum is the set of all z such that C_z is invertible (injective and surjective) but with inverse not continuous;
3) the residual spectrum is the set of all z such that C_z is injective and its image is not dense in the space of tempered distribution.

Since A is an $\mathcal{S}$-diagonalizable operator then there exist a function $a \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ and an $\mathcal{S}$ -basis $\alpha \in \mathcal{SB}(\mathbb{R}^n, \mathcal{S}'_n)$ such that, for every $p \in \mathbb{R}^n$, one has $A(\alpha_p) = a(p)\alpha_p$, i.e., $A(\alpha) = a\alpha$. Assume that z is not an eigenvalue of A, then $C_z = A - zI$ is injective. Moreover

$$C_z(\alpha_p) = A(\alpha_p) - zI(\alpha_p) = a(p)\alpha_p - z\alpha_p = (a(p) - z)\,\alpha_p.$$

Because z is not an eigenvalue of A, the function $a - z$ never vanishes, and then the family $(a - z)\,\alpha$ is yet an $\mathcal{S}$ -basis of the space.

Moreover, it's simple to prove that C_z is surjective. In fact, let $\beta = (a - z)\alpha$, and let u be a tempered distribution, since β is an $\mathcal{S}$-basis of the space $\mathcal{S}'_n$, u is an $\mathcal{S}$-linear superposition of β, moreover

$$u = \int_{\mathbb{R}^n} [u \mid \beta]\,\beta = \int_{\mathbb{R}^n} [u \mid \beta]\,C_z(\alpha) = C_z\left(\int_{\mathbb{R}^n} [u \mid \beta]\,\alpha\right),$$

and so $u \in C_z(\mathcal{S}'_n)$. Consequently C_z is even surjective, and hence the residual spectrum is empty.

Even more, C_z is $\mathcal{S}$-linear and then it is the transpose of a certain weakly continuous (i.e., strongly continuous) endomorphisms on the Frèchet space $\mathcal{S}_n$. This operator is bijective as C_z, so by the Banach inverse operator theorem it is a topological isomorphism. And even more, by the Dieudonnè-Schwartz theorem C_z is a topological isomorphism too. So the continuous spectrum of an $\mathcal{S}$-diagonalizable operator is always empty.

This concludes completely the study of the spectrum of an $\mathcal{S}$-diagonalizable operator.

11. The building of some basic observables of quantum mechanics

Example 11.1 (The position operator in one dimension). A particle moving on the real line can be in a state in which its position is $x \in \mathbb{R}$. It's natural to assume that this state can be represented by the distribution δ_x, so, if we denote by Q the observable "position", we have $Q\delta_x = x\delta_x$, i.e., $Q\delta = \mathbb{I}_{\mathbb{R}}\delta$, applying the above theorem one has

$$Q(u) = \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}} [u \mid \delta] \delta = \int_{\mathbb{R}} (\mathbb{I}_{\mathbb{R}} u) \delta = \mathbb{I}_{\mathbb{R}} u.$$

This justifies the definition of the position operator, which is now possible to define, more naturally, as the only observable that in the state δ_x assume the value x, for every real x.

Example 11.2 (The position operator in three dimensions). A particle moving in the space can be in a state in which its position is the vector $x \in \mathbb{R}^3$. It's natural to assume that this state can be represented by the distribution δ_x. In this state the position has the three components x_1, x_2, x_3. Then, if we denote by $Q = (Q_1, Q_2, Q_3)$ the triple of operators representing the observable "position" in three dimensions, we have $Q\delta_x = (x_1\delta_x, x_2\delta_x, x_3\delta_x)$, i.e., $Q\delta = (\mathbb{I}_1\delta, \mathbb{I}_2\delta, \mathbb{I}_3\delta)$. Let us apply the decomposition theorem to the i-th component, one has

$$Q_i(u) = \int_{\mathbb{R}^3} \mathbb{I}_i [u \mid \delta] \delta = \int_{\mathbb{R}^3} (\mathbb{I}_i u) \delta = \mathbb{I}_i u.$$

This justifies the definition of the position operator, which is now possible to define, more naturally, the only observable that in the state δ_x assume the vector-value x.

Example 11.3 (The momentum operator). Following De Broglie, we assume that the state of a particle moving on the real line with momentum $p \in \mathbb{R}$ be represented by the regular distribution $\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]$. If we denote by P the observable "momentum", we have

$$P\left[e^{\frac{i(p|\cdot)}{\hbar}}\right] = p\left[e^{\frac{i(p|\cdot)}{\hbar}}\right].$$

Putting $f = \left(\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]\right)_{p\in\mathbb{R}}$, we have thus

$$Pf = \mathbb{I}_{\mathbb{R}} f.$$

Applying the above theorem, one has

$$P(u) = \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}} [u \mid f] f = \left(\frac{i}{-1/\hbar}\right)^1 \left(\int_{\mathbb{R}} [u \mid f] f\right)' = -i\hbar u',$$

Example 11.4 (of observable with a continuous degenerate spectrum not coinciding with the whole real line). Following De Broglie, we assume that the state of a particle moving on the real line with momentum $p \in \mathbb{R}$ be represented by the regular distribution $\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]$. If we denote by T the observable "Hamiltonian of a classic free particle in $\mathbb{R}$", we have

$$T\left[e^{\frac{i(p|\cdot)}{\hbar}}\right] = \frac{p^2}{2m}\left[e^{\frac{i(p|\cdot)}{\hbar}}\right].$$

Putting $f = \left(\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]\right)_{p\in\mathbb{R}}$, we have

$$Tf = \frac{p^2}{2m} f.$$

Then, applying the above theorem, one has

$$\begin{aligned} T(u) &= \int_{\mathbb{R}} \frac{(\mathbb{I}_{\mathbb{R}})^2}{2m} [u \mid f] f = \frac{1}{2m} \int_{\mathbb{R}} (\mathbb{I}_{\mathbb{R}})^2 [u \mid f] f = \\ &= \frac{1}{2m} \left(\frac{i}{-1/\hbar}\right)^2 \left(\int_{\mathbb{R}} [u \mid f] f\right)'' = -\frac{\hbar^2}{2m} u''. \end{aligned}$$

Note that the spectrum of T is the set of non-negative real numbers and that the dimension of every eigenspace is 2.

Actually, the spectral theory treated on the paper requires only the concept of $\mathcal{S}$-diagonalizable operator, because the spectral decomposition concerns the $\mathcal{S}$-diagonalizable operators.

Nevertheless, for completeness, we give the definition of $\mathcal{S}$-observable, that is a particular $\mathcal{S}$-diagonalizable operator.

Definition 11.1 (of observable with a continuous range of fundamental eigenstates). *Let $A \in \mathcal{S}\,\mathrm{End}(\mathcal{S}'_n)$. The operator A is said to be an observable with a continuous range of fundamental eigenstates (or an observable with an $\mathcal{S}$ eigenbasis or more simply an $\mathcal{S}$ observable) if it is $\mathcal{S}$ diagonalizable and it is the extension of an adjointable operator on $\mathcal{S}_n$.*

For adjointable operator on $\mathcal{S}_n$, we give the following definition.

Definition 11.2. *A strongly continuous endomorphism $A : \mathcal{S}_n \to \mathcal{S}_n$ is said to be adjointable if there is another strongly continuous endomorphism $B : \mathcal{S}_n \to \mathcal{S}_n$ such that*

$$\langle Ax|y\rangle = \langle x|By\rangle\,,$$

for every x and y in $\mathcal{S}_n$, where $\langle\cdot \mid \cdot\rangle$ is the standard Dirac's scalar product on $\mathcal{S}_n$.

In the above conditions the operator B is uniquely determined and it is denoted by $A^\dagger$. Moreover, it is possible to prove that an adjointable operator A is extendible to an $\mathcal{S}$-linear operator on $\mathcal{S}'_n$.

The most important kind of $\mathcal{S}$observable is the following one. An adjointable operator $A : \mathcal{S}_n \to \mathcal{S}_n$ is said to be symmetric or Hermitian if $A^\dagger = A$.

If an $\mathcal{S}$-observable $A \in \mathcal{S}\,\mathrm{End}(\mathcal{S}'_n)$ is the extension of a symmetric operator it is said a real $\mathcal{S}$ -observable.

Example 11.5 (of observable with a singular spectrum). If we regard the constant function of value c as an observable: $M_c(u) = cu$, we have that M_c has c as unique eigenvalue. On the other hand, every $\mathcal{S}$ -basis is an $\mathcal{S}$ -eigenbasis of M_c. So M_c is an observable with a continuous range of fundamental eigenstates but with a pointwise spectrum.

Now, let v an arbitrary $\mathcal{S}$-basis of the space, we have

$$M_c(u) = cu = c\int_{\mathbb{R}^n} [u \mid v]\, v = \int_{\mathbb{R}^n} c\,[u \mid v]\, v,$$

for every tempered distribution u. The spectral decomposition then holds, note that the superposition is performed on the set indexing the $\mathcal{S}$-basis and not on the spectrum of the operator, moreover it is not an integral decomposition but an expansion via superposition.

Example 11.6 (other observables with a continuous degenerate spectrum not coinciding with the whole real line). Let us consider the energy of a relativistic particle moving on the real line with rest mass m_0 and momentum p:

$$E(x,p) = m_0c^2 + pc.$$

Consider its square

$$E^2(x,p) = m_0^2c^4 + p^2c^2,$$

and the corresponding operator on $\mathcal{S}'_1$

$$H^2 = M_{m_0^2c^4} + c^2\hbar^2\,(\cdot)''\,.$$

It's simple to prove that the distribution

$$f_p = \left[e^{\frac{i(p|\cdot)}{\hbar}}\right]$$

is an eigenvector of H^2 with corresponding eigenvalue $m_0^2c^4 + p^2c^2$. Consequently, being

$$f = \left(\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]\right)_{p\in\mathbb{R}}$$

an $\mathcal{S}$-basis, H^2 is an $\mathcal{S}$-observable. Concerning its spectrum we have

$${}^e\sigma(H^2) = \left[m_0^2c^4, +\infty\right[\,.$$

If we consider the operators on $\mathcal{S}'_1$, defined by

$$H_-\left(f_p\right) = \left(-\sqrt{m_0^2c^4 + p^2c^2}\right) f_p,$$

and

$$H_+\left(f_p\right) = \left(\sqrt{m_0^2c^4 + p^2c^2}\right) f_p,$$

we deduce simply that

$${}^e\sigma(H_-) = \left]-\infty, -m_0c^2\right]$$

and

$${}^e\sigma(H_+) = \left[m_0c^2, +\infty\right[\,.$$

The operators H_- and H_+ are the Hamiltonian of a relativistic antiparticle and particle respectively.

Recall that to define an $\mathcal{S}$-linear operator is enough to give its values on an $\mathcal{S}$-basis.

12. The image of an $\mathcal{S}$-diagonalizable operator under a numerical function

The purpose of this section is to introduce a functional calculus for the $\mathcal{S}$-diagonalizable operators. Our goal is to state and prove a theorem that allow us to give a precise meaning for the action of a numerical function, defined on the spectrum of a certain operator A, on A itself.

We recall that if A is an $\mathcal{S}$-diagonalizable operator, by $EV(A)$ we denote the set of all the eigenvectors of A, $\sigma(A)$ the set of all the eigenvalues of A, $v_A : EV(A) \to \mathbb{C}$ the mapping that sends every eigenvector u of A to its unique eigenvalue. In other words, $v_A(u)$ is the unique c such that $Au = cu$.

First of all we need a lemma.

Lemma 12.1. *Let $M \in \mathbb{N}$, O be an open subset of $\mathbb{K}$, $e \in O$, $r : O \to \mathbb{K}$ be a C^M-function such that $r^{(i)}(e) = 0$ for every integer $i \in \mathbb{N}_0(\leq M)$. Then, for every C^M-function $a : \mathbb{R}^n \to \mathbb{K}$ such that $a(\mathbb{R}^n) \subseteq O$ we have $\partial^p(r \circ a)(x) = 0$, for every $x \in a^-(e)$ and for every multi-index $p \in \mathbb{N}_0^n$ such that* lenght $p \leq M$.

Proof. We see the proof in the case $n = 1$, the general case is wholly similar. Moreover, we shall prove a more general equality, exactly we shall prove that $\left(r^{(j)} \circ a\right)^{(i)}(x) = 0$, for every $i, j \in \mathbb{N}_0(\leq M)$ such that $i + j \leq M$, and for every x such that $a(x) = e$ (in the case $j = 0$ we obtain the statement).

We proceed by induction on $s = i + j$. If $i + j = 0$ then $i = j = 0$, and we have to prove that $r(a(x)) = 0$ for every $x \in a^-(e)$, i.e., $r(e) = 0$, and this is true by assumption. If $i + j = 1$, we have to prove that $r'(a(x)) = 0$ and $(r \circ a)'(x) = 0$. The first is $r'(e) = 0$, true by assumption; the second is $r'(a(x))a'(x) = r'(e)a'(x) = 0$, still by assumption.

Now we assume (by induction) that, fixed a positive integer $k < M$, $\left(r^{(j)} \circ a\right)^{(i)}(x) = 0$ hold true, for every $i, j \in \mathbb{N}_0(\leq M)$ such that $i + j \leq k < M$, and for every x such that $a(x) = e$. We have to prove that $\left(r^{(j)} \circ a\right)^{(i)}(x) = 0$ hold true, for every $i, j \in \mathbb{N}_0(\leq M)$ such that $i + j = k + 1 \leq M$, and for every x such that $a(x) = e$. In fact, if $i + j = k + 1$, we have two possibilities: $i = 0$, and we have nothing to prove; $i > 0$, in this case we have

$$\left(r^{(j)} \circ a\right)^{(i)}(x) \quad = \quad \left(\left(r^{(j)} \circ a\right)'\right)^{(i-1)}(x) = \left(\left(r^{(j+1)} \circ a\right) a'\right)^{(i-1)}(x) =$$

$$= \sum_{w=0}^{i-1} \binom{i-1}{w} \left(r^{(j+1)} \circ a\right)^{(w)} (x) \left(a'\right)^{(i-1-w)} (x) =$$

(by Leibnitz formula)

$$= \binom{i-1}{i-1} \left(r^{(j+1)} \circ a\right)^{(i-1)} (x) a^{(i-i+1)}(x) =$$

(by inductive assumption)

$$= \left(r^{(j+1)} \circ a\right)^{(i-1)} (x) a'(x),$$

note that $j + 1 + w = k + 1 = i + j$ if and only if $w = i - 1$. At this point, if $i = 1$ we can conclude, if $i > 1$, applying yet the previous result, we have

$$\left(r^{(j)} \circ a\right)^{(i)} (x) = \left(r^{(j+2)} \circ a\right)^{(i-2)} (x) \left(a'(x)\right)^2 .$$

In general, if $i \geq q$, for some positive integer q, applying q times the previous result, we have

$$\left(r^{(j)} \circ a\right)^{(i)} (x) = \left(r^{(j+q)} \circ a\right)^{(i-q)} (x) \left(a'(x)\right)^q .$$

In particular,

$$\left(r^{(j)} \circ a\right)^{(i)} (x) = \left(r^{(j+i)} \circ a\right) (x) \left(a'(x)\right)^i = r^{(k+1)}(e) \left(a'(x)\right)^i = 0,$$

as desired. ∎

THEOREM 12.1 (basic lemma on the functions of an $\mathcal{S}$-*diagonalizable operator*). *Let A be an $\mathcal{S}$ -diagonalizable operator with an infinite spectrum, let f be a real or complex smooth function defined on an open set of $\mathbb{K}$ containing the spectrum $\sigma(A)$, such that $f \circ v_A \circ \alpha$ is of class $\mathcal{O}_M$ for some eigenbasis α of A.*

Then, there is a unique $\mathcal{S}$-diagonalizable operator B such that, for every eigenvector η of A, the following relation holds

$$B(\eta) = f(v_A(\eta))\eta.$$

In other words, B is such that $v_B = f \circ v_A$.

Moreover, if α is an eigenbasis of A and $a = v_A \circ \alpha$ is the ordered family of the eigenvalues associated with α, for every tempered distribution u we have

$$B(u) = \int_{\mathbb{R}^n} (f \circ a) [u \mid \alpha] \alpha.$$

PROOF. Because A has an infinite spectrum, v_A is not a constant function.

Existence. Let α be an $\mathcal{S}$-eigenbasis of A. Setting $a = v_A \circ \alpha$, consider the operator defined by

$$Bu = \int_{\mathbb{R}^n} (f \circ a) [u \mid \alpha] \alpha,$$

for every distribution u. It is obviously $\mathcal{S}$-linear.

Concerning the $\mathcal{S}$-diagonalizability, we have

$$\begin{aligned} B\alpha_p &= \int_{\mathbb{R}^n} (f \circ a) [\alpha_p \mid \alpha] \alpha = \\ &= \int_{\mathbb{R}^n} [\alpha_p \mid \alpha] (f \circ a) \alpha = \\ &= \int_{\mathbb{R}^n} \delta_p (f \circ a) \alpha = \\ &= (f \circ a) (p) \alpha_p = \\ &= f(a(p))\alpha_p = \\ &= f(v_A(\alpha_p))\alpha_p. \end{aligned}$$

So α is an eigenbasis for B too, and then B is $\mathcal{S}$ -diagonalizable. More, the defined operator verifies the required property for the basis α.

We shall see that the property holds for every eigenvector.

If η is an eigenvector of A one has

$$\begin{aligned} A\eta &= A \int_{\mathbb{R}^n} [\eta \mid \alpha] \alpha = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha] A\alpha = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha] a\alpha = \\ &= \int_{\mathbb{R}^n} a [\eta \mid \alpha] \alpha, \end{aligned}$$

but on the other hand

$$A\eta = v_A(\eta)\eta = v_A(\eta) \int_{\mathbb{R}^n} [\eta \mid \alpha] \alpha = \int_{\mathbb{R}^n} v_A(\eta) [\eta \mid \alpha] \alpha.$$

from the $\mathcal{S}$-independence of α we have

$$a [\eta \mid \alpha] = v_A(\eta) [\eta \mid \alpha],$$

then, putting $e = v_A(\eta)$, we have

$$(a - e) [\eta \mid \alpha] = 0_{\mathcal{S}'_n}.$$

Because $[\eta \mid \alpha]$ is a tempered distribution, then it is of finite order, say of order $\leq M$. By the Taylor's formula, there is a function r such that

$$r^{(i)}(e) = 0,$$

for every $0 \leq i \leq M$, and such that

$$f(y) = \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(y-e)^k + r(y),$$

for every y in the spectrum of A. Then, for every x, one has

$$f(a(x)) = \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(a(x)-e)^k + r(a(x)),$$

That is,

$$f \circ a = \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k + r \circ a = f(e) + \sum_{k=1}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k + r \circ a.$$

Hence, multiplying by $[\eta \mid \alpha]$, and taking into account that,for $k \geq 1$,

$$(a-e)^k [\eta \mid \alpha] = (a-e)^{k-1}(a-e)[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

we deduce

$$\begin{aligned} (f \circ a)[\eta \mid \alpha] &= f(e)[\eta \mid \alpha] + \sum_{k=1}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k [\eta \mid \alpha] + (r \circ a)[\eta \mid \alpha] = \\ &= f(e)[\eta \mid \alpha] + (r \circ a)[\eta \mid \alpha]. \end{aligned}$$

Note that (by the previous lemma) $r \circ a$ must be vanish with all its derivatives of order $\leq M$, in the closed set $a^{-}(e)$. Moreover, since $[\eta \mid \alpha]$ must vanish in the complement of this set, we have

$$\operatorname{supp}[\eta \mid \alpha] \subseteq a^{-}(e).$$

Thus $r \circ a$ vanishes on the support of $[\eta \mid \alpha]$ with all its derivatives of order $\leq M$, and then, by a classic theorem on the distributions with finite order, we have

$$(r \circ a)[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

and consequently,

$$(f \circ a)[\eta \mid \alpha] = f(e)[\eta \mid \alpha].$$

Finally, we can conclude

$$\begin{aligned} B\eta &= B\int_{\mathbb{R}^n} [\eta \mid \alpha]\,\alpha = \int_{\mathbb{R}^n} [\eta \mid \alpha]\,B\alpha = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha]\,(f \circ a)\,\alpha = \int_{\mathbb{R}^n} (f \circ a)\,[\eta \mid \alpha]\,\alpha = \\ &= \int_{\mathbb{R}^n} f(e)\,[\eta \mid \alpha]\,\alpha = f(e)\int_{\mathbb{R}^n} [\eta \mid \alpha]\,\alpha = \\ &= f(e)\eta. \end{aligned}$$

Uniqueness. Two linear operators coinciding on a same $\mathcal{S}$ -basis are equals. ∎

The preceding theorem allow us to give the following definition

DEFINITION 12.1 (the functions of an $\mathcal{S}$ *-diagonalizable operator*). *Let A be an $\mathcal{S}$-diagonalizable operator. Let $EV(A)$ be the set of the eigenvectors of A, ${}^e\sigma(A)$ the set of all the eigenvalues of A, $v_A : EV(A) \to \mathbb{C}$ the mapping that sends every eigenvector u of A to its unique eigenvalue ($v_A(u)$ is the unique c such that $Au = c$).*

Let f be a complex function defined on the eigenvalues-spectrum ${}^e\sigma(A)$ such that $f \circ v_A \circ \alpha$ is smooth for some eigenbasis α of A.

The unique $\mathcal{S}$-diagonalizable operator B such that, for every eigenvector u of A is

$$Bu = f(v_A(u))u,$$

that is such that $v_B = f \circ v_A$ is called the image of A under f and it is denoted by $f(A)$.

EXAMPLE 12.1. Let t be a real number. Consider the function $f_t : \mathbb{R} \to \mathbb{C}$ defined by

$$f_t(x) = e^{-\frac{it}{\hbar}x}.$$

Let H be an $\mathcal{S}$-diagonalizable operator, and let η be a basis such that

$$H\eta = E\eta,$$

for some smooth real function E. Let $\psi_0 \in \mathcal{S}'_1$ and let $\psi\,(t)$ be the vector state defined by

$$\psi\,(t) = \int_{\mathbb{R}} e^{-\frac{it}{\hbar}E}\,[\psi_0 \mid \eta]\,\eta.$$

Then one has

$$\psi\,(t) = e^{-\frac{it}{\hbar}H}(\psi_0),$$

where with $e^{-\frac{it}{\hbar}H}$ we denoted the operator $f_t(H)$.

ACKNOWLEDGEMENTS.

I desire to thank an anonymous referee whose sharp remarks, questions and requests allow me to enrich greatly the final version of the paper.

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Received: 17.10.2003.
Revised: 26.7.2004. & 19.12.2004.