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# Chaos in Extended Linear Arrays of Josephson Weak Links

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Extended linear arrays of interacting Josephson weak links are studied by numerical simulation using the resistively shunted junction model. The minimum coupling strength for chaotic behavior is determined as a function of the number of links. This strength is found to diminish steadily with increasing number, despite the inclusion of only nearest-neighbor interaction. The implications for Josephson technology are briefly discussed. Mathematically, the results are a confirmation of the Ruelle-Takens scenario for chaos.

## I. INTRODUCTION

In a previous publication[1] the discovery in numerical simulation of chaotic behavior was first reported for the system of three noncapacitive weak links with nearest-neighbor coupling. The links were biased by dc currents, but no ac drive was applied. Two coupled links can have chaotic oscillations but only if there is an ac drive as well.[2] Mathematically these systems have the lowest possible dimension of phase space for which chaos can occur. In the limit of zero coupling, the former is equivalent to three independent one-dimensional systems for which chaos is impossible. Chaos does not immediately occur upon coupling the system arbitrarily weakly, rather a definite threshold for chaos exists, below which the system exhibits multiperiodic (quasiperiodic) motion. The threshold of chaos for *three* weak links was found to be quite high, corresponding to fairly substantial interaction between pairs of weak links.

The question we raise here is whether this threshold in the coupling parameter would drop for larger systems, thus rendering extended arrays of weak links, a technological prospect, more prone to chaotic behavior with its concomitant equivalent high noise level. The general question of whether chaos in extended arrays of coupled nonlinear oscillators is increasingly likely with increasing number may be of larger interest as such systems occur in many biological realms.[3]

## II. THEORY

In the resistively shunted junction model the equation[4] governing a linear array of $N$ capacitance-free junctions is

$$\frac{d\phi_i}{dt}=R_i[(I_i-I_{c_i}\sin\phi_i)+\alpha(I_{i-1}-I_{c_{i-1}}\sin\phi_{i-1})+\alpha(I_{i+1}-I_{c_{i+1}}\sin\phi_{i+1})],\quad i=1,2,\ldots,N, \quad (1)$$

where $R_i$ is the resistance of the $i$th junction, $I_{c_i}$ its critical current, $\phi_i$ is the phase across it, $I_i$ is the dc bias current through it, and $t$ is the normalized time.[4] Variables with subscript 0 or $N+1$ are taken to be zero, consistent with the first and $N$th junction being coupled to only one neighbor. Equation (1) assumes nearest-neighbor coupling only, with $\alpha$ characterizing the strength of that interaction (assumed the same for all pairs), which could have its origin in quasiparticle diffusion between proximal junctions or from intentional coupling in the circuit, e.g., shunting resistors.[2] (All parameters are dimensionless, normalized as in Ref. 4.)

A single junction, for $I_i > I_{c_i}$, is an oscillator with its own characteristic Josephson frequency. When coupled, an array can behave coherently with all the junctions frequency locked,[4] at least for suitable ranges of parameters. This, like chaos, is a peculiarly nonlinear phenomenon. For it to occur, $\alpha$ must be sufficiently large to cause junctions with slightly different frequencies to pull each other into mutual synchronization. For small $\alpha$, this becomes impossible if there is even moderate dispersion in the Josephson frequencies of the various links. Multiperiodic motion, in which there are up to $N$ independent frequencies, will occur instead.

Since the equations are periodic in each of the $\phi_i$, geometrically the system follows a trajectory on an $N$-torus. This path never closes, but ultimately covers the entire torus provided that the frequencies are independent (that is, if there is no rational relation amongst them). If there *are* rational relations the trajectory will cover a lower dimensional subspace. In the case of total frequency locking where there are $N-1$ such relations, the trajectory is a *closed* curve.

In summary, if we imagine $\alpha$ increasing from zero where the system is equivalent to $N$ one-dimensional oscillators, we would first expect to find multiperiodic behavior, then for sufficiently high $\alpha$ either chaotic motion or coherent motion.[1] Since a precondition for coherent motion is that the Josephson frequencies be in close proximity to each other, chaos should prevail with even moderate dispersion in parameter values ($\pm 2\%$). According to a result of Ruelle and Takens,[5] chaos is possible in a nonlinear system having three or more independent frequencies, that is for a system on a torus of dimension greater than two. For the system of $N$ weak links there are a maximum of $N$ independent frequencies involved. We set out to find the minimum $\alpha$ for which this occurs as a function of $N$, for $N$ greater than two.

## III. METHODS

Equation (1) was solved numerically in 17-decimal arithmetic using a fourth-order Runge-Kutta method. Two different situations, with values of $N$ ranging from 3 to 18 in one (Case 1) and from 3 to 144 in the other (Case 2) were considered.

In the previous work[1] it was found that for three weak links, chaos was observed at the lowest values of $\alpha$ when one of the links had across it a low-average dc voltage (frequency) compared to the other two. In Case 1, we therefore chose varying parameters including dc current biases to produce a significantly lower voltage across one junction. For the second series of numerical experiments (Case 2) a more physically realistic array model was used; here all junctions were assumed to have a common bias ($I_i=2$ for all $i$), and the $R_i$ and $I_{c_i}$ were chosen to be $(1+0.02u_i)$, where the $u_i$ were $2N$ randomly chosen numbers in the interval $(-1,1)$. This was intended to simulate an experimental situation in which the junctions are fabricated to a tolerance of 2%.

In both cases the largest Lyapunov exponent $\lambda$ was calculated by the method of Bennetin, Galgani, and Strelcyn.[6] A positive $\lambda$ indicates chaos, while a $\lambda$ of zero implies multiperiodic motion. As this parameter is numerically determined, it is important to distinguish between a small positive exponent (especially likely to occur at the *onset* of chaos) and a truly zero exponent. This determination was verified in two ways. Since $\lambda$ is the time average of the exponential divergence of neighboring trajectories, this divergence was monitored to see whether it was steadily accumulating as the simulation proceeded in time, for approximately $10^5$ time units. To corroborate some of these determinations another aspect of chaos was exploited. Chaotic motion is intrinsically not time reversible in the sense that information about the initial conditions is lost at an exponential rate determined by $\lambda$. For multiperiodic motion this is not the case, so that here solving Eq. (1) from 0 to $T$ followed by solving the time-reversed equations [Eq. (1) with $R_i=-R_i$] with the initial conditions determined by the final values at time $T$ should return us to the initial conditions after a further time $T$, within error bounds determined by the numerical algorithm employed. This method corroborated the conclusions obtained by the Lyapunov exponent since runs of $\sim 10^5$ time units resulted in a return to the original initial conditions within $\sim 10^{-7}$ or $10^{-8}$ when $\lambda=0$ (just below the predicted threshold of chaos), while the deviations from the starting point after time reversal were orders of magnitude higher for $\lambda>0$ (just above), often $\sim 1$ or higher. The loss of accuracy seemed to imply that chaotic oscillations first occur only in a small subgroup of neighboring links, and that as $\alpha$ is increased further, such oscillations then spread to the remainder of the array.

The rate at which the volume of an infinitesimal $N$-dimensional box in phase space evolves with time was monitored to determine whether the chaotic solutions were indeed strange *attractors*. Just above the threshold for chaos this rate was found generally to oscillate around zero indicating no observable volume shrinkage implying either an extremely weak attractor (which is not easily verifiable due to numerical error), or a solution akin to a chaotic trajectory of a Hamiltonian system for which Liouville's theorem precludes attraction. Well above the threshold, however, associated with much larger $\lambda$ values, the volume rate was negative attesting to the attracting nature of the solution.

## IV. RESULTS AND CONCLUSIONS

Figure 1 gives the $\alpha$ threshold for chaos as a function of $N$, the number of links, in both Cases 1 and 2. Because in Case 1 "optimum" conditions for producing chaos were in place, the thresholds are generally lower than for Case 2, the perhaps more realistic simulation of an experimental or technological system. We draw attention to the fact that in both cases the threshold drops steadily with increasing numbers of links, despite the nearest-neighbor aspect of the mutual interaction. The more rapid initial drop at small $N$ fits with the nonexistence of chaos for $N<3$, perhaps in rough analogy to a requirement of infinite coupling there.

One might infer from both cases that the thresholds will drop indefinitely close to $\alpha=0$ as $N\to+\infty$. This is impossible to determine definitively by any numerical calculation involving necessarily finite amounts of computing time. Suffice it to say that already at $N=18$ and $N=144$ in the respective cases, the threshold is small compared to an experimental[7] $\alpha\sim 0.1$. This warrants the assertion that chaos will be virtually generic to any system involving extended arrays of oscillating weak links in close proximity to each other, as envisaged in technological applications, unless, for example, they were contrived by extreme uniformity of manufacture and strong coupling to operate in a totally frequency-locked coherent state. Thus, rela-

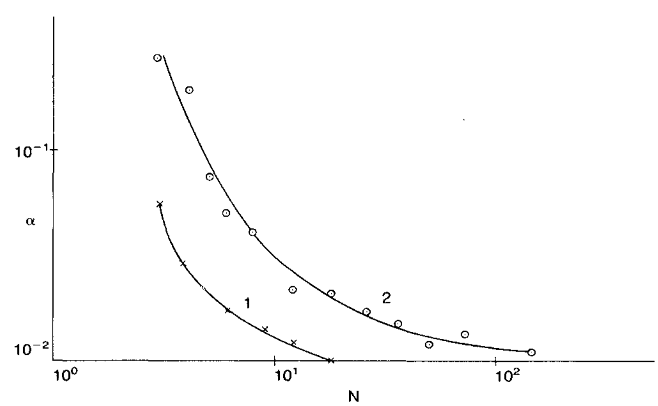


FIG. 1. Logarithmic plots of the threshold values of the dimensionless coupling parameter $\alpha$ against $N$, the number of weak links in the array. The numerals 1 and 2 adjacent to the curves refer to Cases 1 and 2 of the text. Smooth curves are drawn through the actual data at the integer values of $N$ indicated, only as a visual guide to the general trend.

tively high noise levels could be expected in extended Josephson systems.

From a mathematical view, the results are consonant with the Ruelle-Takens scenario which asserts that a nonlinear system with three or more independent frequencies may be unstable against chaotic behavior. Our results indicate that this proclivity for chaos in fact increases steadily with increasing numbers of independent frequencies.

## ACKNOWLEDGMENTS

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