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ON REAL SOLUTIONS OF THE EQUATION $\Phi^t(A) = \frac{1}{n}J_n$*

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Abstract. For a class of $n \times n$-matrices, we get related real solutions to the matrix equation $\Phi^t(A) = \frac{1}{n}J_n$ by generalizing the approach of and applying the results of Zhang, Yang, and Cao [*SIAM J. Matrix Anal. Appl.*, 21 (1999), pp. 642–645]. These solutions contain not only those obtained by Zhang, Yang, and Cao but also some which are neither diagonally nor permutation equivalent to those obtained by Zhang, Yang, and Cao. Therefore, the open problem proposed by Zhang, Yang, and Cao in the cited paper is solved.

Key words. Hadamard product, diagonally equivalent, permutation equivalent

AMS subject classifications. 15A24, 93A99, 65F99

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1. Introduction.

For a given positive integer n, let $M_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ be the sets of all $n \times n$ real matrices and all $n \times n$ real nonsingular matrices, respectively. Two important members of $M_n(\mathbb{R})$ are the $n \times n$ identity and all-one matrix, denoted as I_n and J_n, respectively.

For $A = (a_{ij})$ and $B = (b_{ij})$ in $M_n(\mathbb{R})$, the Hadamard product of A and B is defined as $A \circ B = (a_{ij}b_{ij}) \in M_n(\mathbb{R})$. Then we define

$$\Phi : GL_n(\mathbb{R}) \to M_n(\mathbb{R})$$

by

$$\Phi(A) = A \circ A^{-T}, \qquad A \in GL_n(\mathbb{R}),$$

where A^{-T} means the inverse transpose, $(A^{-1})^T$, of A. The mapping Φ arises in mathematical control theory in chemical engineering design problems. The basic question about Φ is to determine its range.

It is easy to see that every matrix in the range of Φ has row and column sums 1. However, the converse is not true. In fact, Johnson and Shapiro [1] showed that the equation $\Phi(A) = \frac{1}{3}J_3$ has no real solutions. So they asked whether the equation

$$\Phi(A) = \frac{1}{n}J_n$$

has a real solution. This problem was solved by Zhang, Yang, and Cao [2]. In fact, they studied the more general problem, i.e., the existence of real solutions of the equation

$$\Phi^t(A) = \frac{1}{n}J_n \tag{1}$$

for any positive integer t, where Φ^t is the mapping Φ applied t times.

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The purpose of this note is to obtain some new solutions to (1) by generalizing the approach of and applying the results of [2]. Some of the solutions are neither diagonally equivalent nor permutation equivalent to those obtained in [2]. Hence, the open problem proposed in [2] is solved.

The organization of this note is as follows. First, by an example, we partially answer the open problem in [2] and introduce the notion of permutation equivalence. Then we obtain new solutions to (1) which are related to a class of $n \times n$-matrices. As a result, the above-mentioned open problem is solved.

2. An example. First, we recall some results about the mapping Φ.

LEMMA 1 (see [1, Observations 2 and 4]). *For $A \in GL_n(\mathbb{R})$,*

(i) *if D and E in $GL_n(\mathbb{R})$ are diagonal, then $\Phi(DAE) = \Phi(A)$;*

(ii) *if P and Q in $GL_n(\mathbb{R})$ are permutation matrices, then $\Phi(PAQ) = P\Phi(A)Q$.*

When $n = 4$ and $t = 1$, the unique solution to (1) with respect to diagonal equivalence obtained in [2] is

$$A = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}. \tag{2}$$

Let P be the permutation

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then, by Lemma 1,

$$\Phi(PA) = P\Phi(A) = P\left(\frac{1}{4}J_4\right) = \frac{1}{4}J_4.$$

But PA is not diagonally equivalent to A. If not, let $D = \operatorname{diag}(d_1, d_2, d_3, d_4)$ and $E = \operatorname{diag}(e_1, e_2, e_3, e_4)$ be two nonsingular diagonal matrices such that

$$DAE = PA.$$

Particularly, we have $d_1e_1 = -1$, $d_3e_1 = 1$, $d_1e_3 = 1$, and $d_3e_3 = 1$. This is impossible since the first two equations give $d_1 = -d_3$ while the last two give $d_1 = d_3$.

With (1) and the example in mind, Lemma 1 leads us to introduce the notion of permutation equivalence in a similar way to diagonal equivalence (see [2]).

Zhang, Yang, and Cao [2] also proved that, for $n = 2$ and $t = 1$, there is only one nondiagonally equivalent solution to (1) and no solution to (1) when $n = 2$ and $t \geq 2$. This, combined with the result of Johnson and Shapiro [1], intrigues us, and we naturally ask, for $n \geq 4$, whether there exist real solutions to (1) which are neither diagonally equivalent nor permutation equivalent to those found by Zhang, Yang, and Cao [2]. A positive answer will be given in the next section. In this section, we give a partial answer for the case where $n = 4$ and $t = 1$. For $a \neq 0$, let

$$A_a = \begin{pmatrix} 1 & a & -1 & a \\ a & 1 & a & -1 \\ -1 & a & 1 & a \\ a & -1 & a & 1 \end{pmatrix}. \tag{3}$$

Then $|A_a| = -16a^2 \neq 0$ and

$$A_a^{-1} = \frac{1}{4}\begin{pmatrix} 1 & a^{-1} & -1 & a^{-1} \\ a^{-1} & 1 & a^{-1} & -1 \\ -1 & a^{-1} & 1 & a^{-1} \\ a^{-1} & -1 & a^{-1} & 1 \end{pmatrix}.$$

Therefore,

$$\Phi(A_a) = \frac{1}{4}J_4.$$

Now, assume that $a \in \mathbb{R} \setminus \{0, -1, 1\}$. Obviously, A_a is not permutation equivalent to A given in (2). Now, we claim that A_a is not diagonally equivalent to A. In fact, if $D = \operatorname{diag}(d_1, d_2, d_3, d_4)$ and $E = \operatorname{diag}(e_1, e_2, e_3, e_4)$ are two nonsingular diagonal matrices such that

$$DAE = A_a,$$

then we have $e_1 = e_3$, $e_2 = e_4 = -ae_1$, $d_1 = d_3$, $d_2 = d_4 = -ad_1$, $d_1e_1 = -1$, and $d_2e_2 = -1$. Thus $-1 = d_2e_2 = a^2d_1e_1 = -a^2$ or $a^2 = 1$, a contradiction.

Remark 1. To obtain the form of A_a in (3), we tried the cyclic matrix generated by $(1, a, b, c)$. By requiring it satisfy $\Phi(A) = \frac{1}{4}J_4$, we get $b = 1$ or -1. But $b = 1$ is not suitable. Taking $b = -1$, we have $a = c$. Fortunately, the matrix we get satisfies $\Phi(A) = \frac{1}{4}J_4$. We believe that this approach is also applicable to deal with the general case. However, in the following section, by generalizing the approach of and using the results of Zhang, Yang, and Cao [2], we give a new approach to the general case.

3. New solutions to $\boldsymbol{\Phi^t(A) = \frac{1}{n}J_n}$. In this section we always assume $n \geq 4$. Note that what is important in the presentation of Zhang, Yang, and Cao [2] are some special properties of J_n such as $J_n^2 = nJ_n$ and $J_n \circ J_n^T = J_n$. Inspired by this observation, for $k \in \mathbb{R}$ such that $k \leq 0$ or $k \geq 2n-4$ (these restrictions on k will be clear later), we introduce a subset $M_{n,k}(\mathbb{R})$ of $M_n(\mathbb{R})$ as follows:

$$M_{n,k}(\mathbb{R}) = \{A \in M_n(\mathbb{R}); A \circ A^T = J_n, a_{ii} = 1 \text{ for } i = 1, \ldots, n,\ A^2 = kI_n + (n-k)A\}.$$

Generalizing the approach of and applying some results of Zhang, Yang, and Cao [2], we show that for each $A \in M_{n,k}(\mathbb{R})$ there exist corresponding solutions to (1).

Examples. Let $a = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}$ with $a_i \neq 0$ for $i \in \{1, \ldots, n-1\}$. Define $A^a \in M_n(\mathbb{R})$ by

$$A^a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \prod\limits_{k=i}^{j-1} a_k & \text{if } i < j, \\ \dfrac{1}{\prod\limits_{k=j}^{i-1} a_k} & \text{if } i > j. \end{cases}$$

Then it is easy to show that $A^a \in M_{n,0}(\mathbb{R})$. Particularly, when $a_i = 1$ for $i \in \{1, \ldots, n-1\}$, $A^a = J_n$. Furthermore, define $\tilde{A}^a \in M_n(\mathbb{R})$ by

$$\tilde{A}^a_{ij} = \begin{cases} A^a_{ij} & \text{if } i = j, \\ (-1)^{i-j+1}A^a_{ij} & \text{if } i \neq j. \end{cases}$$

Then $\tilde{A}^a \in M_{n,2n-4}(\mathbb{R})$. Note that neither A^a nor $\tilde{A}^a$ is symmetric if there exists an $i_0 \in \{1, \ldots, n-1\}$ such that $a_{i_0}^2 \neq 1$.

The following results can be easily proved and hence the proofs are omitted.

LEMMA 2. *Let $A \in M_{n,k}(\mathbb{R})$. If $a[a+b(n-k)]-b^2k \neq 0$, then $aI_n+bA \in GL_n(\mathbb{R})$ with*

$$(aI_n + bA)^{-1} = \frac{1}{a[a+b(n-k)] - b^2k}\{[a+b(n-k)]I_n - bA\}$$

and therefore

$$\Phi(aI_n + bA) = \frac{1}{a[a+b(n-k)] - b^2k}\{[a^2 + ab(n-k) + b^2(n-k)]I_n - b^2 J_n\}.$$

LEMMA 3. *Let $A \in M_{n,k}(\mathbb{R})$ and $\lambda \neq k$. Denote*

$$A(\lambda) = \frac{1}{\lambda - k}[(\lambda + n - k)I_n - A]. \tag{4}$$

Then

(i) *$A(\lambda) \in GL_n(\mathbb{R})$ if $\lambda \neq \frac{-(n-k)\pm\sqrt{(n-k)^2+4k}}{2}$;*

(ii) *if $\lambda \neq \frac{-(n-k)\pm\sqrt{(n-k)^2+4k}}{2}$, we have $\Phi(A(\lambda)) = J_n(\mu)$, where $\mu = \lambda(\lambda+n-k)-k$;*

(iii) *$A(\alpha)$ and $A(\beta)$ are not diagonally equivalent if $\alpha \neq \beta$.*

Proof. Since $J_n \in M_{n,0}(\mathbb{R})$, it follows from (4) that

$$J_n(\lambda) = \frac{1}{\lambda}[(\lambda + n)I_n - J_n] \tag{5}$$

for $\lambda \neq 0$. Now, (i) and (ii) follow easily from Lemma 2. The proof of (iii) is similar to that of (iii) of Lemma 2 of Zhang, Yang, and Cao [2]. This completes the proof of the lemma. □

THEOREM 1. *Let $A \in M_{n,k}(\mathbb{R})$ and t be a positive integer. Then if $n > 4$, there are 2^t distinct, real values of λ such that $\Phi^t(A(\lambda)) = \frac{1}{n}J_n$ and hence (1) has at least 2^t nondiagonally equivalent solutions. When $n = 4$, there are 2^{t-1} distinct, real values of λ such that $\Phi^t(A(\lambda)) = \frac{1}{4}J_4$ and hence (1) has at least 2^{t-1} nondiagonally equivalent solutions.*

Proof. We prove only the theorem for the case where $n > 4$. The proof is similar for the case where $n = 4$. First note that the nondiagonal equivalence follows from (iii) of Lemma 3. Second, it follows from (5) that

$$J_n(-n) = \frac{1}{n}J_n. \tag{6}$$

Now we distinguish two cases to complete the proof.

Case 1. $t = 1$. Consider the equation

$$\lambda(\lambda + n - k) - k = -n. \tag{7}$$

Note $\Delta = (n-k)^2 - 4(n-k) = (n-k)(n-k-4) > 0$ since $k \leq 0$ or $k \geq 2n-4$. Thus (7) has two distinct real solutions λ_1 and λ_2. By (6) and (ii) of Lemma 3,

$$\Phi(A(\lambda_1)) = \Phi(A(\lambda_2)) = J_n(-n) = \frac{1}{n}J_n.$$

Case 2. $t > 1$. It follows from (ii) of Lemma 3, for $\mu \neq -n$, that

$$\Phi(J_n(\mu)) = J_n(f(\mu)),$$

where $f(\mu) = \mu(\mu + n)$. Thus, if $\lambda \neq \frac{-(n-k)\pm\sqrt{(n-k)^2+4k}}{2}$, using (ii) of Lemma 3 again, we have

$$\Phi^t(A(\lambda)) = \Phi^{t-1}(\Phi(A(\lambda))) = \Phi^{t-1}(J_n(\mu)) = J_n(f^{t-1}(\mu)), \tag{8}$$

where $\mu = \lambda(\lambda+n-k)-k$. Lemma 3 of Zhang, Yang, and Cao [2] tells us that in the interval $(-\frac{n^2}{4}, 0]$ there are 2^{t-1} distinct, real solutions to the equation $f^{t-1}(\mu) = -n$, say $\mu_1, \dots, \mu_{2^{t-1}}$. For $i = 1, \dots, 2^{t-1}$, consider

$$\lambda(\lambda + n - k) - k = \mu_i. \tag{9}$$

Noting

$$\begin{array}{rcl} \Delta & = & (n-k)^2 + 4(k + \mu_i) \\ & = & (n^2 + 4\mu_i) + (k^2 - 2nk + 4k) \\ & > & k^2 - 2nk + 4k \\ & = & k[k - (2n - 4)] \\ & \geq & 0 \end{array}$$

(from here you see why we require $k \leq 0$ or $k \geq 2n - 4$), we know that (9) has two distinct, real solutions, say $\lambda_{i,1}$ and $\lambda_{i,2}$. Moreover, it is easy to see that all $\lambda_{1,1}$, $\lambda_{1,2}$, $\dots$, $\lambda_{2^{t-1},1}$, and $\lambda_{2^{t-1},2}$ are distinct. Thus it follows from (6) and (8) that

$$\Phi^t(A(\lambda_{i,j})) = J_n(f^{t-1}(\mu_i)) = J_n(-n) = \frac{1}{n}J_n, \qquad i = 1, \dots, 2^{t-1}, j = 1, 2,$$

and the proof is complete. $\square$

Remark 2. For $A \in M_{n,k}(\mathbb{R})$, we can find 2^t and 2^{t-1} mutually nondiagonally equivalent real solutions $A(\lambda_0)$ to (1) for $n > 4$ and $n = 4$, respectively, where λ_0 satisfies the following inverted iteration (see Theorem 1 here and Remark 1 of Zhang, Yang, and Cao [2]):

$$\begin{cases} \lambda_t = -n, \\ \lambda_k = \frac{-n\pm\sqrt{n^2+4\lambda_{k+1}}}{2}, \quad k = 1, \dots, t-1, \\ \lambda_0 = \frac{-(n-k)\pm\sqrt{(n-k)^2+4(\lambda_1+k)}}{2}. \end{cases}$$

Remark 3. For $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$ with $a_i \neq 0$ for $i \in \{1, \dots, n-1\}$, let A^a and $\tilde{A}^a$ be defined as in the examples above. Then we can easily show that the solutions associated with A^a and $\tilde{A}^a$ are diagonally equivalent to those associated with $A^{\mathbf{1}}$ and $\tilde{A}^{\mathbf{1}}$, respectively, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{n-1}$. But the solutions associated with $A^{\mathbf{1}}$ and $\tilde{A}^{\mathbf{1}}$ are neither diagonally equivalent nor permutation equivalent. Note that the solutions associated with $A^{\mathbf{1}}$ are just those obtained by Zhang, Yang, and Cao [2] and hence the open problem, whether there are real solutions to (1) which are not diagonally equivalent to those found in [2] when $n \geq 4$, proposed by them is solved. Moreover, when $n = 4$, $t = 1$ and $a \in \mathbb{R} \setminus \{0, -1, 1\}$, the solutions given in section 2 are not associated with any $A \in M_{n,k}(\mathbb{R})$.

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