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# STABILITY IMPLICATIONS OF BENDIXSON'S CRITERION*

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**Abstract.** This note presents a proof that the omega limit set of a solution to a planar system satisfying the Bendixson criterion is either empty or is a single equilibrium. The proof involves elementary techniques which should be accessible to senior undergraduates and graduate students.

**Key words.** Bendixson's criterion, omega limit set

**AMS subject classification.** 3401

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We consider a planar differential equation

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $P$ and $Q$ are real-valued, continuously differentiable functions on $\mathbb{R}^2$. A solution $(x(t),\, y(t))$ is uniquely determined by its initial value $(x_o,\, y_o) = (x(0),\, y(0))$. If the solution exists for all $t \geq 0$, its positive semiorbit is $C^+(x_o,\, y_o) = \{(x(t),\, y(t)) : t \in [0,\infty)\}$, and if the solution exists for all real $t$, its orbit is $C(x_o,\, y_o) = \{(x(t),\, y(t)) : t \in (-\infty,\infty)\}$.

In senior undergraduate courses and introductory graduate courses on ordinary differential equations, the questions of existence and stability of periodic solutions of (1) are frequently studied. A solution $(x(t),\, y(t))$ is periodic with period $\omega > 0$ if $x(t+\omega) = x(t)$ and $y(t+\omega) = y(t)$ for all $t$. Clearly, the orbit of such a solution is either a simple closed curve or, in the case of constant solutions, a single point called an equilibrium.

The *Poincaré–Bendixson theory* for system (1) shows that if $C^+(x_o,\, y_o)$ is bounded and the omega limit set $\Omega(x_o,\, y_o) = \bigcap_{t\geq 0} \overline{C^+(x(t),\, y(t))}$ contains no equilibria, then $\Omega(x_o,\, y_o)$ is the orbit of a nonconstant periodic solution of (1), cf [4, p. 46]. On the negative side, if *Bendixson's criterion* $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ on $\mathbb{R}^2$ is satisfied, then no nonconstant periodic solutions of (1) exist, cf [4, p. 39]. Thus, every semiorbit of a system satisfying Bendixson's criterion is either unbounded or its omega limit set contains an equilibrium. In fact, a stronger assertion holds. The omega limit set is a single equilibrium or is empty.

*Suppose that* $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ *on* $\mathbb{R}^2$ *and that* $\Omega(x_o,\, y_o)$ *is nonempty. Then*

$$\lim_{t\to\infty} \big(x(t),\, y(t)\big) = \big(\bar{x},\, \bar{y}\big),$$

*where* $P(\bar{x},\, \bar{y}) = Q(\bar{x},\, \bar{y}) = 0$.

This statement is a special case of a result for higher-dimensional systems satisfying generalized forms of Bendixson's criterion established by Smith [3] and by Li and Muldowney [2]. The proofs are nonelementary in that they rely heavily on the Pugh closing lemma and results such as the centre manifold theorem. We present here a more accessible proof for 2-dimensional systems which relies only on the content of a typical introductory course.

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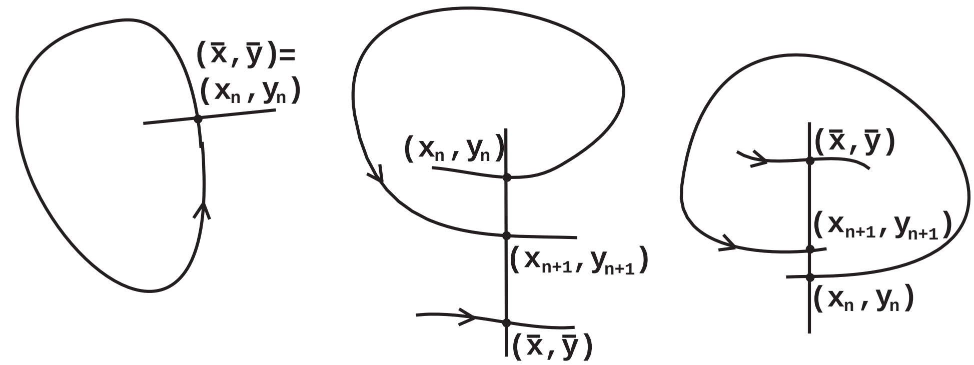


**Case 1** **Case 2** **Case 3**

Suppose that $(\bar{x},\, \bar{y})$ is an omega limit point which is not an equilibrium, and let $T$ be a transversal through $(\bar{x},\, \bar{y})$. That is, let $T$ be a straight line segment through $(\bar{x},\, \bar{y})$ such that the vector $(P(x,\, y),\, Q(x,\, y))$ is neither zero nor parallel to $T$ at any $(x,\, y) \in T$. Thus there is a neighborhood of $(\bar{x}, \bar{y})$ such that the orbit of any point in this neighborhood crosses $T$ and all crossings of $T$ are in the same direction.

Uniqueness of solutions implies that successive intersections of $T$ by an orbit are monotone on $T$, and so one of the situations displayed in the diagrams as Case 1, Case 2, and Case 3 must occur where $t_n < t_{n+1}$ and $(x_n,\, y_n) = (x(t_n),\, y(t_n)) \in T$.

Let $T_n$ be the segment of $T$ joining $(x_n,\, y_n)$ and $(x_{n+1},\, y_{n+1})$. Let $C_n$ be the segment of $C^+(x_o,\, y_o)$ joining $(x_n,\, y_n)$ and $(x_{n+1},\, y_{n+1})$. Let $D_n$ be the region bounded by $T_n$ and $C_n$. Then, by Green's theorem, we have

$$
\begin{aligned}
\int_{D_n} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy &= \int_{T_n} (P\, dy - Q\, dx) \;+ \int_{C_n} (P\, dy - Q\, dx) \\
&= \int_{T_n} (P\, dy - Q\, dx)
\end{aligned} \tag{2}
$$

since $\int_{C_n} (P\, dy - Q\, dx) = \pm \int_{t_n}^{t_{n+1}} (P\, \frac{dy}{dt} - Q\, \frac{dx}{dt})\, dt = 0$ from (1) where the sign of the expression is determined by the orientation of the curve.

We can now rule out Case 1. In this case, $T_n$ is merely the point $\big(x_n,\, y_n\big)$ and so the right-hand side of (2) is zero. The left-hand side, however, is nonzero since the integrand is of constant sign and the domain of integration is nontrivial. This contradiction shows that $C^+(x_o,\, y_o)$ does not self-intersect; there are no nonconstant periodic orbits as asserted by Bendixson's criterion. In fact, this is the classical proof of Bendixson's result.

Consider Case 2. By Case 1, $C^+(x_o,\, y_o)$ does not self-intersect. Since the $(x_n, y_n)$ are monotone on $T$ we find $D_1 \subset D_2 \subset \dots$. The integrand has constant sign so that the left-hand side of (2) increases in magnitude as $n$ increases. The sequence $\{(x_n,\, y_n)\}$ converges to $(\bar{x},\, \bar{y})$, so the length of segment $T_n$ approaches zero as $n$ tends to infinity. Combining this with the fact that $P$ and $Q$ are continuous and so must be bounded in a neighborhood of $(\bar{x},\, \bar{y})$, we see that $\lim_{n\to\infty} \int_{T_n} (P\, dy - Q\, dx) = 0$. This, from (2), gives us a contradiction, ruling out Case 2.

Finally, consider Case 3. In this case $D_1 \supset D_2 \supset \dots$, and we will show that there is a nonempty open set $U$ such that $U \subset D_n$ for all $n$, and therefore, from (2),

$$0 < \Big|\int_U \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dx\,dy\Big| \le \Big|\int_{D_n} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dx\,dy\Big| = \Big|\int_{T_n} \big(P\,dy - Q\,dx\big)\Big|.$$

But, $\lim_{n\to\infty} \int_{T_n} (P\,dy - Q\,dx) = 0$ as before, so again we have a contradiction ruling out Case 3. To construct $U$, observe that the transversal $T$ is divided into two separate line segments by $(\bar{x},\,\bar{y})$. One of these line segments $T'$ contains the sequence $\{(x_n,\,y_n)\}$. The other line segment $T''$ does not intersect $C^+(x_o,\,y_o)$, so for all $n$, we have $T'' \subset D_n$. Each $D_n$ is positively invariant: the positive semiorbit of each point in $D_n$ lies in $D_n$. Thus semiorbits that begin in $T''$ lie in each $D_n$ for all $t > 0$. Let $I$ be an open ended subsegment of $T''$. Then $U = \{(x(t),\,y(t)) : t \in (0,1),\,(x(0),\,y(0)) \in I\}$ is a nonempty subset of $D_n$ as asserted. This resolves Case 3.

Thus we see that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \ne 0$ implies that the omega limit set of a bounded orbit consists entirely of equilibria. The final step is to show that this omega limit set is a single equilibrium.

Suppose that $\Omega(x_o,\,y_o)$ contains more than a single point. Either $\Omega(x_o,\,y_o)$ is connected or each connected component of $\Omega(x_o,\,y_o)$ is unbounded, as can be seen by slightly modifying the proof of Theorem 1.1 in [1, p. 145]. In either case, there can be no isolated omega limit points. Let $(\bar{x},\,\bar{y}) \in \Omega(x_o,\,y_o)$. Let $M(x,y)$ denote the Jacobian matrix at $(x,y)$ of the map $f : (x,y) \mapsto (P(x,y), Q(x,y))$. This matrix is singular at each $(x,y) \in \Omega(x_o,y_o)$ since the solutions $(x,y)$ of $P(x,y) = 0,\ Q(x,y) = 0$ are not isolated, and therefore $f$ is not locally one-to-one. Thus one of the eigenvalues $\lambda_1, \lambda_2$ of $M(\bar{x},\bar{y})$ is zero. Since

$$\lambda_1 + \lambda_2 = \operatorname{Tr} M = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \ne 0,$$

it follows that there is a nonzero eigenvalue of $M(\bar{x},\bar{y})$. Without loss of generality, it may be assumed that $(\bar{x},\bar{y}) = (0,0)$ and

$$P(x,y) = \lambda x + p(x,y), \quad Q(x,y) = q(x,y),$$

where $\lambda$ is the nonzero eigenvalue of $M(0,0)$ and $p(0,0) = q(0,0) = 0$, $Dp(0,0) = Dq(0,0) = 0$. This can always be achieved by an affine transformation. The implicit function theorem implies $P(x,y) = 0$ in a neighborhood $V$ of $(0,0)$ if and only if $x = g(y)$, where $g$ is a continuously differentiable function on a neighborhood of 0 such that $g(0) = 0$ and $g'(y) = -\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x}$. Moreover, $Q(x,y) = 0$ if $x = g(y)$ since $Q(0,0) = 0$ and $\frac{d}{dy} Q(g(y),y) = \frac{\partial Q}{\partial x} g'(y) + \frac{\partial Q}{\partial y} = -\frac{\partial Q}{\partial x}(\frac{\partial P}{\partial y}/\frac{\partial P}{\partial x}) + \frac{\partial Q}{\partial y} = \det M(x,y) \,/\, \frac{\partial P}{\partial x} = 0$ since $M(x,y)$ is singular when $x = g(y)$.

Each equilibrium $(g(y),y) \in V$ has a one-dimensional stable manifold if $\lambda < 0$ and has a one-dimensional unstable manifold if $\lambda > 0$. Consider a neighborhood $B$ of $(0,0)$ such that the boundary of $B$ is a simple closed curve formed by arcs from the stable or unstable manifolds of $(g(y_*), y_*)$ and $(g(-y_*), -y_*)$ and from the curves $x = g(y) + c$ and $x = g(y) - c$. If $c > 0$ and $y_* > 0$ are chosen sufficiently small, then $B \subset V$ and there is a point in $\Omega(x_o, y_o)$ which is not in $B$. Thus the interior of $B$ and that of its complement both intersect $\Omega(x_o, y_o)$, so $C^+(x_o, y_o)$ enters and exits $B$ infinitely many times; it must do so through the arcs $x = g(y) + c$ and $x = g(y) - c$ since stable and unstable manifolds are invariant. At least one of these arcs must therefore contain a point of $\Omega(x_o, y_o)$ and so is an equilibrium. This contradicts the fact that all equilibria in $B$ are in the curve $x = g(y)$.

This establishes that $\Omega(x_o, y_o)$ contains at most one point, and so, as asserted, $\lim_{t\to\infty}(x(t), y(t)) = (\bar{x},\,\bar{y})$ since $(\bar{x},\bar{y})$ is the only omega limit point.

This result can be easily extended to Dulac's criterion. Namely, if there exists a scalar function $\alpha$ defined on $\mathbb{R}^2$ such that $\frac{\partial(\alpha P)}{\partial x} + \frac{\partial(\alpha Q)}{\partial y} \neq 0$ on $\mathbb{R}^2$, then every omega limit set is either empty or a single point.

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