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Yucai Su
Shanghai Jiaotong University

Kaiming Zhao
Wilfrid Laurier University, kzhao@wlu.ca

Linsheng Zhu
Changshu College

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Yucai Su
Department of Mathematics, Shanghai Jiaotong University, Shanghai 200030, China

Kaiming Zhao
Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada N2L 3C5, and Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China

Linsheng Zhu
Department of Mathematics, Changshu College, Jiangsu 215500, China

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We classify the pairs $(\mathcal{A},\mathcal{D})$ consisting of an (ϵ,Γ)-color-commutative associative algebra $\mathcal{A}$ with an identity element over an algebraically closed field $\mathbb{F}$ of characteristic zero and a finite dimensional subspace $\mathcal{D}$ of (ϵ,Γ)-color-commutative locally finite color-derivations of $\mathcal{A}$ such that $\mathcal{A}$ is Γ-graded $\mathcal{D}$-simple and the eigenspaces for elements of $\mathcal{D}$ are Γ-graded. Such pairs are the important ingredients in constructing some simple Lie color algebras which are in general not finitely-graded. As some applications, using such pairs, we construct new explicit simple Lie color algebras of generalized Witt type, Weyl type. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1628837]

I. INTRODUCTION

Lie color algebras, a notion first appeared in mathematical physics,[1,3,5–7,15] are generalizations of Lie algebras and Lie superalgebras. Let us start with the definition. Let $\mathbb{F}$ be an algebraically closed field of characteristic zero and let Γ be an additive group. A *skew-symmetric bicharacter* of Γ is a map $\epsilon:\Gamma\times\Gamma\to\mathbb{F}^\times=\mathbb{F}\backslash\{0\}$ satisfying

$$\epsilon(\lambda,\mu)=\epsilon(\mu,\lambda)^{-1},\quad \epsilon(\lambda,\mu+\nu)=\epsilon(\lambda,\mu)\epsilon(\lambda,\nu),\quad \forall\,\lambda,\mu,\nu\in\Gamma. \tag{1.1}$$

It is clear that

$$\epsilon(\lambda,0)=1,\quad \forall\,\lambda\in\Gamma. \tag{1.2}$$

Let $L=\oplus_{\lambda\in\Gamma}L_\lambda$ be a Γ-graded $\mathbb{F}$-vector space. For a nonzero homogeneous element a, denote by $\overline{a}$ the unique group element in Γ such that $a\in L_{\overline{a}}$. We shall call $\overline{a}$ the *color* of a. The $\mathbb{F}$-bilinear map $[\cdot,\cdot]:L\times L\to L$ is called a Lie color bracket on L if the following conditions are satisfied:

$$[a,b]=-\epsilon(\overline{a},\overline{b})[b,a]\quad \text{(skew symmetry)},$$

$$[a,[b,c]]=[[a,b],c]+\epsilon(\overline{a},\overline{b})[b,[a,c]]\quad \text{(Jacobi identity)},$$

for all homogeneous elements $a,b,c\in L$. The algebra structure $(L,[\cdot,\cdot])$ is called an (ϵ,Γ)-*Lie color algebra or simply a Lie color algebra*. If $\Gamma=\mathbb{Z}/2\mathbb{Z}$ and $\epsilon(i,j)=(-1)^{ij},\forall\, i,j\in\mathbb{Z}/2\mathbb{Z}$, then (ϵ,Γ)-Lie color algebras are simply Lie superalgebras. For Lie color algebras, we refer the reader to Ref. 1.

For any Γ-graded $\mathbb{F}$-vector space V, we denote

$$H(V)=\{\text{all homogeneous elements in } V\}.$$

Let $\mathcal{A}=\oplus_{\lambda\in\Gamma}\mathcal{A}_\lambda$ be a Γ-graded associative $\mathbb{F}$-algebra with an identity element 1, i.e., $\mathcal{A}_\lambda\mathcal{A}_\mu\subset\mathcal{A}_{\lambda+\mu}$ for all $\lambda,\mu\in\Gamma$. So $1\in\mathcal{A}_0$. We say that $\mathcal{A}$ is *graded simple* if $\mathcal{A}$ does not have nontrivial Γ-graded ideals. If we define the bilinear product $[\cdot,\cdot]$ on $\mathcal{A}$ by

$$[x,y]=xy-\epsilon(\overline{x},\overline{y})yx,\quad \forall\, x,y\in H(\mathcal{A}),\tag{1.3}$$

then $(\mathcal{A},[\cdot,\cdot])$ becomes a Lie color algebra.

A *Lie color ideal* U of $\mathcal{A}$ is a Γ-graded vector space U of $\mathcal{A}$ such that $[\mathcal{A},U]\subset U$. Sometimes it is called an (ϵ,Γ)-Lie ideal. The *ϵ-center* $Z_\epsilon(\mathcal{A})$ of $\mathcal{A}$ is defined as

$$Z_\epsilon=Z_\epsilon(\mathcal{A})=\{x\in\mathcal{A}\,|\,[x,\mathcal{A}]=0\}.$$

It is easy to see that $Z_\epsilon(\mathcal{A})$ is Γ-graded. We say that $\mathcal{A}$ is *color-commutative* (or ϵ-color-commutative) if $Z_\epsilon(\mathcal{A})=\mathcal{A}$, i.e., $[\mathcal{A},\mathcal{A}]=0$.

Let $\mathcal{A}$ be an (ϵ,Γ)-color-commutative associative algebra with an identity element 1. A nonzero $\mathbb{F}$-linear transformation $\partial:\mathcal{A}\to\mathcal{A}$ is called a *homogeneous color-derivation* of degree $\lambda\in\Gamma$ if

$$\begin{aligned}&\partial(a)\in\mathcal{A}_{\lambda+\mu},\quad \forall\, a\in\mathcal{A}_\mu,\quad \mu\in\Gamma,\\ &\partial(ab)=\partial(a)b+\epsilon(\lambda,\overline{a})a\partial(b),\quad \forall\, a,b\in H(\mathcal{A}).\end{aligned}\tag{1.4}$$

For convenience, we shall often denote $\overline{\partial}=\lambda$ if ∂ has degree λ. Clearly $\partial(c)=0$ for all $c\in\mathbb{F}$. Denote $\mathrm{Der}^\epsilon(\mathcal{A})=\oplus_{\lambda\in\Gamma}\mathrm{Der}^\epsilon_\lambda(\mathcal{A})$, where $\mathrm{Der}^\epsilon_\lambda(\mathcal{A})$ is the $\mathbb{F}$-vector space spanned by all homogeneous color derivations of degree λ. Similar to the Lie algebra case, it is easy to verify that $\mathrm{Der}^\epsilon_\lambda(\mathcal{A})$ becomes a Lie color algebra under the Lie color bracket

$$[\partial,\partial']=\partial\partial'-\epsilon(\overline{\partial},\overline{\partial'})\partial'\partial,\quad \forall\,\partial,\partial'\in H(\mathrm{Der}^\epsilon(\mathcal{A})),$$

where $\partial\partial'$ is the composition of the operators ∂ and ∂'.

Let $\mathcal{D}=\oplus_{\lambda\in\Gamma}\mathcal{D}_\lambda$ be an (ϵ,Γ)-color-commutative subspace of $\mathrm{Der}^\epsilon(\mathcal{A})$, i.e.,

$$\partial\partial'=\epsilon(\overline{\partial},\overline{\partial'})\partial'\partial,\quad \forall\,\partial,\partial'\in H(\mathcal{D}).\tag{1.5}$$

Recall that the associative algebra $\mathcal{A}$ is called *graded $\mathcal{D}$-simple* if $\mathcal{A}$ has no nontrivial graded $\mathcal{D}$-stable ideals.[4]

A linear transformation T on a vector space V is called *locally finite* if

$$\dim(\mathrm{span}\{T^m(v)\,|\,m\in\mathbb{N}\})<\infty,$$

for any $v\in V$. The transformation T is called *locally nilpotent* if for any $v\in V$, we have $T^n(v)=0$ for some $n\in\mathbb{N}$, and T is called *semisimple* if it acts diagonalizably on V.

For a pair $(\mathcal{A},\mathcal{D})$ of an (ϵ,Γ)-color-commutative associative algebra with an identity element and an (ϵ,Γ)-color-commutative subspace $\mathcal{D}$ of $\mathrm{Der}^\epsilon(\mathcal{A})$, Passman[4] proved that the Lie color algebra (including the Lie algebra case) $\mathcal{A}\mathcal{D}=\mathcal{A}\otimes\mathcal{D}$ is simple if and only if $\mathcal{A}$ is graded $\mathcal{D}$-simple and $\mathcal{A}\mathcal{D}$ acts faithfully on $\mathcal{A}$ (except a minor case). The authors of the present paper[11] (see also Refs. 9, 10, and 14) constructed (associative and Lie) color algebras of Weyl type $\mathcal{A}[\mathcal{D}]$, which is the color commutative algebra generated by $\mathcal{A}$ and $\mathcal{D}$ (as operators on $\mathcal{A}$), and proved that $\mathcal{A}[\mathcal{D}]$ is simple as an associative algebra or is *central simple* as a Lie color algebra (i.e., the derived subalgebra modulo its ϵ-center is simple) if and only if $\mathcal{A}$ is graded $\mathcal{D}$-simple (except a minor case in Lie case). However, it is still a question of how to construct new explicit simple Lie color algebras of generalized Witt type or Weyl type.

The problem of classifying all the pairs $(\mathcal{A},\mathcal{D})$ of a commutative associative algebra $\mathcal{A}$ with an identity element and a finite-dimensional locally finite commutative derivation subalgebra $\mathcal{D}$ such that $\mathcal{A}$ is *$\mathcal{D}$-simple* (i.e., $\mathcal{A}$ does not have $\mathcal{D}$-stable ideals), was settled in Ref. 8 (using the pairs

$(\mathcal{A},\mathcal{D})$, Xu constructed explicit simple Lie algebras of generalized Cartan type[12] and of generalized Block type[13]). However, this problem becomes much more complicated in color case.

In order to construct explicit new simple Lie color algebras of generalized Witt, Weyl types, the first aim of the present paper is to give a classification of all the pairs $(\mathcal{A},\mathcal{D})$ of an (ϵ,Γ)-color-commutative associative algebra $\mathcal{A}$ with an identity element over an algebraically closed field $\mathbb{F}$ of characteristic zero and a finite-dimensional subspace $\mathcal{D}$ of (ϵ,Γ)-color-commutative locally finite color derivations of $\mathcal{A}$ such that $\mathcal{A}$ is Γ-graded $\mathcal{D}$-simple and the eigenspaces for elements of $\mathcal{D}$ are Γ-graded (see Theorem 2.2). Then in Sec. III, as some applications, using the pairs $(\mathcal{A},\mathcal{D})$, we construct explicit new simple Lie color algebras (including Lie superalgebras) of generalized Witt, Weyl types (see Theorem 3.1).

II. $\mathcal{D}$-SIMPLE COLOR ALGEBRAS

In this section, we shall classify the pairs $(\mathcal{A},\mathcal{D})$ of an (ϵ,Γ)-commutating associative algebra $\mathcal{A}$ with an identity element 1 and a finite-dimensional subspace $\mathcal{D}$ of (ϵ,Γ)-commutative locally finite color derivations of $\mathcal{A}$ such that $\mathcal{A}$ is graded $\mathcal{D}$-simple and the eigenspaces for elements of $\mathcal{D}$ are Γ-graded.

First we would like to remark that the eigenspace of a derivation is not necessarily Γ-graded. Since we are considering Γ-graded algebras, it is natural that we require the eigenspaces for elements of $\mathcal{D}$ are Γ-graded.

We shall start with constructing explicitly such pairs $(\mathcal{A},\mathcal{D})$. The motivation to construct such pairs will become clear in the proof of Theorem 2.2 below. Actually, the proof of Theorem 2.2 leads us to the way to construct such pairs.

Set

$$\Gamma_+=\{\lambda\in\Gamma\mid\epsilon(\lambda,\lambda)=1\},\quad \Gamma_-=\{\lambda\in\Gamma\mid\epsilon(\lambda,\lambda)=-1\}.$$

Then by (1.1), Γ_+ is a subgroup of Γ with index $\leqslant 2$. For any graded subspace $\mathcal{B}$ of $\mathcal{A}$, we define $\mathcal{B}_+=\oplus_{\lambda\in\Gamma_+}\mathcal{B}_\lambda$, then $\mathcal{B}_+$ is Γ-graded. Similarly we can define $\mathcal{B}_-$. Since $\Gamma=\Gamma_+\cup\Gamma_-$, it follows that $\mathcal{B}=\mathcal{B}_+\oplus\mathcal{B}_-$. By (1.5), we have

$$a^2=0\quad\text{or}\quad\partial^2=0\quad\text{if}\ \overline{a}\in\Gamma_-\quad\text{or}\quad\overline{\partial}\in\Gamma_-. \tag{2.1}$$

For $m,n\in\mathbb{Z}$, we denote

$$\overline{m,n}=\{m,m+1,\ldots,n\}.$$

To construct the pair $(\mathcal{A},\mathcal{D})$, first we construct a Γ-*graded* ϵ-*commutative field extension* $\mathbb{E}$ of $\mathbb{F}$ (i.e., each nonzero homogeneous element of $\mathbb{E}$ is invertible). To do this, let $\Gamma^0\subset\Gamma_+$ be a subgroup of Γ and let $\mathbb{E}_0$ be a field extension of $\mathbb{F}$. Let $e:\Gamma^0\times\Gamma^0\to\mathbb{E}_0^\times=\mathbb{E}_0\backslash\{0\}$ be a 2-variable function $e:(\alpha,\beta)\mapsto e_{\alpha,\beta}$ such that

$$e_{\alpha,\beta}=\epsilon(\alpha,\beta)e_{\beta,\alpha},\quad e_{\alpha,0}=1,\quad e_{\alpha,\beta}e_{\alpha+\beta,\gamma}=e_{\alpha,\beta+\gamma}e_{\beta,\gamma},\quad\forall\ \alpha,\beta,\gamma\in\Gamma^0. \tag{2.2}$$

You will see that these are required by the associativity of the algebra we are going to construct. Let $\mathbb{E}=\mathbb{E}_0[\Gamma^0]=\mathrm{span}_{\mathbb{E}_0}\{E_\alpha\mid\alpha\in\Gamma^0\}$ be a Γ^0-graded ϵ-commutative associative algebra over $\mathbb{E}_0$ such that E_α has color $\overline{E_\alpha}=\alpha$, with the multiplication

$$E_\alpha\cdot E_\beta=e_{\alpha,\beta}E_{\alpha+\beta},\quad\forall\ \alpha,\beta\in\Gamma^0. \tag{2.3}$$

From (2.2) it is easy to see that $\mathbb{E}$ is a Γ-graded ϵ-commutative field extension of $\mathbb{F}$.

Let

$$\underline{k}=(k_1,k_2,k_3,k_4)\in\mathbb{N}^4\quad\text{such that}\ k=k_1+k_2+k_3+k_4>0.$$

We also require that $k_4=0$ if $\Gamma_-=\varnothing$. We shall construct $\mathcal{D}$ which will be spanned by color derivations $\partial_p, p\in\overline{1,k}$ such that

$$\partial_p \text{ is semisimple with color } \overline{\partial}_p=0, \quad \forall\ p\in\overline{1,k_1}, \tag{2.4}$$

$$\partial_{k_1+p} \text{ is locally finite but not semisimple with color } \overline{\partial}_{k_1+p}=0, \quad \forall\ p\in\overline{1,k_2}, \tag{2.5}$$

$$\partial_{k_1+k_2+p} \text{ is locally nilpotent with color } \overline{\partial}_{k_1+k_2+p}\in\Gamma_+, \quad \forall\ p\in\overline{1,k_3}, \tag{2.6}$$

$$\partial_{k_1+k_2+k_3+p} \text{ is locally nilpotent with color } \overline{\partial}_{k_1+k_2+k_3+p}\in\Gamma_-, \quad \forall\ p\in\overline{1,k_4} \tag{2.7}$$

[cf. (2.21) and (2.22)]. To this end, we first need to construct $\mathcal{A}$ which will be the tensor product of two algebras $\mathcal{A}=\mathcal{A}_1\otimes\mathcal{A}_2$ [cf. (2.19)] such that $\mathcal{A}_1$ is a "group-algebra-like" algebra [cf. (2.12)] and $\mathcal{A}_2$ is a "polynomial-like" algebra [cf. (2.16)].

Now we construct $\mathcal{A}_1$ such that $\partial_p|_{\mathcal{A}_1}$ are nonzero semisimple operators for $p\in\overline{1,k_1+k_2}$ and $\partial_{k_1+k_2+p}|_{\mathcal{A}_1}$ are zero operators for $p\in\overline{1,k_3+k_4}$ [cf. (2.4)–(2.7) and (2.21)–(2.22)]. To do this, let G be a *nondegenerate* additive subgroup of $\mathbb{F}^{k_1+k_2}$, i.e., G contains an $\mathbb{F}$-basis of $\mathbb{F}^{k_1+k_2}$. If $k_1+k_2=0$, we take $G=\{0\}$. An element in G is usually denoted by

$$\underline{a}=(a_1,a_2,\ldots,a_k) \quad \text{with } a_p=0,\ \forall\ p>k_1+k_2. \tag{2.8}$$

Let $\hat{}:G\to\Gamma_+$ be a map $\hat{}:\underline{a}\mapsto\hat{\underline{a}}$ satisfying

$$\hat{0}=0, \quad \theta_{\underline{a},\underline{b}}:=\hat{\underline{a}}+\hat{\underline{b}}-\widehat{\underline{a}+\underline{b}}\in\Gamma^0, \quad \forall\ \underline{a},\underline{b}\in G. \tag{2.9}$$

Let $f(\cdot,\cdot):G\times G\to\mathbb{E}_0^\times$ be a map such that

$$f(\underline{a},\underline{b})=\epsilon(\hat{\underline{a}},\hat{\underline{b}})f(\underline{b},\underline{a}), \quad f(\underline{a},0)=1, \tag{2.10}$$

$$e_{\theta_{\underline{a},\underline{b}},\theta_{\underline{a}+\underline{b},\underline{c}}}f(\underline{a},\underline{b})f(\underline{a}+\underline{b},\underline{c})=\epsilon(\hat{\underline{a}},\theta_{\underline{b},\underline{c}})e_{\theta_{\underline{b},\underline{c}},\theta_{\underline{a},\underline{b}+\underline{c}}}f(\underline{b},\underline{c})f(\underline{a},\underline{b}+\underline{c}), \tag{2.11}$$

for $\underline{a},\underline{b},\underline{c}\in G$. Denote by $\mathcal{A}_1=\mathcal{A}(G,\mathbb{E},f)$ the (ϵ,Γ)-color commutative associative algebra with $\mathbb{E}$-basis $\{x^{\underline{a}}\mid\underline{a}\in G\}$ or $\mathbb{E}_0$-basis $\{E_\alpha x^{\underline{a}}\mid(\alpha,\underline{a})\in\Gamma^0\times G\}$ such that $x^{\underline{a}}$ has color $\hat{\underline{a}}$ and

$$x^{\underline{a}}\cdot x^{\underline{b}}=f(\underline{a},\underline{b})E_{\theta_{\underline{a},\underline{b}}}x^{\underline{a}+\underline{b}}, \quad \forall\ \underline{a},\underline{b}\in G, \tag{2.12}$$

and in general

$$E_\alpha x^{\underline{a}}\cdot E_\beta x^{\underline{b}}=\epsilon(\hat{\underline{a}},\beta)e_{\alpha,\beta}e_{\alpha+\beta,\theta_{\underline{a},\underline{b}}}f(\underline{a},\underline{b})E_{\alpha+\beta+\theta_{\underline{a},\underline{b}}}x^{\underline{a}+\underline{b}}, \quad \forall\ \alpha,\beta\in\Gamma^0,\ \underline{a},\underline{b}\in G \tag{2.13}$$

[cf. (2.3)]. The ϵ-commutativity and associativity of $\mathcal{A}_1$ are guaranteed by conditions (2.10) and (2.11).

Now we shall construct $\mathcal{A}_2$ such that $\partial_p|_{\mathcal{A}_2}=0$ for $p\in\overline{1,k_1}$ and $\partial_p|_{\mathcal{A}_2}$ are nonzero locally nilpotent operators for $p\in\overline{k_1+1,k}$ [cf. (2.4)–(2.7) and (2.21)–(2.22)]. To this end, let $t_{k_1+1},\ldots,t_k$ be $k_2+k_3+k_4$ variables such that each t_p has color $\overline{t_p}$ satisfying

$$\overline{t}_{k_1+p}=0, \quad \overline{t}_{k_1+k_2+q}\in\Gamma_+, \quad \overline{t}_{k_1+k_2+k_3+r}\in\Gamma_-, \tag{2.14}$$

for $p\in\overline{1,k_2}$, $q\in\overline{1,k_3}$, $r\in\overline{1,k_4}$. For convenience, we denote $t_p=0$ if $p\leqslant k_1$. Denote $\mathcal{J}=\{0\}^{k_1}\times\mathbb{N}^{k_2+k_3}\times\mathbb{Z}_2^{k_4}$, where $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$, i.e., $\mathcal{J}$ is the subset of $\mathbb{F}^k$ consisting of the following elements:

$$\underline{i}=(i_1,i_2,\ldots,i_k), \tag{2.15}$$

with $i_p=0$ for $p\leqslant k_1$, and $i_q\in\mathbb{N}$ for $q\in\overline{k_1+1,k_1+k_2+k_3}$, and $i_r=0,1$ for $q>k_1+k_2+k_3$ [(2.1) and (2.16) explain why we shall have $i_q=0,1$ for $q>k_1+k_2+k_3$]. Let $\mathcal{A}_2=\mathbb{E}[t_{k_1+1},\dots,t_k]$ be the ϵ-commutative algebra of polynomials in $k_2+k_3+k_4$ variables with an $\mathbb{E}$-basis consisting of the elements

$$t^{\underline{i}}=t_{k_1+1}^{i_{k_1+1}}\cdots t_k^{i_k},\quad \forall\ \underline{i}\in\mathcal{J}, \tag{2.16}$$

or $\mathbb{E}_0$-basis $\{E_\alpha t^{\underline{i}}\mid(\alpha,\underline{i})\in\Gamma^0\times\mathcal{J}\}$ such that

$$E_\alpha t^{\underline{i}}\cdot E_\beta t^{\underline{j}}=e_{\alpha,\beta}\prod_{p=k_1+1}^{k}\epsilon(\overline{t_p},\beta)^{i_p}\prod_{k_1<p<q\leqslant k}\epsilon(\overline{t_q},\overline{t_p})^{i_qj_p}E_{\alpha+\beta}t^{\underline{i}+\underline{j}},\quad \forall\ \alpha,\beta\in\Gamma^0,\ \underline{i},\underline{j}\in\mathcal{J} \tag{2.17}$$

[cf. (2.3)], where we use the convention that $t^{\underline{i}}=0$ if $\underline{i}\notin\mathcal{J}$. For convenience, we shall denote

$$\epsilon_{\underline{i},\beta}=\prod_{p=k_1+1}^{k}\epsilon(\overline{t_p},\beta)^{i_p},\quad \widetilde{\epsilon}_{\underline{i},\underline{j}}=\prod_{k_1<p<q\leqslant k}\epsilon(\overline{t_q},\overline{t_p})^{i_qj_p},\quad \forall\ \underline{i},\underline{j}\in\mathcal{J},\ \beta\in\Gamma^0. \tag{2.18}$$

Definition 2.1: We define $\mathcal{A}=\mathcal{A}(\underline{k},G,\mathbb{E},f)$ to be the (ϵ,Γ)-commutative associative algebra with the identity element $1=E_0=x^0$, which is the tensor product of algebras $\mathcal{A}=\mathcal{A}_1\otimes_{\mathbb{E}}\mathcal{A}_2$, having $\mathbb{E}_0$-basis

$$E_\alpha x^{\underline{a},\underline{i}}=E_\alpha x^{\underline{a}}t^{\underline{i}},\quad \forall\ (\alpha,\underline{a},\underline{i})\in\Gamma^0\times G\times\mathcal{J}, \tag{2.19}$$

with the multiplication

$$E_\alpha x^{\underline{a},\underline{i}}\cdot E_\beta x^{\underline{b},\underline{j}}=\epsilon_{\underline{i},\beta}e_{\alpha,\beta}\epsilon_{\underline{i},\underline{b}}\widetilde{\epsilon}_{\underline{i},\underline{j}}\epsilon(\hat{\underline{a}},\beta)e_{\alpha+\beta,\theta_{\underline{a},\underline{b}}}f(\underline{a},\underline{b})E_{\alpha+\beta+\theta_{\underline{a},\underline{b}}}x^{\underline{a}+\underline{b},\underline{i}+\underline{j}}, \tag{2.20}$$

for $\alpha,\beta\in\Gamma^0$, $\underline{a}$, $\underline{b}\in G$, $\underline{i},\underline{j}\in\mathcal{J}$ [cf. (2.3), (2.13), (2.17), and (2.18)].

For $a\in\mathbb{F}^k,p\in\overline{1,k}$, we denote

$$a_{[p]}=(0,\dots,0,\overset{p}{a},0,\dots,0)\in\mathbb{F}^k.$$

For $p\in\overline{1,k}$, we define the linear transformations $\partial_p,\partial_{t_p},\partial_p^*$ on $\mathcal{A}$ such that they have color $-\overline{t_p}$ [in particular, they have color 0 if $p\leqslant k_1+k_2$, cf. (2.14)], and

$$\partial_p=\partial_p^*+\partial_{t_p}, \tag{2.21}$$

$$\partial_p^*(E_\alpha x^{\underline{a},\underline{i}})=a_pE_\alpha x^{\underline{a},\underline{i}},\quad \partial_{t_p}(E_\alpha x^{\underline{a},\underline{i}})=\epsilon(\overline{\partial_{t_p}},\alpha+\hat{\underline{a}})\prod_{q=1}^{p-1}\epsilon(\overline{\partial_{t_p}},\overline{t_q})^{i_q}\,i_pE_\alpha x^{\underline{a},\underline{i}-1_{[p]}}, \tag{2.22}$$

for $(\alpha,\underline{a},\underline{i})\in\Gamma^0\times G\times\mathcal{J}$. Clearly, $\partial_p^*=0$ if $p>\underline{k_1}+k_2$ by (2.8), and $\partial_q=0$ if $q\leqslant k_1$ by (2.15). Then $\partial_p,\partial_p^*,\partial_{t_p}$ are ϵ-derivations of $\mathcal{A}$ for $p\in\overline{1,k}$. We call ∂_p^* a *grading operator* (or *degree operator*), ∂_{t_p} a *down-grading operator*, and $\partial_p=\partial_p^*+\partial_{t_p}$ a *mixed operator* if both p_p^* and ∂_{t_p} are nonzero. Then

$$\mathcal{D}=\mathrm{span}_{\mathbb{F}}\{\partial_p\mid p\in\overline{1,k}\} \tag{2.23}$$

is a finite-dimensional subspace of ϵ-commutative locally finite color derivations of $\mathcal{A}$ such that the eigenspaces for elements of $\mathcal{D}$ are Γ-graded.

Theorem 2.2: Let $\mathcal{A}=\Sigma_{\alpha\in\Gamma}\mathcal{A}_\alpha$ be an ϵ-commutative associative graded algebra with an identity element over an algebraically closed field $\mathbb{F}$ of characteristic zero and let $\mathcal{D}=\Sigma_{\alpha\in\Gamma}\mathcal{D}_\alpha$ be

a finite-dimensional Γ-graded $\mathbb{F}$-subspace of ϵ-commutative locally finite color-derivations of $\mathcal{A}$ such that the eigenspaces for elements of $\mathcal{D}$ are Γ-graded. Then $\mathcal{A}$ is graded $\mathcal{D}$-simple if and only if $\mathcal{A}$ is isomorphic to the algebra of the form $\mathcal{A}(\underline{k},G,\mathbb{E},f)$ defined in (2.19) and (2.20), and $\mathcal{D}$ is of the form (2.21)–(2.23).

Proof: "⇐:" Let $\mathcal{I}$ be a Γ-graded $\mathcal{D}$-stable nonzero ideal of $\mathcal{A}=\mathcal{A}(\underline{k},G,\mathbb{E},f)$. By (2.21) and (2.22), we see that

$$\left(\bigcup_{(\alpha,\underline{a})\in\Gamma^0\times G}\mathbb{F}(E_\alpha x^{\underline{a}})\right)\Big\backslash\{0\},$$

is the set of the common eigenvectors of $\mathcal{D}$. We also see that if a homogeneous element $\partial\in H(\mathcal{D})$ has a nonzero eigenvalue, then $\partial\in\mathcal{D}_0$. Thus $\Sigma_{0\neq\alpha\in\Gamma}\mathcal{D}_\alpha$ acts locally nilpotently on $\mathcal{I}$. Since $\mathcal{D}_0$ is commutative [cf. (1.2) and (1.5)], and $\mathcal{D}_0$ commutes with $\Sigma_{0\neq\alpha\in\Gamma}\mathcal{D}_\alpha$, and $\Sigma_{0\neq\alpha\in\Gamma}\mathcal{D}_\alpha$ is color-commutative, by linear algebra, $\mathcal{I}$ must contain a common eigenvector of $\mathcal{D}$. Thus $E_\alpha x^{\underline{a}}\in\mathcal{I}$ for some $(\alpha,\underline{a})\in\Gamma^0\times G$. Then

$$1=e_{-\alpha,\alpha}^{-1}f(-\underline{a},\underline{a})^{-1}(E_{-\alpha}x^{-\underline{a}})\cdot(E_\alpha x^{\underline{a}})\in\mathcal{I}$$

[cf. (2.13)]. Hence $\mathcal{I}=\mathcal{A}$. This proves that $\mathcal{A}$ is graded $\mathcal{D}$-simple.

"⇒:" Suppose $\partial\in H(\mathcal{D})$ has a nonzero eigenvalue $\underline{a}\in\mathbb{F}$ such that $u_{\underline{a}}\in H(\mathcal{A})$ is a corresponding eigenvector. Then we have $\partial(u_{\underline{a}})=\underline{a}u_{\underline{a}}$, and so $\overline{\partial}+\overline{u_{\underline{a}}}=\overline{u_{\underline{a}}}$ by (1.4). Thus $\overline{\partial}=0$. In other words, we have

$$\partial\in H(\mathcal{D}),\ \overline{\partial}\neq 0\Rightarrow\ \partial\quad\text{acts locally nilpotent on}\quad\mathcal{A}. \tag{2.24}$$

Since $\mathbb{F}$ is algebraically closed and $\mathcal{D}$ is a finite-dimensional subspace of ϵ-commutative locally finite color derivations of $\mathcal{A}$, from linear algebra, we have

$$\mathcal{A}=\bigoplus_{\underline{a}\in\mathcal{D}^*}\mathcal{A}(\underline{a}),$$

where $\mathcal{D}^*$ is the dual space of $\mathcal{D}$, and

$$\mathcal{A}(\underline{a})=\{u\in\mathcal{A}\,|\,(\partial-\underline{a}(\partial))^m(u)=0\ \text{ for }\ \partial\in H(\mathcal{D})\ \text{ and some } m\in\mathbb{N}\},$$

for $\underline{a}\in\mathcal{D}^*$ [note that $\underline{a}(\partial)=0$ if $\overline{\partial}\neq 0$ by (2.24)]. Denote

$$G=\{\underline{a}\in\mathcal{D}^*\,|\,\mathcal{A}(\underline{a})\neq 0\}.$$

By (2.24), G can be viewed as a subset of $\mathcal{D}_0^*$ by the restriction $\underline{a}\mapsto\underline{a}|_{\mathcal{D}_0}$. For any $\underline{a}\in G$, $n\in\mathbb{N}$, we define

$$\mathcal{A}(\underline{a})^{(n)}=\{u\in\mathcal{A}\,|\,(d_1-\underline{a}(d_1))\cdots(d_{n+1}-\underline{a}(d_{n+1}))(u)=0,\ \forall\, d_1,\cdots,d_{n+1}\in H(\mathcal{D})\}. \tag{2.25}$$

Then

$$\mathcal{A}(\underline{a})=\bigcup_{n=0}^{\infty}\mathcal{A}(\underline{a})^{(n)},\quad\forall\,\underline{a}\in G.$$

A nonzero vector in $\mathcal{A}(\underline{a})^{(0)}$ is called a *root vector* with root $\underline{a}$. For any homogeneous root vector $u\in\mathcal{A}(\underline{a})^{(0)}$, clearly $\mathcal{A}u$ is a Γ-graded $\mathcal{D}$-stable ideal of $\mathcal{A}$. Thus $\mathcal{A}u=\mathcal{A}$. In particular, $vu=1$ for some $v\in\mathcal{A}$. So any homogeneous root vector is invertible. For a root vector $u\in H(\mathcal{A}(\underline{a})^{(0)})$ with $\underline{a}\in G$ and any $\partial\in H(\mathcal{D})$, we have

$$0=\partial(1)=\partial(uu^{-1})=\partial(u)u^{-1}+\epsilon(\overline{\partial},\overline{u})u\partial(u^{-1})=\underline{a}(\partial)uu^{-1}+\epsilon(\overline{\partial},\overline{u})u\partial(u^{-1})$$
$$=\begin{cases}\epsilon(\overline{\partial},\overline{u})u\partial(u^{-1}) & \text{if} \quad \overline{\partial}\neq 0,\\ \underline{a}(\partial)+u\partial(u^{-1}) & \text{if} \quad \overline{\partial}=0,\end{cases}$$

because $\underline{a}(\partial)=0$ if $\overline{\partial}\neq 0$ by (2.24). This implies

$$\partial(u^{-1})=-\underline{a}(\partial)u^{-1}, \tag{2.26}$$

by (2.24). Hence

$$-\underline{a}\in G, \quad \forall \quad \underline{a}\in G. \tag{2.27}$$

For any $x\in H(\mathcal{A}(\underline{a})^{(0)})$, $y\in H(\mathcal{A}(\underline{b})^{(0)})$, and $\partial\in H(\mathcal{D})$, we have

$$\partial(xy)=\partial(x)y+\epsilon(\overline{\partial},\overline{x})x\partial(y)=\begin{cases}0 & \text{if} \quad \overline{\partial}\neq 0,\\ (\underline{a}(\partial)+\underline{b}(\partial))xy & \text{if} \quad \overline{\partial}=0.\end{cases}$$

Hence

$$\mathcal{A}(\underline{a})^{(0)}\cdot\mathcal{A}(\underline{b})^{(0)}\subset\mathcal{A}(\underline{a}+\underline{b})^{(0)}, \quad \forall \quad \underline{a},\underline{b}\in G.$$

Considering the invertibility of root vectors, we have

$$\mathcal{A}(\underline{a})^{(0)}\cdot\mathcal{A}(\underline{b})^{(0)}=\mathcal{A}(\underline{a}+\underline{b})^{(0)}, \quad \forall \quad \underline{a},\underline{b}\in G.$$

In particular, we obtain

$$\underline{a}+\underline{b}\in G, \quad \forall \quad \underline{a},\underline{b}\in G. \tag{2.28}$$

Thus by (2.27) and (2.28), G is an additive subgroup of D^*. Set

$$\mathbb{E}=\mathcal{A}(0)^{(0)}. \tag{2.29}$$

Then $\mathbb{E}$ is a Γ-graded field extension of $\mathbb{F}$ such that $\mathbb{E}_0$ is a field extension of $\mathbb{F}$. We set

$$\Gamma^0=\{\alpha\in\Gamma \mid \mathbb{E}_\alpha\neq\{0\}\}.$$

Clearly, Γ^0 is a subgroup of Γ and $\Gamma^0\subset\Gamma_+$ by (2.1). For any $\alpha\in\Gamma^0$, choose $E_\alpha=1$ if $\alpha=0$, and $E_\alpha\in\mathbb{E}_\alpha\backslash\{0\}$ if $\alpha\neq 0$. Then $\{E_\alpha \mid \alpha\in\Gamma^0\}$ forms an $\mathbb{E}_0$-basis of $\mathbb{E}$. Thus we have (2.3) such that the coefficient $e_{\alpha,\beta}$ satisfies (2.2) by color commutativity and associativity.

First assume that $\mathcal{A}(0)\neq\mathbb{E}$. Since $\underline{a}(\partial)=0$ for any homogeneous derivation ∂ with $\overline{\partial}\neq 0$, for $u\in\mathcal{A}(\underline{a})^{(m)}$, $v\in\mathcal{A}(\underline{b})^{(n)}$ and $d_1,\ldots,d_{m+n+1}\in H(\mathcal{D})$, by induction on $m+n+1$, we can write

$$(d_1-(\underline{a}+\underline{b})(d_1))\cdots(d_{m+n+1}-(\underline{a}+\underline{b})(d_{m+n+1}))(uv), \tag{2.30}$$

as a linear combination of the forms

$$(d_{i_1}-\underline{a}(d_{i_1}))\cdots(d_{i_r}-\underline{a}(d_{i_r}))(u)\cdot(d_{j_1}-\underline{a}(d_{j_1}))\cdots(d_{j_s}-\underline{a}(d_{j_s}))(v), \tag{2.31}$$

where

$$r+s=m+n+1, \quad \{i_1,\ldots,i_r,j_1,\ldots,j_s\}=\{1,\ldots,m+n+1\}.$$

By definition (2.25), we obtain that (2.31) is zero, and so is (2.30). It follows that $uv\in\mathcal{A}(\underline{a}+\underline{b})^{(m+n)}$. Thus

$$\mathcal{A}(\underline{a})^{(m)}\cdot\mathcal{A}(\underline{b})^{(n)}\subset\mathcal{A}(\underline{a}+\underline{b})^{(m+n)},\quad \forall\ \ \underline{a},\underline{b}\in G,\ \ m,n\in\mathbb{N}. \tag{2.32}$$

In particular (since homogeneous root vectors are invertible),

$$\mathbb{E}\mathcal{A}(\underline{a})^{(m)}=\mathcal{A}(\underline{a})^{(m)}=\mathcal{A}(\underline{a})^{(0)}\mathcal{A}(0)^{(m)},\quad \forall\ \ \underline{a}\in G,\ \ m\in\mathbb{N} \tag{2.33}$$

[cf. (2.29)]. Hence each $\mathcal{A}(\underline{a})^{(m)}$ is a vector space over the graded field $\mathbb{E}$. For any $v\in\mathcal{A}(0)^{(1)}$, we have $\mathcal{D}(v)\subset\mathbb{E}$ and

$$\mathcal{D}(v)=0\ \ \Leftrightarrow\ \ v\in\mathbb{E}. \tag{2.34}$$

Set

$$\mathcal{H}=\mathbb{E}\mathcal{D},\quad \mathcal{H}_1=\{\partial\in\mathcal{H}\mid\partial(\mathcal{A}(0)^{(1)})=\{0\}\},\quad k_1=\dim_{\mathbb{E}}\mathcal{H}_1. \tag{2.35}$$

Expression (2.34) implies that $\mathcal{A}(0)^{(1)}/\mathbb{E}$ is isomorphic to a subspace of the space $\mathrm{Hom}_{\mathbb{E}}(\mathcal{H},\mathbb{E})$ over $\mathbb{E}$. By linear algebra, there exist subsets

$$\{\partial_{k_1+1},\partial_{k_1+2},\cdots,\partial_k\}\subset H(\mathcal{D}),\quad \{t_{k_1+1},t_{k_1+2},\cdots,t_k\}\subset H(\mathcal{A}(0)^{(1)}), \tag{2.36}$$

for some $k\in\mathbb{N}$, such that

$$\mathcal{A}(0)^{(1)}=\mathbb{E}+\sum_{l=k_1+1}^{k}\mathbb{E}t_l,\quad \partial_p(t_q)=\delta_{p,q},\quad \forall\ \ p,q\in\overline{k_1+1,k}. \tag{2.37}$$

Set

$$\mathcal{H}_2=\sum_{p=k_1+1}^{k}\mathbb{E}\partial_p.$$

Then we have

$$\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2.$$

For convenience, denote

$$t^{\underline{i}}=t_{k_1+1}^{i_{k_1+1}}\cdots t_k^{i_k}\quad \text{for}\ \ \underline{i}=(i_{k_1+1},\ldots,i_k)\in\mathbb{N}^{\ell},$$

where $\ell=k-k_1$. By (2.1) then

$$t^{\underline{i}}=0\ \text{if}\ \ i_p\geqslant 2\ \text{with}\ \ \overline{t_p}\in\Gamma_-\ \text{for some}\ \ p\in\overline{k_1+1,k},$$

and

$$t^{\underline{i}}\cdot t^{\underline{j}}=\prod_{k_1+1\leqslant p<q\leqslant k}\epsilon(\overline{t_q},\overline{t_p})^{i_q j_p}t^{\underline{i}+\underline{j}},\quad \forall\ \ \underline{i},\underline{j}\in\mathbb{N}^{\ell}.$$

Furthermore, by (2.37), we can deduce by induction on the level $|\underline{i}|:=\Sigma_{p=k_1+1}^{k}i_p$ that

$$t^{\underline{i}}\in\mathcal{A}(0)^{(|\underline{i}|)},\quad \partial_p(t^{\underline{i}})=\prod_{k_1<q<p}\epsilon(\overline{p_p},t_q)^{i_q}\,i_p t^{\underline{i}-1_{[p]}}, \tag{2.38}$$

for $\underline{i}\in\mathbb{N}^{\ell},p\in\overline{k_1+1,k}$. Set

$$\tilde{\mathcal{A}}(0)=\sum_{\underline{i}\in\mathbb{N}^{\ell}}\mathbb{E}t^{\underline{i}}\subset\mathcal{A}(0). \tag{2.39}$$

Then $\tilde{\mathcal{A}}(0)$ forms a subalgebra of $\mathcal{A}$. We want to prove that $\mathcal{A}(0)=\tilde{\mathcal{A}}(0)$. By (2.37), $\mathcal{A}(0)^{(1)}\subset\tilde{\mathcal{A}}(0)$. Suppose $\mathcal{A}(0)^{(m)}\subset\tilde{\mathcal{A}}(0)$ for some $1\leqslant m\in\mathbb{N}$. By (2.25), $\partial(\mathcal{A}(0)^{(m+1)})\subset\mathcal{A}(0)^{(m)}\subset\tilde{\mathcal{A}}(0)$ for any $\partial\in\mathcal{H}$. Thus, for $\partial_{k_1+1}\in H(\mathcal{H})$ and $u\in H(\mathcal{A}(0)^{(m)})$, we may assume that

$$\partial_{k_1+1}(u)=\sum_{\underline{i}\in\mathbb{N}^{\ell}}c_{\underline{i}}t^{\underline{i}}, \tag{2.40}$$

where $c_{\underline{i}}\in H(\mathbb{E})$ and $c_{\underline{i}}=0$ for all but a finite number of $\underline{i}$. If $\overline{\partial}_{k_1+1}\in\Gamma_+$, then we set

$$u_1=\sum_{\underline{i}\in\mathbb{N}^{\ell}}c_{\underline{i}}\epsilon(\overline{\partial}_{k_1+1},\overline{c_{\underline{i}}})^{-1}(i_{k_1+1}+1)^{-1}t^{\underline{i}+1_{[k_1+1]}}\in H(\tilde{\mathcal{A}}(0)), \tag{2.41}$$

and we obtain

$$\partial_{k_1+1}(u)=\partial_{k_1+1}(u_1). \tag{2.42}$$

If $\overline{\partial}_{k_1+1}\in\Gamma_-$, then by (2.1), $\partial_{k_1+1}^2=0$, we must have

$$i_{k_1+1}=0 \quad \text{if } c_{\underline{i}}\neq 0, \tag{2.43}$$

otherwise if (2.43) does not hold, then by (2.38) and (2.40) we would have $\partial_{k_1+1}^2(u)\neq 0$, leading to a contradiction to the fact that $\partial_{k_1+1}^2=0$. Thus we can still choose u_1 as in (2.41) to give (2.42). Similarly, since $\partial_{k_1+2}(u-u_1)\in\mathcal{A}(0)^{(m)}\subset\tilde{\mathcal{A}}(0)$, there exists $u_2\in H(\tilde{\mathcal{A}}(0))$ such that

$$\partial_{k_1+2}(u-u_1)=\partial_{k_1+2}(u_2). \tag{2.44}$$

Assume that $u_2=\Sigma_{\underline{i}\in\mathbb{N}^{\ell}}c'_{\underline{i}}t^{\underline{i}}$, where $c'_{\underline{i}}\in H(\mathbb{E})$. Since $\mathcal{H}$ is color commutative, by (2.42) and (2.44), we have

$$0=\partial_{k_1+1}\partial_{k_1+2}(u_2)=\sum_{\underline{i}\in\mathbb{N}^{\ell}}c'_{\underline{i}}\epsilon(\overline{\partial}_{k_1+1}+\overline{\partial}_{k_1+2},\overline{c'_{\underline{i}}})e(\overline{\partial}_{k_1+2},t_{k_1+1})^{i_{k_1+1}}i_{k_1+1}i_{k_1+2}t^{\underline{i}-1_{[k_1+1]}-1_{[k_1+2]}}.$$

Thus $i_{k_1+1}i_{k_1+2}=0$ if $c'_{\underline{i}}\neq 0$. Hence we can re-choose $u_2\in H(\tilde{\mathcal{A}}(0))$ such that

$$\partial_{k_1+1}(u_2)=0, \quad \partial_{k_1+2}(u-u_1)=\partial_{k_1+2}(u_2).$$

Similarly, we can find $u_2,\ldots,u_{\ell}\in H(\tilde{\mathcal{A}}(0))$ such that

$$\partial_{k_1+p}\left(u-\sum_{q=1}^{p}u_q\right)=0, \quad \partial_{k_1+1}(u_p)=\partial_{k_1+2}(u_p)=\cdots=\partial_{k_1+p-1}(u_p)=0, \quad \forall\ \ p\in\overline{2,\ell},$$

by induction on p. Thus we have

$$\partial_{k_1+p}\left(u-\sum_{q=1}^{\ell}u_q\right)=0, \quad \forall\ \ p\in\overline{1,\ell}. \tag{2.45}$$

For any $\partial,\partial'\in H(\mathcal{H}_1)$, using (2.35) and (2.39) we deduce

$$\partial\partial'\left(u-\sum_{p=1}^{\ell}u_p\right)\in\partial(\mathcal{A}(0)^{(m)})+\partial'(\mathcal{A}(0)^{(m)})\subset\partial(\tilde{\mathcal{A}}(0))+\partial'(\tilde{\mathcal{A}}(0))=\{0\}. \tag{2.46}$$

Now (2.45) and (2.46) show that $u-\Sigma_{p=1}^{\ell}u_p\in\mathcal{A}(0)^{(1)}$. Thus by (2.35),

$$\partial\left(u-\sum_{p=1}^{\ell}u_p\right)=0,\quad \forall\ \ \partial\in H(\mathcal{H}_1). \tag{2.47}$$

Then (2.45), (2.47) and the definition (2.25) show that

$$u-\sum_{p=1}^{\ell}u_p\in\mathcal{A}(0)^{(0)}=\mathbb{E}.$$

Thus $u\in\tilde{\mathcal{A}}(0)$. This proves

$$\mathcal{A}(0)=\tilde{\mathcal{A}}(0).$$

The case $\mathcal{A}(0)=\mathbb{E}$ can be viewed as in the general case $\mathcal{A}(0)=\tilde{\mathcal{A}}(0)$ with $\ell=0$.

We re-choose $\partial_p, t_p, p\in\overline{1,k}$ as follows: Choose a homogeneous $\mathbb{F}$-basis $\{\partial_1,\ldots,\partial_{k_1}\}$ of $\mathcal{D}\cap\mathcal{H}_1$, and set $t_p=0$ for $p\in\overline{1,k_1}$, then ∂_p are semi-simple derivations on $\mathcal{A}$ by (2.33) and (2.35). Let ℓ_1 be the dimension of the maximal locally nilpotent $\mathbb{F}$-subspace of $\mathcal{D}$. Clearly $\ell_1\leqslant\ell=k-k_1$. Let $k_2=\ell-\ell_1$. Now we choose $\partial_{k_1+k_2+1},\ldots,\partial_k$ to be homogeneous locally nilpotent derivations of $\mathcal{D}$ such that the first k_3 derivations have colors in Γ_+ and the last k_4 derivations have colors in Γ_- for some k_3,k_4 with $k_3+k_4=\ell_1$. Extend $\{\partial_p\mid p\in\overline{1,k_1}\cup\overline{k_1+k_2+1,k}\}$ to a homogeneous $\mathbb{F}$-basis $\{\partial_p\mid p\in\overline{1,k}\}$ of $\mathcal{D}$. By the choices of ∂_p, then there exists $t_p\in\mathcal{A}(0)^{(1)}$ for each $p\in\overline{k_1+1,k}$ satisfying (2.36) and (2.37).

For any $\underline{a}\in G$, we identify

$$\underline{a}\ \leftrightarrow\ (\underline{a}(\partial_1),\ldots,\underline{a}(\partial_{k_1+k_2}))\in\mathbb{F}^{k_1+k_2}.$$

Then G is a nondegenerate subgroup of $\mathbb{F}^{k_1+k_2}$ (otherwise, there exists $\partial\in\Sigma_{p=1}^{k_1+k_2}\mathbb{F}\partial_p$ such that $\underline{a}(\partial)=0$ for all $\underline{a}\in G$ and so ∂ is locally nilpotent, which contradicts the maximality of ℓ_1). Taking homogeneous root vector $u\in\mathcal{A}^{(\underline{a})}$, by (2.26) and (2.32), we have

$$u^{-1}\mathcal{A}(\underline{a})\subset\mathcal{A}(0),\quad u\mathcal{A}(0)\subset\mathcal{A}(\underline{a}).$$

Hence

$$u\mathcal{A}(0)=\mathcal{A}(\underline{a}). \tag{2.48}$$

In particular,

$$\mathcal{A}(\underline{a})^{(0)}=\mathbb{E}u \tag{2.49}$$

is one-dimensional over $\mathbb{E}$. Choose

$$x^0=1,\quad 0\neq x^{\underline{a}}\in\mathcal{A}(\underline{a})^{(0)}\quad\text{for } 0\neq\underline{a}\in G,$$

such that $x^{\underline{a}}$ is homogeneous with color denoted by $\hat{\underline{a}}$. Since $x^{\underline{a}}$ is invertible, we have $\hat{\underline{a}}\in\Gamma_+$. Then we have a map $\hat{}$ satisfying (2.9). By (2.32) and (2.49), we have (2.12) with $f(\underline{a},\underline{b})$ satisfying (2.10) and (2.11) by color commutativity and associativity. By (2.48), we obtain

$$\mathcal{A}=\mathcal{A}^{(0)}\mathcal{A}(0)\cong\mathcal{A}^{(0)}\otimes\mathcal{A}(0),$$

where $\mathcal{A}^{(0)}=\oplus_{\underline{a}\in G}\mathcal{A}(\underline{a})^{(0)}$ is isomorphic to the algebra $\mathcal{A}_1$ defined in (2.12) and (2.13), and $\mathcal{A}(0)=\tilde{\mathcal{A}}(0)$ is isomorphic to the algebra $\mathcal{A}_2$ defined in (2.16) and (2.17). Therefore, the algebra $\mathcal{A}$ is isomorphic to the algebra $\mathcal{A}(\underline{k},G,\mathbb{E},f)$ defined in (2.19) and (2.20), and $\mathcal{D}$ is of the form (2.23). This completes the proof of Theorem 2.2. □

III. CONSTRUCTING SIMPLE LIE COLOR ALGEBRAS FROM $\mathcal{D}$-SIMPLE COLOR ALGEBRAS

In this section, as applications, we shall construct some explicit simple Lie color algebras using the pairs $(\mathcal{A},\mathcal{D})$ given in the last section. For simplicity, we assume that the pairs $(\mathcal{A},\mathcal{D})$ in (2.19) and (2.23) satisfies

$$\{u\in\mathcal{A}\mid\mathcal{D}(u)=0\}=\mathbb{F}.$$

This is equivalent to that $\mathbb{E}_0=\mathbb{F}$ and $\Gamma^0=\{0\}$. So the map $\hat{}:G\to\Gamma_+$ in (2.9) is a group homomorphism and $\theta_{\underline{a},\underline{b}}=0$ for all $\underline{a},\underline{b}\in G$. In this case, noting that $\mathbb{F}$ is algebraically closed, we prove that we can choose suitable basis $\{x^{\underline{a}}\mid\underline{a}\in G\}$ such that the coefficient $f(\underline{a},\underline{b})$ determined by (2.12), which satisfies (2.10) and (2.11), has the following form:

$$f(\underline{a},\underline{b})=\epsilon(\hat{\underline{a}},\hat{\underline{b}})^{1/2},\quad\forall\ \ \underline{a},\underline{b}\in G,\tag{3.1}$$

where the right-hand side is a fixed square root such that (2.10) and (2.11) hold.

Let G' be a maximal subgroup of G such that $x^{\underline{a}}$, $\underline{a}\in G'$ can be chosen so that (3.1) holds for $\underline{a},\underline{b}\in G'$. Suppose $G'\neq G$. Let $\underline{c}\in G\backslash G'$ and set $G''=G'+\mathbb{Z}\underline{c}$. If $G'\cap\mathbb{Z}\underline{c}=\{0\}$, we choose any $x^{\underline{c}}\neq 0$, and set $x^{\underline{a}+k\underline{c}}=\epsilon(\hat{\underline{a}},\hat{\underline{c}})^{-k/2}x^{\underline{a}}\cdot(x^{\underline{c}})^k$ for $\underline{a}+k\underline{c}\in G''$. If $G'\cap\mathbb{Z}\neq\{0\}$, then $G'\cap\mathbb{Z}\underline{c}=\mathbb{Z}\underline{d}$ for some $\underline{d}=m\underline{c}$, $m>1$. In this case, since $\mathbb{F}$ is algebraically closed, we can choose $x^{\underline{c}}$ such that $(x^{\underline{c}})^m=x^{\underline{d}}$, and set $x^{\underline{a}+k\underline{c}}$ as above. In any case, the coefficient $f(\underline{a},\underline{b})$ determined by (2.12) satisfies (3.1) for $\underline{a},\underline{b}\in G''$. But $G'\neq G''\supset G'$. This contradicts the maximality of G'. This proves (3.1).

Let $\mathbb{F}[\mathcal{D}]$ be the (ϵ,Γ)-commutative associative algebra with basis

$$\{\partial^{\mu}=\partial_1^{\mu_1}\cdots\partial_k^{\mu_k}\mid\mu=(\mu_1,\ldots,\mu_k)\in\mathcal{M}\},$$

where $\mathcal{M}=\mathbb{N}^{k_1+k_2+k_3}\times\mathbb{Z}_2^{k_4}$. For convenience, we denote $\partial^{\mu}=0$ if $\mu\notin\mathcal{M}$. Denote

$$W=W(\underline{k},G)=\mathcal{A}\otimes\mathcal{D}=\operatorname{span}\{x^{\underline{a},\underline{i}}\partial_p\mid(\underline{a},\underline{i})\in G\times\mathcal{J},p\in\overline{1,k}\},$$

$$\mathcal{W}=\mathcal{W}(\underline{k},G)=\mathcal{A}\otimes\mathbb{F}[\mathcal{D}]=\operatorname{span}\{x^{\underline{a},\underline{i}}\partial^{\mu}\mid(\underline{a},\underline{i},\mu)\in G\times\mathcal{J}\times\mathcal{M}\}.$$

Then as spaces, we have $W\subset\mathcal{W}$. By regarding $\mathcal{W}$ as operators on $\mathcal{A}$, $\mathcal{W}$ becomes a Γ-graded associative algebra whose multiplication is the composition of operators. Thus $\mathcal{W}$ forms an (ϵ,Γ)-Lie color algebra under the bracket (1.3). We call $\mathcal{W}$ a *Lie color algebra of (generalized) Weyl type*. Clearly $\mathbb{F}$ is the center of $\mathcal{W}$. Let $\tilde{\mathcal{W}}=\mathcal{W}/\mathbb{F}$ and let $\overline{\mathcal{W}}=[\tilde{\mathcal{W}},\tilde{\mathcal{W}}]$ the derived algebra of $\tilde{\mathcal{W}}$. Obviously, W forms an (ϵ,Γ)-Lie color subalgebra of $\mathcal{W}$, called a *Lie color algebra of (generalized) Witt type*. Using results in Refs. 4 and 11, we obtain the following.

Theorem 3.1: The Lie color algebras $\overline{\mathcal{W}}$ and W are simple if $k_1+k_2+k_3>0$ or $k_4>1$. Furthermore, $\overline{\mathcal{W}}=\tilde{\mathcal{W}}$ if $k_1+k_2+k_3>0$ or otherwise, $\tilde{\mathcal{W}}=\overline{\mathcal{W}}+\mathbb{F}t^{\underline{n}}\partial^{\lambda}$, where $\underline{n}$ and λ are the largest elements respectively in $\mathcal{J}$ and in $\mathcal{M}$. □

Note that in case $k=k_4=1$, $\overline{\mathcal{W}}=0$ and $W=\mathbb{F}t_1\partial_1$ are not simple. If $k=k_4>1$, then we obtain finite dimensional simple Lie color algebras $\overline{\mathcal{W}}$ and W of dimensions $2^{2n}-2$ and $n2^n$. In particular, if $\Gamma=\mathbb{Z}_2,\epsilon(i,j)=(-1)^{ij},i,j\in\mathbb{Z}_2$, we obtain the finite dimensional simple Lie superalgebras $\overline{\mathcal{W}}=H(2n)$ and $W=W(n)$ (see Ref. 2).

Using the pair $(\mathcal{A},\mathcal{D})$, one might construct other simple Lie color algebras, for example, other series of Lie color algebras of Cartan type.

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