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Regularity and Products of Idemopotents in Endmorphism Monoids of Projective Acts

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REGULARITY AND PRODUCTS OF IDEMPOTENTS IN ENDOMORPHISM MONOIDS OF PROJECTIVE ACTS

SYDNEY BULMAN-FLEMING

§1. *Introduction.* That the monoid of all transformations of any set and the monoid of all endomorphisms of any vector space over a division ring are regular (in the sense of von Neumann) has been known for many years (see **[6]** and **[16]**, respectively). A common generalization of these results to the endomorphism monoid of an independence algebra can be found in **[13]**. It also follows from **[13]** that the endomorphism monoid of a free G-act is regular, where G is any group. In the present paper we use a version of the wreath product construction of **[8]**, **[9]** to determine the projective right S-acts (S any monoid) whose endomorphism monoid is regular.

Another line of investigation began when in **[1]** (and, independently, in **[14]**), it was shown that every non-invertible transformation of a finite set can be written as a product (that is, composite) of idempotent transformations of that set. This result inspired a great deal of subsequent research, leading to analogous or related theorems for the non-invertible endomorphisms of finite-dimensional vector spaces **[5]**, **[7]**, **[19]**, infinite-dimensional vector spaces **[20]**, finite chains **[2]**, **[15]**, independence algebras of finite rank **[11]**, independence algebras of infinite rank **[12]**, and finitely-generated modules over principal ideal domains **[10]**. A by-product of the investigation in **[11]** is the fact that every non-invertible endomorphism of a finitely-generated G-act (G a group) is a product of idempotent endomorphisms. In the present work, again relying on the wreath product idea mentioned above, we determine the projective S-acts P with the property that every non-invertible element of $End_S(P)$ is a product of idempotents. In addition, we are able to give a precise description of the elements of $End_S(F_n)$ which are products of idempotents, where F_n is a free S-act on $n > 1$ free generators.

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§2. *Preliminaries.* Let S denote any monoid, with identity element 1. In **Act**-S, the category of all right S-acts, the coproduct of a non-empty family $\{A_i : i \in I\}$ is simply the disjoint union and is denoted $\coprod_{i\in I} A_i$. It is well known that every S-act is the coproduct of its maximal indecomposable S-subacts, that A is free in **Act**-S if, and only if, $A \cong \coprod_{i \in I} A_i$ where each A_i is isomorphic to the right S-act S, and that A is projective if, and only if, $A \cong \coprod_{i \in I} e_i S$ where $e_i^2 = e_i \in S$ for each $i \in I$.

If $P = \coprod_{i \in I} e_i S$ ($e_i^2 = e_i \in S$ for each $i \in I$) is a projective act, we let $W(P)$ denote the monoid whose universe is

$$\{(\alpha, f) : \alpha \in T(I), f \in S^I, \text{ and } f(i) \in e_{\alpha(i)} S e_i \text{ for all } i \in I\},$$

where $T(I)$ denotes the set of all self-maps of I. Multiplication in $W(P)$ is given by

$$(\alpha, f)(\beta, g) = (\alpha\beta, f_\beta g),$$

where $\alpha\beta$ is the composition (from right to left) in $T(I)$ and where $(f_\beta g)(i) := f(\beta(i))g(i)$, $i \in I$. The identity element of $W(P)$ is $(\iota, \mathbf{1})$ where ι is the identity selfmap of I and $\mathbf{1}(i) = e_i$ for each $i \in I$. This is a special case of a construction given in [**9**]. It is quite easy to see that the monoids $End_S(P)$ and $W(P)$ are isomorphic: $\varphi \in End_S(P)$ corresponds to the element $(\alpha, f) \in W(P)$ such that $\varphi(i, e_i) = (\alpha(i), f(i))$ (for clarity we here momentarily regard P as being $\bigcup_{i \in I} (\{i\} \times e_i S)$. In the sequel, we shall deal almost exclusively with $W(P)$ rather than with $End_S(P)$.

It is easy to check that $(\alpha, f) \in W(P)$ is idempotent if, and only if, $\alpha^2 = \alpha$ and $f(\alpha(i))f(i) = f(i)$ for all $i \in I$. Thus, given any $\alpha^2 = \alpha \in T(I)$, readily follows that defining

$$f(j) = \begin{cases} e_j, & \text{if } j = \alpha(j) \in \operatorname{im} \alpha, \\ \text{any} \quad s \in e_{\alpha(j)} S e_j, & \text{otherwise}, \end{cases}$$

produces an idempotent $(\alpha, f) \in W(P)$. An idempotent $\alpha \in T(I)$ such that $|I - \operatorname{im} \alpha| = 1$ is called a *defect*-1 *idempotent*, and is symbolized $\binom{i}{j}$ in case $i \neq j$, $\alpha(i) = j$, and $\alpha(k) = k$ for all $k \neq i$. An idempotent of $W(P)$ of the form $(\binom{i}{j}, f)$ is also said to have defect 1, and is called a *special idempotent* of $W(P)$ in the case when $f(i) = s \in e_j S e_i$ and $f(k) = e_k$, $k \neq i$: the notation $\binom{i}{j\,s}$ will be used in the latter situation.

For future use, we quote the following result from [**14**], to end this section.

PROPOSITION 2.1. *Let $T(I)$ denote the monoid of self-maps of a set I, and let T_n denote $T(\{1, 2, \ldots, n\})$. Then*:

(a) *every non-invertible element of T_n is a product of defect-1 idempotents; and*

(b) *if I is infinite, then not every non-invertible element of $T(I)$ is a product of idempotents.*

§3. *Projective acts every non-invertible endomorphism of which is a product of idempotents.* We begin with a simple lemma.

LEMMA 3.1 *Let $e^2 = e$, $f^2 = f \in S$. The following statements are equivalent.*

(a) *Every element of eSe is right invertible and every element of fSf is left invertible.*

(b) *For every $t \in eSf$ there exists $s \in fSe$ such that $st = f$ and $ts = e$.*

Proof. (a) *implies* (b). Let $t \in eSf$. Then $te \in eSe$ so $teu = tfeu = e$ for some $u \in eSe$. Let $s = feu = fu$. We also have $ft \in fSf$, so $vft = vfet = f$ for some $v \in fSf$. As $vfe = ve = vteu = vtu = vftu = fu = s$, we thus have $ts = e$ and $st = f$ as required.

(b) *implies* (a). Let $a = eae \in eSe$. Then $af \in eSf$, so $afs = e$ for some $s \in fSe$ and so efs is a right inverse of a in eSe. If $b = fbf \in fSf$, then $eb \in eSf$, so $s'eb = f$ for some $s' \in fSe$ and hence $s'ef$ is a left inverse of b in fSf.

COROLLARY 3.2. *If $e^2 = e, f^2 = f \in S$ and if eSe and fSf are both groups, then for every $t \in eSf$ there exists a unique $s \in fSe$ such that $ts = e$ and $st = f$.*

Note. In the situation described in the corollary above, we will write $s = t^{-1}$. Also, as is shown in [4], p. 88, for example, eSe is a group if, and only if, eS is a minimal right ideal of S.

PROPOSITION 3.3. *Let S be a monoid and let $P = \coprod_{i \in I} e_i S$ $(|I| > 1)$ be a projective S-act, where $e_i^2 = e_i \in S$ for each i. Suppose every non-invertible element of $W(P)$ is a product of idempotents. Then:*

(a) *I is finite; and*

(b) *each $e_i S e_i$ is a group.*

Proof. (a) If I is infinite, let $\alpha \in T1(I)$ be non-invertible but not a product of idempotents in $T(I)$ (see Proposition 2.1). For each $i \in I$ let $f(i) = e_{\alpha(i)} e_i$. Then $(\alpha, f) \in W(P)$ is non-invertible, but not a product of idempotents.

(b) Let $i \in I$. Since $|I| > 1$ there exists $j \in I$, $j \neq i$. Let t be any element of $e_j S e_i$, let α denote the transposition (ij), and define $f \in S^I$ by

$$f(k) = \begin{cases} t, & \text{if } k = i, \\ e_i e_j, & \text{if } k = j, \\ e_k, & \text{if } k \notin \{i, j\}. \end{cases}$$

Then $(\alpha, f) \in W(P)$ is not a product of idempotents (as $\alpha \in T(I)$ isn't) and so by hypothesis (α, f) must be invertible. Suppose $(\alpha, f)(\beta, g) = (\iota, \mathbf{1})$ where $(\beta, g) \in W(P)$. Then $\beta = \alpha$ and $f(\alpha(j))g(j) = e_j$, which means $f(i)g(j) = e_j$, or $tg(j) = e_j$. Let $s = g(j) \in e_i S e_j$. Then $ts = e_j$. By the same token, there exists $u \in e_j S e_i$ such that $su = e_i$, from which we get $t = te_i = tsu = e_j u = u$, and so $st = e_i$. By Lemma 3.1, every element of $e_j S e_j$ is right invertible and every element of $e_i S e_i$ is left invertible, and so both $e_j S e_j$ and $e_j S e_i$ are groups.

Our next goal is to show that (a) and (b) of Proposition 3.3 are also sufficient for every non-invertible element of $W(P)$ (P a non-cyclic projective act) to be a product of idempotents. This is done *via* a series of lemmas.

LEMMA 3.4. *Let $P = \coprod_{i=1}^{n} e_i S$ $(n > 1)$ be a finitely-generated, non-cyclic, projective S-act, and assume $e_i S e_i$ is a group for each i. If $(\alpha, f) \in W(P)$ is non-invertible, then α is a product of defect-1 idempotents in T_n.*

Proof. By Proposition 2.1 we need only show that if $\alpha \in T_n$ is invertible, then so is any $(\alpha, f) \in W(P)$. Assuming $\alpha \in T_n$ is a permutation and

$(\alpha,f)\in W(P)$, observe that, for each $i\in\{1,\ldots,n\}$, $f(i)$ belongs to $e_{\alpha(i)}Se_i$, so $f(\alpha^{-1}(i))\in e_iSe_{\alpha^{-1}(i)}$. Using the convention announced earlier, let $g(i)=(f(\alpha^{-1}(i)))^{-1}\in e_{\alpha^{-1}(i)}Se_i$. Then $(\alpha^{-1},g)\in W(P)$ is the inverse of (α,f).

LEMMA 3.5. *Let $P=\coprod_{i=1}^n e_iS$ $(n>1)$ be a finitely-generated, non-cyclic, projective S-act, and assume e_iSe_i is a group for each i. If $(\alpha\beta,f)\in W(P)$ and $\alpha^2=\alpha$, then there exist $g,h\in S^I$ such that (α,g), $(\beta,h)\in W(P)$, $(\alpha\beta,f)=(\alpha,g)(\beta,h)$, and (α,g) is idempotent. If α has defect 1, furthermore, then (α,g) may be chosen to be a special idempotent $\binom{i}{js}$.*

Proof. Let $g(i)=e_{\alpha(i)}e_i$ for each $i\in\{1,\ldots,n\}$. Then $(\alpha,g)\in W(P)$ is idempotent since $(g_\alpha g)(i)=g(\alpha(i))g(i)=e_{\alpha^2(i)}e_{\alpha(i)}e_{\alpha(i)}e_i=g(i)$ for each i. Moreover, if α has defect 1 then (α,g) is a special idempotent.

For each $i\in\{1,\ldots,n\}$ let $h(i)=(g(\beta(i)))^{-1}f(i)\in e_{\beta(i)}Se_i$. Then $(\beta,h)\in W(P)$, and $(\alpha,g)(\beta,h)=(\alpha\beta,f)$ since, for each i, $(g_\beta h)(i)=g(\beta(i))h(i)=(g(\beta(i))(g(\beta(i)))^{-1}f(i)=e_{\alpha\beta(i)}f(i)=f(i)$.

COROLLARY 3.6. *Let $P=\coprod_{i=1}^n e_iS$ $(n>1)$ be a finitely-generated, non-cyclic, projective S-act, and assume e_iSe_i is a group for each i. If $(\alpha,f)\in W(P)$ is non-invertible, then*

$$(\alpha,f)=(\alpha_1,g_1)\ldots(\alpha_{m-1},g_{m-1})(\alpha_m,h),$$

where each $(\alpha_i,g_i)\in W(P)$ is a special idempotent, (α_m,h) belongs to $W(P)$, and $\alpha_m^2=\alpha_m$ has defect 1.

The next lemma guarantees that the factor (α_m,h) mentioned in the corollary above is also a product of special idempotents in W(P).

LEMMA 3.7. *Let $P=\coprod_{i=1}^n e_iS$ $(n>1)$ be a finitely-generated, non-cyclic, projective S-act, and assume e_iSe_i is a group for each i. Then every element of $W(P)$ of the form $(\binom{i}{j},h)$, $i\neq j$, is a product of special idempotents in $W(P)$.*

Proof. Without loss of generality we assume $\binom{i}{j}=\binom{1}{2}$. For $k\in\{2,\ldots,n\}$ let

$$g_k=\begin{pmatrix}1 & 2 & \ldots & k & k+1 & \ldots & n\\ h(1) & h(2) & \ldots & h(k) & e_{k+1} & \ldots & e_n\end{pmatrix},$$

so $g_n=h$. We prove by induction on k that $(\binom{1}{2},g_k)\in$W(P) is a product of special idempotents, noting first that

$$\left(\begin{pmatrix}1\\2\end{pmatrix},g_2\right)=\left(\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1 & 2 & 3 & \ldots & n\\ h(1) & h(2) & e_3 & \ldots & e_n\end{pmatrix}\right)=\begin{pmatrix}1\\2 & h(1)\end{pmatrix}\begin{pmatrix}2\\1 & (h(1))^{-1}h(2)\end{pmatrix}.$$

Assuming that $(\binom{1}{2}, g_k)$ is a product of special idempotents for some $k \in \{2, \ldots, n-1\}$, take any $s \in e_{k+1}Se_1$ and note that

$$\begin{pmatrix} 1 \\ k+1 & s \end{pmatrix}\begin{pmatrix} k+1 \\ 1 & s^{-1}h(k+1) \end{pmatrix}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, g_k\right)$$

$$= \left(\begin{pmatrix} 1 \\ k+1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & n \\ s & e_2 & \ldots & e_k & h(k+1) & \ldots & e_n \end{pmatrix}\right)$$

$$\times \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & \ldots & k & k+1 \ldots n \\ h(1) & h(2) & \ldots & h(k) & e_{k+1} \ldots e_n \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & \ldots & k & k+1 & \ldots & n \\ h(1) & h(2) & \ldots & h(k) & h(k+1) & \ldots & e_n \end{pmatrix}\right) = \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, g_{k+1}\right),$$

The induction is now complete.

THEOREM 3.8. *Let $P = \coprod_{i \in I} e_i S$ be a projective right S-act, where $|I| > 1$. Then the following statements are equivalent.*

(a) *Each non-invertible element of $W(P)$ is a product of idempotents.*

(b) *Each non-invertible element of $W(P)$ is a product of* special *idempotents.*

(c) *I is finite and each $e_i S$ is a minimal right ideal of S (or, equivalently, each $e_i S e_i$ is a group).*

If $P = eS$ is a cyclic projective act, then it is no longer necessary for eSe to be a group in order for every non-invertible element of $W(P)$ to be a product of idempotents. Just take $S = \{0, 1\}$ and note that, although each non-invertible element of $End_S(1S) \cong S$ is a product of idempotents, $1S$ is not a minimal right ideal of S.

COROLLARY 3.9. *If F is a free non-cyclic right S-act, then each non-invertible element of $W(F)$ is a product of idempotents if, and only if, F is finitely-generated and S is a group.*

Proof. Take each $e_i = 1$ in Theorem 3.8.

(The sufficiency part of Corollary 3.9 is contained in **[11]**, where the proof is done in the context of idependence algebras.)

Problem 1. Under what conditions is every $(\alpha, f) \in W(P)$ *with α non-invertible* a product of idempotents? Necessarily P must be finitely-generated, as before, but the argument that each $e_i S e_i$ must be a group can no longer be used. If $P = F_n$ $(n > 1)$ is *free* we shall discover in Section 5 that every $(\alpha, f) \in W(F_n)$ of positive defect is a product of defect-1 idempotents if, and only if, the principal right ideals of S form a chain.

§4. *Regularity of $W(P)$, where P is projective.* In this section we generalize to projective acts the result of Skornjakov **[21]** characterizing the free acts whose endomorphism monoid is regular. As usual, we assume $P=\coprod_{i\in I} e_iS$, where $e_i^2=e_i\in S$ for each i. Also, E will stand for $\{e_i: i\in I\}$ throughout this section.

PROPOSITION 4.1. *Let $P=\coprod_{i\in I} e_iS$, where $e_i^2=e_i\in S$ for each i be a projective act and let $E=\{e_i: i\in I\}$. If $W(P)$ ($\cong End_S(P)$ as discussed earlier) is a regular monoid, then*:

(a) *ESE is a regular semigroup*; *and*

(b) *for any $(\alpha,f)\in W(P)$ there exists $\beta\in T(I)$ such that $\alpha\beta\alpha=\alpha$ and $f(i)\in f(\beta\alpha(i))$ for each $i\in I$.*

Proof. (a) Take any $e_ise_j\in ESE$. Define $\alpha\in T(I)$ by

$$\alpha(k)=\begin{cases} k, & \text{if } k\notin\{i,j\},\\ i, & \text{if } k=j,\\ j, & \text{if } k=i,\end{cases}$$

and define $f\in S^I$ by

$$f(k)=\begin{cases} e_ise_j, & \text{if } k=j,\\ \text{any element of } e_{\alpha(k)}Se_k, & \text{if } k\neq j.\end{cases}$$

Since (α,f) belongs to the regular monoid $W(P)$, there exists $(\beta,g)\in W(P)$ such that $(\alpha,f)(\beta,g)(\alpha,f)=(\alpha,f)$. Because $\alpha\beta\alpha=\alpha$, $\alpha\beta(i)=i$ and so $\beta(i)=j$. Therefore,

$$\begin{aligned} e_ise_j&=f(j)=((f_\beta g)_\alpha f)(j)=f(\beta\alpha(j))g(\alpha(j))f(j)=f(j)g(i)f(j)\\ &=(e_ise_j)g(i)(e_ise_j)\end{aligned}$$

where $g(i)\in e_jSe_i\subseteq ESE$.

(b) Take any $(\alpha,f)\in W(P)$. By assumption there exists $(\beta,g)\in W(P)$ such that $\alpha\beta\alpha=\alpha$ and, for each $i\in I$, $f(i)=((f_\beta g)_\alpha f)(i)=f(\beta\alpha(i))g(\alpha(i))f(i)\in f(\beta\alpha(i))S$, as required.

Before demonstrating that the conditions of Proposition 4.1 are sufficient for regularity of $W(P)$, we present an alternative form of (b) which is perhaps more applicable.

LEMMA 4.2. *Condition* (b) *of Proposition* 4.1 *above is equivalent to condition* (b′) *below*:

(b′) *for any fixed $i\in I$ and any I-indexed family $\{s_j : j\in I\}\subseteq S$ the set $T=\{e_is_je_j : j\in I\}$ generates a principal right ideal in ESE.*

Proof. (*b*) implies (*b′*). Assume (b), let $i\in I$, let $\{s_j : j\in I\}$ be any I-indexed family of elements of S, and let $T=\{e_is_je_j : j\in I\}$. Let $(\alpha,f)\in W(P)$ be such that $\alpha\in T(I)$ is the constant map with image $\{i\}$ and $f(j)=e_is_je_j$ for each $j\in I$. By (b) there exists $\beta\in T(i)$ such that $\alpha\beta\alpha=\alpha$ and $f(j)\in f(\beta\alpha(j))S$ for each $j\in I$. Thus, $TESE\subseteq f(\beta\alpha(j))ESE\subseteq TESE$ and so $TESE$ is a principal right ideal of ESE.

(b') implies (b). Let $(\alpha, f) \in W(P)$ be given. For each $l \in \operatorname{im} \alpha$ let j_l denote a fixed element of $\alpha^{-1}(l)$ and observe that $f(j_l) \in e_l S e_{j_l}$. Now let

$$T_l = \{f(j_l)e_j : j \in I - \alpha^{-1}(l)\} \cup \{f(j) : j \in \alpha^{-1}(l)\}.$$

By assumption, $T_l ESE$ is a principal right ideal of ESE (either of form $f(j_l)ESE$ for some $j \in I - \alpha^{-1}(l)$ or else of form $f(j)ESE$ for some $j \in \alpha^{-1}(l)$). Thus, for each $l \in \operatorname{im} \alpha$ there exists $k_l \in \alpha^{-1}(l)$ such that $T_l ESE \subseteq f(k_l)ESE$. Define $\beta \in T(I)$ by

$$\beta(i) = \begin{cases} k_i, & \text{if } i \in \operatorname{im} \alpha, \\ i, & \text{otherwise.} \end{cases}$$

Then, $\alpha\beta\alpha(i) = \alpha(k_{\alpha(i)}) = \alpha(i)$ for all $i \in I$, and as $i \in \alpha^{-1}(\alpha(i))$ for each i, we have $f(i) \in T_{\alpha(i)} \subseteq f(k_{\alpha(i)})ESE = f(\beta\alpha(i))ESE \subseteq f(\beta\alpha(i))S$.

THEOREM 4.3. *Let $P = \coprod_{i \in I} e_i S$, where $e_i^2 = e_i \in S$ for each i, be a projective act and let $E = \{e_i : i \in I\}$. Then $W(P)$ ($\cong End_S(P)$ as discussed earlier) is a regular monoid if, and only if,*

(a) *ESE is a regular semigroup, and*

(b) *for any fixed $i \in I$ and any I-indexed family $\{s_j : j \in I\} \subseteq S$ the set $\{e_i S_j e_j : j \in I\}$ generates a principal right ideal of ESE.*

Proof. From 4.1 and 4.2 it is only the sufficiency of (a) and (b) that must be shown, and for this purpose we may use (b) of 4.1 in place of (b) of 4.3. So let $(\alpha, f) \in W(P)$ be given. Using 4.1(b), choose $\beta \in T(I)$ such that $\alpha\beta\alpha = \alpha$, and for each $i \in I$ select $s_i \in e_{\beta\alpha(i)} S e_i$ such that $f(i) = f(\beta\alpha(i))s_i$. Using (a) above, for each $j \in \operatorname{im} \alpha$ choose $t_j \in e_{\beta(j)} S e_j$ so that $f(\beta(j))t_j f(\beta(j)) = f(\beta(j))$ (observing that, for such j, $f(\beta(j)) \in e_j S e_{\beta(j)}$).

Now define $g \in S^I$ by

$$g(k) = \begin{cases} t_k, & \text{if } k \in \operatorname{im} \alpha, \\ e_{\beta(k)} e_k, & \text{otherwise.} \end{cases}$$

Note that $(\beta, g) \in W(P)$, $\alpha\beta\alpha = \alpha$, and for each $i \in I$,

$$((f_\beta g)_\alpha f)(i) = f(\beta\alpha(i))g(\alpha(i))f(i) = f(\beta\alpha(i))t_{\alpha(i)} f(\beta\alpha(i))s_i$$
$$= f(\beta\alpha(i))s_i = f(i).$$

In other words, $(\alpha, f)(\beta, g)(\alpha, f) = (\alpha, f)$ as desired.

COROLLARY 4.4, [**21**]. *If $F \cong S^{(I)}$ is a free right S-act on free generating set I, then $End_S(F)$ is a regular monoid if, and only if, S is regular and each I-indexed family of elements of S generates a principal right ideal of S.*

Proof. Take each $e_i = 1$ in Theorem 4.3.

§5. *Products of idempotents in $W(F)$, F a finitely-generated free S-act.* If F_n is a free right S-act on n free generators, then from above $End_S(F_n)$ is isomorphic to the monoid $W(F_n)=\{(\alpha,f)\colon \alpha\in T_n, f\in S^n\}$, where $(\alpha,f)(\beta,g)=(\alpha\beta,f_\beta g)$ as described earlier. Special idempotents in $W(F_n)$ have the form

$$\begin{pmatrix} & i & \\ j & & s\end{pmatrix}=\left(\begin{pmatrix} i \\ j\end{pmatrix}, f\right)$$

where $i\neq j$, $\binom{i}{j}$ is the element of T_n mapping i to j and fixing all other elements, $f(i)=s\in S$, and $f(k)=1$ if $k\neq i$. In this section we are going to investigate just which elements of $W(F_n)$ can be written as products of idempotents.

LEMMA 5.1. *If $(\alpha,f)\in W(F_n)$ has a right factor which is a special idempotent $\binom{i}{j\ s}$, then $f(i)=f(j)s\in f(j)S$ and $\alpha(i)=\alpha(j)$.*

Proof. The conclusions follow from $(\alpha,f)\binom{i}{j\ s}=(\alpha,f)$.

Our next main goal is to show that every element $(\alpha,f)\in W(F_n)$ for which $i\neq j$ exist with $\alpha(i)=\alpha(j)$ and $f(i)\in f(j)S$ is in fact a product of special idempotents. For any set I define $W(F_I):=\{(\alpha,f)\colon \alpha\in T(I), f\in S^I\}$, with multiplication $(\alpha,f)(\beta,g)=(\alpha\beta,f_\beta g)$ as usual: $W(F_I)$ is of course isomorphic to the endomorphism monoid of the free S-act with free generating set I. If $I\subseteq J$ and $(\alpha,f)\in W(F_I)$, then the *natural extension of* (α,f) to $W(F_J)$ is the element $(\bar{\alpha},\bar{f})$ of $W(F_J)$ defined by

$$\bar{\alpha}(k)=\begin{cases}\alpha(k), & \text{if } k\in I,\\ k, & \text{if } k\in J-I,\end{cases}$$

and

$$\bar{f}(k)=\begin{cases} f(k), & \text{if } k\in I,\\ 1, & \text{if } k\in J-1.\end{cases}$$

It is easy to see that if (α,f) is a product of special idempotents in $W(F_I)$, then $(\bar{\alpha},\bar{f})$ is a product of special idempotents in $W(F_J)$: simply interpret each $\binom{i}{j\ s}$ appearing in the factorization of (α,f) as a special idempotent in $W(F_J)$ and note that the resulting product in $W(F_J)$ is equal to $(\bar{\alpha},\bar{f})$. For example, if $I=\{1,2,4\}$ and $J=\{1,2,3,4,5\}$, then

$$\text{in } W(F_I)\colon \begin{pmatrix} & 2 & \\ 1 & & a\end{pmatrix}\begin{pmatrix} & 1 & \\ 4 & & b\end{pmatrix}\begin{pmatrix} & 4 & \\ 2 & & c\end{pmatrix}=\left(\begin{pmatrix}1 & 2 & 4\\ 4 & 1 & 1\end{pmatrix},\begin{pmatrix}1 & 2 & 4\\ b & a & ac\end{pmatrix}\right)=(\alpha,f);$$

in $W(F_J)$:

$$\begin{pmatrix} & 2 & \\ 1 & & a\end{pmatrix}\begin{pmatrix} & 1 & \\ 4 & & b\end{pmatrix}\begin{pmatrix} & 4 & \\ 2 & & c\end{pmatrix}=\left(\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 4 & 1 & 3 & 1 & 5\end{pmatrix},\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ b & a & 1 & ac & 1\end{pmatrix}\right)=(\bar{\alpha},\bar{f}).$$

According to Lemma 5.1, the elements of $W(F_2)$ having defect 1 which could possibly be products of special idempotents are those elements of form

$$\left(\begin{pmatrix}2\\1\end{pmatrix},(as,a)\right),\quad \left(\begin{pmatrix}2\\1\end{pmatrix},(a,as)\right),\quad \left(\begin{pmatrix}1\\2\end{pmatrix},(a,as)\right),\quad \text{or}\quad \left(\begin{pmatrix}1\\2\end{pmatrix},(as,a)\right).$$

(Here and in the remainder of this paper we use the convention of indicating an element f of S^n by simply listing its range as a sequence $(f(1), \ldots, f(n))$.) The next lemma gives explicit factorizations of each of these as products of special idempotents.

LEMMA 5.2. *For any $a, s \in S$:*

$$\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, (as, a)\right) = \begin{pmatrix} & 2 & \\ 1 & & a \end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & s \end{pmatrix};$$

$$\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, (a, as)\right) = \begin{pmatrix} & 2 & \\ 1 & & a \end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & 1 \end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & s \end{pmatrix};$$

$$\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, (a, as)\right) = \begin{pmatrix} & 1 & \\ 2 & & a \end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & s \end{pmatrix}; \quad \text{and}$$

$$\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, (as, a)\right) = \begin{pmatrix} & 1 & \\ 2 & & a \end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & 1 \end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & s \end{pmatrix}.$$

Proof. Direct calculation.

The next lemma is combinatorial in nature, and is used in the proposition which immediately follows it.

LEMMA 5.3. *Let $n \in \mathbf{N}$, $n \geqslant 4$, and suppose $i_3, \ldots, i_n \in \{1, \ldots, n\}$ are such that, for any $T \subseteq \{3, \ldots, n\}$ with $|T| = n-3$ there exists $t \in T$ such that $i_t \notin \{1, 2\} \cup T$. Then $(i_3, \ldots, i_n)$ is a permutation of $(3, \ldots, n)$.*

Proof. Pick any $j \in \{3, \ldots, n\}$ and let $T = \{3, \ldots, n\} - \{j\}$. By assumption there exists $t \in T$ such that $i_t \notin \{1, 2\} \cup T = \{1, \ldots, n\} - \{j\}$, *i.e.* $i_t = j$ for some $t \in T$.

PROPOSITION 5.4. *For any $n \geqslant 2$, if $(\alpha, f) \in W(F_n)$ is such that there exist $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, $\alpha(i) = \alpha(j)$, and $f(i) \in f(j)S$, then (α, f) is a product of special idempotents in $W(F_n)$.*

Proof. We employ induction on n, noting that Lemma 5.2 covers the case $n = 2$. Assume $n \geqslant 3$, and that elements of $W(F_m)(m < n)$ having the prescribed form can be written as products of idempotents in $W(F_m)$. Consider $(\alpha, f) \in W(F_n)$ which satisfies the condition above. Without loss of generality we may assume $i = 1$, $j = 2$, and $\alpha(1) = \alpha(2) = k \in \{1, 2, 3\}$, so

$$(\alpha, f) = ((k, k, i_3, \ldots, i_n), (as, a, b_3, \ldots, b_n))$$

where $k \in \{1, 2, 3\}$, $i_3, \ldots, i_n \in \{1, \ldots, n\}$, and $a, s, b_3, \ldots, b_n \in S$. Noting that

$$((2, 2, i_3, \ldots, i_n), (as, a, b_3, \ldots, b_n)) = ((2, 2, i_3, \ldots, i_n), (a, as, b_3, \ldots, b_n))\begin{pmatrix} & 2 & \\ 1 & & 1\end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & s\end{pmatrix},$$

we can in fact assume henceforth that $k \in \{1, 3\}$.

Next, we explicitly handle the case $n=3$. Note that both $((1, 1, 3), (as, a, 1))$ and, if $i_3 \in \{1, 3\}$, $((i_3, 2, i_3), (1, 1, b_3))$ are natural extensions of elements which belong essentially to $W(F_2)$ and which are by Lemma 5.2 products of special idempotents there. Hence, as noted earlier, this implies $((1, 1, 3), (as, a, 1))$ and $((i_3, 2, i_3), (1, 1, b_3))$ $(i_3 \in \{1, 3\})$ are products of special idempotents in $W(F_3)$. Thus, the factorizations

$$((1, 1, i_3), (as, a, b_3)) = \begin{cases} \begin{pmatrix} & 3 & \\ i_3 & & b_3\end{pmatrix}((1, 1, 3), (as, a, 1)), & \text{if } i_3 \neq 3, \\ \begin{pmatrix} & 2 & \\ 3 & & 1\end{pmatrix}\begin{pmatrix} & 3 & \\ 2 & & b_3\end{pmatrix}((1, 1, 3), (as, a, 1)), & \text{if } i_3 = 3, \end{cases}$$

and

$$((3, 3, i_3), (as, a, b_3)) = \begin{cases} \begin{pmatrix} & 2 & \\ 3 & & a\end{pmatrix}((i_3, 2, i_3), (1, 1, b_3))\begin{pmatrix} & 1 & \\ 2 & & s\end{pmatrix}, & \text{if } i_3 \in \{1, 3\}, \\ \begin{pmatrix} & 1 & \\ 3 & & 1\end{pmatrix}\begin{pmatrix} & 3 & \\ 2 & & b_3\end{pmatrix}\begin{pmatrix} & 2 & \\ 1 & & a\end{pmatrix}\begin{pmatrix} & 1 & \\ 2 & & s\end{pmatrix}, & \text{if } i_3 = 2, \end{cases}$$

yield the desired conclusion.

Henceforth we assume $n \geqslant 4$. Recall that we are also assuming

$$(\alpha, f) = ((k, k, i_3, \ldots, i_n), (as, a, b_3, \ldots, b_n))$$

where $k \in \{1, 3\}$. If $\{i_3, \ldots, i_n\} \subseteq \{1, 2, \ldots, n-1\}$ then by inductive hypothesis and natural extension the element $(\beta, g) := ((k, k, i_3, \ldots, i_{n-1}, n)$, $(as, a, b_3, \ldots, b_{n-1}, 1))$ is a product of special idempotents in $W(F_n)$. So also then is (α, f): if $i_n \neq n$ this is by virtue of $(\alpha, f) = (\,{}^{n}_{i_n}\;{}_{b_n})(\beta, g)$, and if $i_n = n$, it is because $(\alpha, f) = (\,{}^{j}_{n}\;{}_{1})(\,{}^{n}_{j}\;{}_{b_n})(\beta, g)$, where j is selected so that $j \in \{1, \ldots, n\}$ and $j \notin \{k, i_3, \ldots, i_{n-1}, n\}$.

Similarly, suppose $T \subseteq \{3, \ldots, n\}$ exists with $|T| = n-3$ such that $\{i_t : t \in T\} \subseteq \{1, 2\} \cup T$. If $k \in \{1, 2\} \cup T$ then T may be used in place of $\{3, \ldots, n-1\}$ in the previous paragraph to complete the proof. Otherwise, we have $k = 3 \notin T = \{4, \ldots, n\}$ where also $3 \notin \{i_4, \ldots, i_n\}$. In this case, we note that for any $j \notin \{3, i_3, i_4, \ldots, i_n\}$,

$$(\gamma, h) = ((j, j, 3, i_4, \ldots, i_n), (as, a, 1, b_4, \ldots, b_n))$$

is a product of special idempotents in $W(F_n)$, and the factorizations

$$(\alpha, f) = ((k, k, i_3, i_4, \ldots, i_n), (as, a, b_3, b_4, \ldots, b_n))$$
$$= \begin{cases} \begin{pmatrix} & j \\ 3 & 1 \end{pmatrix}\begin{pmatrix} & 3 \\ i_3 & b_3 \end{pmatrix}(y, h), & \text{if } i_3 \neq 3, \\ \begin{pmatrix} & j \\ 3 & 1 \end{pmatrix}\begin{pmatrix} & 3 \\ j & b_3 \end{pmatrix}(\gamma, h), & \text{if } i_3 = 3, \end{cases}$$

do the job.

Finally, if no such T exists, then for every $T \subseteq \{3, \ldots, n\}$ with $|T| = n-3$ there exists $t \in T$ such that $i_t \notin \{1, 2\} \cup T$ and so, by Lemma 5.3, $(i_3, \ldots, i_n)$ is a permutation of $(3, \ldots, n)$. We may thus write

$$((1, 1, i_3, \ldots, i_n), (as, a, b_3, \ldots, b_n))$$
$$= ((1, i_3, i_3, i_4, \ldots, i_n), (1, 1, b_3, b_4, \ldots, b_n))\begin{pmatrix} & 2 \\ 1 & a \end{pmatrix}\begin{pmatrix} & 1 \\ 2 & s \end{pmatrix}$$

and

$$((3, 3, i_3, \ldots, i_n), (as, a, b_3, \ldots, b_n))$$
$$= \begin{pmatrix} & 2 \\ 3 & a \end{pmatrix}((3, 2, i_3, \ldots, i_n), (1, 1, b_3, \ldots, b_n))\begin{pmatrix} & 2 \\ 2 & s \end{pmatrix},$$

and the proof is complete, using the inductive hypothesis and again the natural extension idea.

THEOREM 5.5. *Let S be any monoid, $n \in \mathbf{N}$, $n \geqslant 2$, and let $W(F_n) \cong End_S(F_n)$ be as defined earlier. Then for any $(\alpha, f) \in W(F_n)$ the following assertions are equivalent.*

(a) *(α, f) is a product of idempotents of positive defect in $W(F_n)$.*
(b) *(α, f) is a product of special idempotents $\binom{\ \ i}{j\ \ s}$ in $W(F_n)$.*
(c) *(α, f) has a right factor which is a special idempotent $\binom{\ \ i}{j\ \ s}$ in $W(F_n)$.*
(d) *There exist $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $\alpha(i) = \alpha(j)$ and $f(i) \in f(j)S$.*

Proof. That (c) implies (d) follows from Lemma 5.1. That (d) implies (b) follows from Proposition 5.4. That (b) implies (c) and that each of (b)–(d) implies (a) are obvious. We show finally that (a) implies (d), and for this it suffices to show that any idempotent $(\beta, g) \in W(F_n)$ of positive defect satisfies (d). But this is easy: $\beta(i) = \beta(j) = j$ for some $i, j \in \{1, \ldots, n\}$, $i \neq j$. Since $(\beta, g)(\beta, g) = (\beta, g)$ we obtain $g(i) = (g_\beta g)(i) = g(j)g(i) \in g(j)S$ as desired.

Of course, $W(F_n)$ may also contain non-trivial idempotents of defect 0, so Theorem 5.5 does not yet as a rule describe all products of idempotents in $W(F_n)$. We now address this matter.

Observe that $(\alpha, f) \in W(F_n)$ of defect 0 is a product of idempotents if, and only if, $\alpha = \iota$ (the identity permutation) and $f(i)$ is a product of indempotents in S for each $i\in(1,\ldots,n\}$. If (α,f) is a positive defect and has a right factor which is an idempotent of positive defect, then (α,f) is as described by Theorem 5.5. The remaining possibility is that (α,f) is of positive defect and is equal to $(\alpha,g)(\iota,h)$ where (α,g) is a product of defect-1 idempotents and (ι,h) is a product of defect-0 idempotents. (If $(\alpha,f)=(\varepsilon_k,g_k)\ldots(\varepsilon_1,g_1)$ where each (ε_i,g_i) is idempotent, α has positive defect, and $\varepsilon_1=\iota$, let r be the smallest index such that $\varepsilon_r\neq\iota$. Then we may take $(\alpha,g)=(\varepsilon_k,g_k)\ldots(\varepsilon_r,g_r)$ and $(\iota,h)=(\varepsilon_{r-1},g_{r-1})\ldots(\varepsilon_1,g_1)$.) In this case, there exist $i,j\in\{1,\ldots,n\}$, $i\neq j$, a, $s\in S$, and products e, e' of idempotents in S such that $\alpha(i)=\alpha(j)$, $f(i)=ase$, and $f(j)=ae'$. Conversely if $(\alpha,f)\in W(F_n)$ satisfies this condition, we can define g, $h\in S^n$ by

$$g(l)=\begin{cases} f(l), & l\notin\{i,j\},\\ as, & l=i,\\ a, & l=j,\end{cases}\qquad h(l)=\begin{cases} 1, & l\notin\{i,j\},\\ e, & l=i,\\ e', & l=j,\end{cases}$$

and then verify that $(\alpha,f)=(\alpha,g)(\iota,h)$, a product of idempotents. Thus, we have

THEOREM 5.6. *Let S be any monoid, $n\in\mathbf{N}$, $n\geqslant 2$, and let $W(F_n)\cong End_S(F_n)$ be as defined earlier. Then $(\alpha,f)\in W(F_n)$ is a product of idempotents in $W(F_n)$ if, and only if, either*

(a) *$\alpha=\iota$ and $f(i)$ is a product of idempotents in S for each $i\in\{1,\ldots,n\}$, or*

(b) *there exist $i,j\in\{1,\ldots,n\}$, $i\neq j$, a, $s\in S$, and products e, e' of idempotents in S such that $\alpha(i)=\alpha(j)$, $f(i)=ase$, and $f(j)=ae'$.*

We now apply Theorem 5.6 to characterize those monoids S having the property that, for every $n>1$, every $(\alpha,f)\in W(F_n)$ of positive defect is a product of idempotents. In Corollary 3.10, recall, we obtained the corresponding result where it was demanded that every non-invertible (α,f), even one having defect 0, be a product of idempotents: this forced S to be a group. In the present situation, a larger class of monoids is delineated.

THEOREM 5.7. *Let F be a free, non-cyclic, right S-act, and consider $W(F)\cong End_S(F)$ as defined earlier.*

(a) *Every element of $W(F)$ of positive defect is a product of idempotents if, and only if, F is finitely-generated and, for every u, v, $\in S$, there exist a, s, e, $e'\in S$ (with e and e' products of idempotents) such that $\{u,v\}=\{ase, ae'\}$.*

(b) *Every element of $W(F)$ of positive defect is a product of idempotents of positive defect if, and only if, F is finitely-generated and the principal right ideals of S form a chain.*

Proof. (a) Assume every element of $W(F)$ of positive defect is a product of idempotents. The proof of Proposition 3.3(a) shows F is finitely-generated, so $F\cong F_n$ for some $n>1$. By Theorem 5.6, as $((1,1,3,\ldots,n),(u,v,1,\ldots,1))$ is a product of idempotents, there exist a, s, e, $e'\in S$ with e and e' products of idempotents such that $\{u, v\} = \{ase, ae'\}$. Conversely, if we assume that the condition holds and if we take any $(\alpha, f) \in W(F) = W(F_n)$ of positive defect, then $\alpha(i) = \alpha(j)$ for some $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Using the condition, $f(i) = ase$ and $f(j) = ae'$ for some $a, s, e, e' \in S$ with e, e' products of idempotents. By Theorem 5.6, (α, f) is a product of idempotents.

(b) is proven in exactly the same way, using Theorem 5.5 instead of Theorem 5.6.

COROLLARY 5.8. *Let S be an idempotent-generated monoid. Then, for every $n > 1$, every element of $W(F_n)$ having positive defect is a product of idempotents. Every such element is a product of idempotents of positive defect if, and only if, the principal right ideals of S form a chain.*

Proof. Only the first assertion requires proof. For this we apply Theorem 5.7(a). Take any $u, v \in S$, and let $a = 1, s = u, e = 1, e' = v$, so that $u = ase$ and $v = ae'$. The result follows.

If we take S to be the semilattice $\{0, e, f, 1\}$ with e and f incomparable elements, then for every $n > 1$ every element of $W(F_n)$ having positive defect is a product of idempotents, but not necessarily of defect-1 idempotents. Moreover, since S is not a group, there exist non-invertible elements of $W(F_n)$ (necessarily of defect 0) that are not products of idempotents at all.

Problem 2. Extend the results of Section 5 to other situations, such as when F is not finitely-generated or to describe products of idempotents in $W(P)$ where P is projective but not necessarily free. (Assume at first the P is finitely-generated.)

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