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Flatness Properties of Acts over Commutative, Cancellative Monoids

Sydney Bulman-Fleming
Wilfrid Laurier University, sbulman@wlu.ca

FLATNESS PROPERTIES OF ACTS OVER COMMUTATIVE, CANCELLATIVE MONOIDS

SYDNEY BULMAN-FLEMING

Abstract. This note presents a classification of commutative, cancellative monoids S by flatness properties of their associated S-acts.

§1. *Introduction.* For almost three decades, an active area of research in semigroup theory has been the classification of monoids S by so-called flatness properties of their associated S-acts. The properties in question, arranged in strictly decreasing order of strength, are as follows:

free ⇒ *projective* ⇒ *strongly flat* ⇒ *condition* (P) ⇒ *flat*

⇒ *weakly flat* ⇒ *principally weakly flat* ⇒ *torsion-free.*

The general problem is to determine, for each pair of these properties, the class of monoids S over which the two chosen properties in fact coincide for all S-acts. One can vary the problem by considering S-acts of specified types (*e.g.*, cyclic or monocyclic acts), and, for reasons of tractability, one often limits the class of monoids S being considered: in this paper, for example, we consider only commutative, cancellative monoids.

We will refrain for the moment from giving detailed definitions of the flatness properties. It is sufficient to note that freeness and projectivity have the usual category-theoretic definitions, and, for acts over commutative, cancellative monoids, they are in fact identical with each other. The strongly flat acts form a class lying strictly between projective acts and acts satisfying condition (P), and it is shown in [**8**] that (over any monoid S) all strongly flat right S-acts are free if, and only if, S is a group. From [**3**] one can deduce that, over a commutative, cancellative monoid, condition (P), flatness, and weak flatness all coincide, as do principal weak flatness and torsion-freeness. Furthermore, from [**7**] it follows that in the same context, condition (P) and strong flatness coincide if, and only if, the monoid is trivial. Finally, it is shown in [**10**] and [**8**] respectively that, for all S-acts over a commutative, cancellative monoid to be flat or torsion-free, it is necessary and sufficient that S be a group. These results are summarized in Table 1.

If one instead considers only *cyclic* acts over a commutative, cancellative monoid S, the results above change in the following respects: strong flatness and projectivity of S-acts now coincide (see [**5**]) and are the same as condition (P) just when S is trivial (see [**2**]). The fact that all *cyclic* S-acts are flat if, and only if, S is a group is shown in [**9**]. For completeness, we summarize these results in Table 2.

Table 1. Classification of commutative, cancellative monoids by flatness properties of acts.

⇒	free = projective [8]	strongly flat	(*P*) = flat = weakly flat	principally weakly flat = torsion-free
strongly flat	Groups [8]			
(*P*) = flat = weakly flat [3]	{1}	{1} [7]		
principally weakly flat = torsion-free [3]	{1}	{1}	?	
all	{1}	{1}	Groups [10]	Groups [8]

The situation for monocyclic acts (that is, acts of the form $S/\rho(s, t)$ where $\rho(s, t)$ is the smallest congruence on S containing the pair (s, t), for $s, t \in S$) is identical with that just described for cyclic acts.

The purpose of this paper is to complete the classification of commutative, cancellative monoids S by flatness properties of their S-acts, and also by flatness properties of their cyclic or monocyclic acts. From the preceding paragraphs, we see that the remaining problem is *to determine, relative to each of these classes of acts, the conditions on S under which all torsion-free acts satisfy condition (P).*

§2. *Preliminaries.* For reasons of brevity, we will define only those terms to be used directly in this paper. For more complete information, we refer the reader to [4] and its bibliography, or to the survey article [1].

Let S be any monoid. A *right S-act* is a non-empty set A together with a mapping

$$A \times S \to A, \quad (a, s) \rightsquigarrow as,$$

satisfying the conditions $a1 = a$ and $(as)t = a(st)$ for all $a \in A$ and $s, t \in S$. Cyclic right S-acts are isomorphic to acts of the form S/ρ where ρ is a right congruence on S, and such acts are termed *monocyclic* in case $\rho = \rho(s, t)$ for some $s, t \in S$.

Because flatness properties of acts of the form $S/\rho(s, t)$ will play a large role in the sequel, the following description of $\rho(s, t)$ in the commutative,

Table 2. Classification of commutative, cancellative monoids by flatness properties of cyclic acts.

⇒	free = projective = strongly flat [5]	(*P*) = flat = weakly flat	principally weakly flat = torsion-free
(*P*) = flat = weakly flat [3]	{1} [2]		
principally weakly flat = torsion-free [3]	{1}	?	
all	{1}	Groups [9]	Groups [8]

cancellative setting will be important. (Its straightforward proof, and that of the ensuing corollary, are omitted.)

PROPOSITION 2.1. *Let S be a commutative, cancellative monoid and let $s, t \in S$. Define the relation α on S by*

$$(x, y) \in \alpha \text{ if there exists } u \in S \text{ such that } x = su \text{ and } tu = y.$$

Then

(1) *$(x, y) \in \alpha^n$ (for $n \in \mathbf{N}$) if, and only if, there exist $u_1, \ldots, u_n \in S$ such that*

$$\begin{aligned} x &= su_1, \\ tu_1 &= su_2, \\ &\vdots \\ tu_{n-1} &= su_n, \\ tu_n &= y; \end{aligned}$$

(2) *for each $m, n \in \mathbf{Z}$, $\alpha^m \circ \alpha^n \subseteq \alpha^{m+n}$ (where we define α^0 to be Δ, the equality relation on S);*

(3)

$$\rho(s, t) = \bigcup_{n \in \mathbf{Z}} \alpha^n.$$

COROLLARY 2.2. *If s and t are distinct elements of a commutative, cancellative monoid S and if $x\ \rho(s, t)\ y$ for elements $x, y \in S$, then there exists a non-negative integer n such that $s^n x = t^n y$ or $t^n x = s^n y$.*

A right S-act A is said to *satisfy condition* (P) if, for every $a, a' \in A$ and $s, s' \in S$, $as = a's'$ implies $a = a''u$, $a' = a''u'$, and $us = u's'$ for some $a'' \in A$ and some $u, u' \in S$. The act A is called *torsion-free* if, whenever $ac = a'c$, with $a, a' \in A$ and c a right cancellable element of S, it follows that $a = a'$. (As seen in the Introduction, it is only with these two properties that we need be concerned, for the present purpose.) For cyclic acts these properties assume the following forms.

PROPOSITION 2.3. (*See* [4].) *If ρ is a right congruence on a monoid S, then*

(1) *S/ρ is torsion-free if, and only if, for all $x, y, c \in S$ with c right cancellable, $xc\ \rho\ yc$ implies $x\ \rho\ y$;*

(2) *S/ρ satisfies condition (P) if, and only if, for all $x, y \in S$, $x\ \rho\ y$ implies $ux = vy$ for some $u, v \in S$ such that $u\ \rho\ 1\ \rho\ v$.*

Let S be any monoid. We call a right congruence ρ on S *torsion-free* if the cyclic right act S/ρ is torsion-free. It is clear that the universal relation ∇ is torsion-free, and that the intersection of any family of torsion-free right congruences is again torsion-free. Thus, every right congruence ρ is contained in a smallest torsion-free right congruence that we designate $\bar{\rho}$. For commutative, cancellative monoids we can give an explicit description of $\bar{\rho}$ in case $\rho = \rho(s, t)$ for some $s, t \in S$.

PROPOSITION 2.4. *Let S be a commutative, cancellative monoid, let $s, t \in S$, and let $\rho = \rho(s, t)$. Then $\bar{\rho}$ is the congruence on S defined by*

$$(x, y) \in \bar{\rho} \Leftrightarrow (x = y) \text{ or } (\exists n \in \mathbf{N})(s^n x = t^n y \text{ or } s^n y = t^n x).$$

Proof. Let τ denote the relation given by

$$(x, y) \in \tau \Leftrightarrow (x = y) \text{ or } (\exists n \in \mathbf{N})(s^n x = t^n y \text{ or } s^n y = t^n x).$$

The verification that τ is a congruence is routine. From the equality $st = ts$ it follows that $\rho \subseteq \tau$. To see that τ is torsion-free, suppose that $xc \; \tau \; yc$ for some $x, y, c \in S$. If $xc = yc$ then $x = y$, and $x \; \tau \; y$ follows at once. Otherwise, either $s^n xc = t^n yc$ or $s^n yc = t^n xc$ for some $n \in \mathbf{N}$, from which we get $s^n x = t^n y$ or $s^n y = t^n x$, so again $x \; \tau \; y$. Finally, suppose that θ is any torsion-free congruence containing ρ. If $s^n x = t^n y$, then because $s^n \; \rho \; t^n$ we see that $s^n x = t^n y \; \theta \; s^n y$, and so $x \; \theta \; y$. A similar proof handles the case where $s^n y = t^n x$. Thus $\tau \subseteq \theta$. It follows that $\tau = \bar{\rho}$, as required.

PROPOSITION 2.5. *Let S be a commutative, cancellative monoid and let s and t be distinct elements of S. Then*

(1) *$S/\bar{\rho}(s, t)$ satisfies condition (P) if, and only if, $us = vt$ for some $u, v \in S$ such that, for some $m \geqslant 0$, either $u^{m+1} = v^m$ or $v^{m+1} = u^m$;*
(2) *$S/\rho(s, t)$ is torsion-free if, and only if, $\rho(s, t) = \bar{\rho}(s, t)$;*
(3) *$S/\rho(s, t)$ satisfies condition (P) if, and only if, at least one of s, t is a unit.*

Proof. (1) Let us denote $\bar{\rho}(s, t)$ by simply $\bar{\rho}$. Suppose first that $S/\bar{\rho}$ satisfies (P). Then, by Proposition 2.3, because $s \; \bar{\rho} \; t$, there exist $w, z \in S$ such that $ws = zt$ and $w \; \bar{\rho} \; 1 \bar{\rho} \; z$. If $w = 1$, we may take $u = 1$, $v = z$, $m = 0$ and obtain $us = vt$ and $u^{m+1} = v^m$ as required. If $z = 1$, we let $u = w$, $v = 1$, and $m = 0$, obtaining $us = vt$ and $v^{m+1} = u^m$.

Assume now that neither w nor z is 1. Then, from Proposition 2.4, either $s^n w = t^n$ or $t^n w = s^n$ for some $n \in \mathbf{N}$, and either $s^m z = t^m$ or $t^m z = s^m$ for some $m \in \mathbf{N}$. Let us examine the possible cases.

Suppose that $s^n w = t^n$ and $s^m z = t^m$ for $n, m \in \mathbf{N}$. If $m = n$, then we obtain $w = z$, and therefore $s = t$, contrary to assumption. If $m < n$, from the equality $ws = zt$ we obtain $w^n s^n = z^n t^n$ and so $w^n s^n = z^n s^n w$, which yields $w^{n-1} = z^n$. If $m > n$, we instead use $w^m s^m = z^m t^m$ to get $w^m s^m = z^m s^m z$, and thence $w^m = z^{m+1}$.

Now suppose $s^n w = t^n$ and $t^m z = s^m$ for $n, m \in \mathbf{N}$. If $m = n$ then $t^m = s^m w = t^m zw$, $zw = 1$, s and t divide each other, and we may proceed as in the first paragraph. If $m > n$, from $w^m s^m = z^m t^m$ we obtain $w^m t^m z = z^m t^m$ and thus $w^m = z^{m-1}$, whereas if $m < n$, we use $w^n s^n = z^n t^n$ to get $w^n s^n = z^n s^n w$, yielding $w^{n-1} = z^n$.

The remaining two cases can be handled using symmetry with the two just presented.

Finally, let us show that, if distinct elements s and t satisfy the stated condition, then $S/\bar{\rho}$ indeed satisfies (P). To this end, suppose that x and y are distinct elements related by $\bar{\rho}$, so that $s^p x = t^p y$ or $t^p x = s^p y$ for some $p \in \mathbf{N}$. Let us assume first that $us = vt$ and $u^{m+1} = v^m$ for some $m \geqslant 0$. Then

$$u^{m+1} s^{m+1} = v^{m+1} t^{m+1} \Rightarrow v^m s^{m+1} = v^{m+1} t^{m+1} \Rightarrow s^{m+1} = vt^{m+1}.$$

By Proposition 2.4, this shows that $1\ \bar{\rho}\ v$. Similarly, the chain of implications

$$u^m s^m = v^m t^m \Rightarrow u^m s^m = u^{m+1} t^m \Rightarrow s^m = u t^m$$

shows that $1\ \bar{\rho}\ u$. If $s^p x = t^p y$, the calculation

$$u^p s^p x = v^p t^p x = u^p t^p y \Rightarrow v^p x = u^p y,$$

together with the observation $v^p\ \bar{\rho}\ 1\ \bar{\rho}\ u^p$ yields the desired conclusion by Proposition 2.3. The calculation is parallel in case $t^p x = s^p y$. A similar development, in the case $v^{m+1} = u^m$, finishes the proof.

(2) This is immediate from Proposition 2.4.

(3) If $s = t$ then $S/\rho(s, t)$ is in fact free, so satisfies condition (P). If s is a unit, then $\rho(s, t) = \rho(1, s^{-1}t)$, and it is well-known (see [**4**]) that, in this case, $S/\rho(s, t)$ also satisfies condition (P). On the other hand, if we are given that $S/\rho(s, t)$ satisfies (P), then by Proposition 2.3, $us = vt$ for some $u, v \in S$ such that $u\ \rho(s, t)\ 1\ \rho(s, t)\ v$. If $u = 1 = v$ then $s = t$ follows, whereas if at least one of u, v is different from 1, then at least one of s, t is necessarily a unit.

§3. *Classification of commutative, cancellative monoids S by flatness properties of S-acts.* As observed earlier, the only unresolved item in Table 1 is the description of the monoids over which all torsion-free acts satisfy condition (P). The following proposition provides this description.

PROPOSITION 3.1. *Let S be commutative, cancellative monoid. Then every torsion-free (right) S-act satisfies condition (P) if, and only if, the principal ideals of S form a chain (under inclusion).*

Proof. Suppose that every torsion-free S-act satisfies condition (P). For any $a, b \in S$ the ideal $aS \cup bS$, being torsion-free, satisfies condition (P) and so, from $ab = ba$ we know that there exist $c \in aS \cup bS$ and $u, v \in S$ such that $cu = a$, $cv = b$, and $ub = va$. If $c \in aS$ then $bS \subseteq aS$ results, whereas if $c \in bS$ we have $bS \subseteq aS$.

Now assume that the principal ideals of S form a chain. Suppose that A_S is a torsion-free right S-act, and suppose that $as = bt$ for $a, b \in A$ and $s, t \in S$. Without loss of generality, let us assume that $s = ut$ for some $u \in S$. Then, from $aut = bt$, using the fact that A is torsion-free, we obtain $au = b$. Thus, from $as = bt$, we have found an element u such that $a = a1$, $b = au$, and $1s = ut$. This shows that A satisfies condition (P).

§4. *Classification of commutative, cancellative monoids by flatness properties of cyclic S-acts.* We now provide the missing item in the classification of commutative, cancellative monoids by flatness properties of cyclic acts (see Table 2 earlier).

THEOREM 4.1. *Let S be a commutative, cancellative monoid. Then the following statements are equivalent.*

(1) *Every torsion-free cyclic S-act satisfies condition (P).*

(2) *$S/\bar{\rho}(s, t)$ satisfies condition (P), for all $s, t \in S$.*

(3) *For all $x, y \in S$, either $x = y$ or there exist $u, v \in S$ such that $ux = vy$, and either $u^{m+1} = v^m$ or $v^{m+1} = u^m$ for some $m \geqslant 0$.*

Proof. Only the implication (3) $\Rightarrow$ (1) requires attention, in view of what has gone before. Suppose that S/ρ is torsion-free, and suppose that $x \rho y$ for some elements $x, y \in S$. If $x = y$, it is easy to see that taking $u = 1 = v$ fulfils condition (2) of Proposition 2.3. If $x \neq y$, take u and v as provided by condition (3). Let us assume first that $ux = vy$ and $u^{m+1} = v^m$ for some $m \geqslant 0$. Then

$$u^{m+1}x^{m+1} = v^{m+1}y^{m+1} \Rightarrow v^m x^{m+1} = v^{m+1}y^{m+1} \Rightarrow vy^{m+1} = x^{m+1}.$$

But $x^{m+1} \rho \, y^{m+1}$, and so $vy^{m+1} \, \rho \, 1y^{m+1}$. Because ρ is torsion-free, $v \, \rho \, 1$. On the other hand,

$$u^m x^m = v^m y^m \Rightarrow u^m x^m = u^{m+1} y^m \Rightarrow uy^m = x^m \, \rho \, 1y^m$$

and so, again using torsion-freeness, $u \, \rho \, 1$. The argument is similar in case $v^{m+1} = u^m$. This completes the proof.

COROLLARY 4.2. *Let S be a commutative cancellative monoid having the property that every torsion-free, cyclic S-act satisfies condition (P). Then, for every $s, t \in S$ there exists a positive integer n such that $s^n \mid t^n$ or $t^n \mid s^n$.*

Proof. Without loss of generality we may assume that $s \neq t$, so that there exist $u, v \in S$ and $m \geqslant 0$ such that $us = vt$ and either $u^{m+1} = v^m$ or $v^{m+1} = u^m$. From $u^{m+1}s^{m+1} = v^{m+1}t^{m+1}$, in the first case we derive $s^{m+1} = vt^{m+1}$, and in the second case we obtain $t^{m+1} = us^{m+1}$. Thus the choice $n = m + 1$ does the job.

We will see shortly that the necessary condition presented in the corollary above is not sufficient even for all torsion-free *monocyclic* S-acts to satisfy condition (P).

It was shown earlier that in order for *every* torsion-free S-act to satisfy condition (P), it is necessary and sufficient that the principal ideals of S be linearly ordered (*i.e.*, that one can always take $n = 1$ in the corollary above). We now give an example of a commutative, cancellative monoid S over which every torsion-free *cyclic* act satisfies (P), yet over which not every torsion-free act satisfies (P).

PROPOSITION 4.3. *Let S denote the monoid having presentation*

$$S = \langle u, v \mid u^2 = v^3, uv = vu \rangle^1.$$

Then S is a commutative, cancellative monoid with the property that every torsion-free cyclic *S-act satisfies condition (P). However, not every torsion-free S-act satisfies condition (P).*

Proof. Every element of S can be written uniquely as $u^i v^j$ where $i=0$ or 1 and $j \geqslant 0$. Using the rules

$$v^i \cdot v^j = v^{i+j},$$
$$uv^i \cdot v^j = uv^{i+j},$$
$$uv^i \cdot uv^j = v^{i+j+3},$$

one can show that S is cancellative, and that the ideals uS and vS are incomparable. Thus, from Proposition 3.1, we know that not all torsion-free acts satisfy condition (P).

We now show that S satisfies condition (3) of Theorem 4.1 to complete the proof. We must show that

$$(\forall s \neq t \in S)(\exists x, y \in S)(xs = yt,\ x^{m+1} = y^m \text{ or } y^{m+1} = x^m \text{ for some } m \geqslant 0).$$

We observe first that if sS and tS are comparable, then the condition above is easily satisfied: for example, if $s = pt$, take $x = 1$ and $y = p$, obtaining $xs = yt$, $x^1 = y^0$. This immediately reduces our problem to showing that the condition is satisfied when $s = uv^i$, $t = v^j$ (without loss of generality) for some $i, j \geqslant 0$.

Because $uv^k \cdot uv^i = v^{i+k+3}$, we have $v^j \in uv^i S$ whenever $j \geqslant i+3$. Also if $j \leqslant i$ we have $uv^i = uv^{i-j} \cdot v^j$, implying that $uv^i \in v^j S$. Thus we are left with the two cases $j = i+1$ and $j = i+2$. If $s = uv^i$, $t = v^{i+1}$, we observe that $v \cdot uv^i = u \cdot v^{i+1}$, and $v^3 = u^2$. On the other hand, if $s = uv^i$, $t = v^{i+2}$, then we note that $u \cdot uv^i = v \cdot v^{i+2}$ and $u^2 = v^3$. This completes the proof.

§5. *Classification of commutative, cancellative monoids by flatness properties of monocyclic S-acts.*

THEOREM 5.1. *Let S be a commutative, cancellative monoid. Then the following statements are equivalent.*

(1) *Every torsion-free monocyclic S-act satisfies condition (P).*

(2) *For every pair s, t of distinct non-units of S, $\bar{\rho}(s, t) \neq \rho(s, t)$.*

Proof. First assume (1). If s and t are distinct non-units of S, then the monocyclic act $S/\rho(s, t)$ does not satisfy condition (P), by part (3) of Proposition 2.5, and so by assumption it is not torsion-free. From part (2) of the same proposition it follows that $\bar{\rho}(s, t) \neq \rho(s, t)$, as required.

Now assume (2). Let $S/\rho(s, t)$ be a torsion-free monocyclic S-act. If $S/\rho(s, t)$ did not satisfy condition (P) then, using part (3) of Proposition 2.5 again, s and t would be distinct non-units of S. Yet, by part (2) of the same proposition, $\rho(s, t) = \bar{\rho}(s, t)$. This is in contradiction to our assumption (2).

The characterization given in Theorem 5.1 above is not as applicable as we would like. We can however give a version of it that handles many familiar situations. Let us say that elements s and t of a commutative, cancellative monoid S are *power-cancellative* if whenever $s^n = t^n$ for some $n \in \mathbf{N}$, then s and t are associates in S (that is, they differ by a unit factor). The monoid S

itself will be called power-cancellative if every pair of elements of S is power-cancellative. Define s and t to be *coprime* if they are non-units, and if, for all $u \in S$, $s \mid tu \Rightarrow s \mid u$, and $t \mid su \Rightarrow t \mid u$. Observe that if s and t form a coprime pair, then neither element is a divisor of the other.

PROPOSITION 5.2. *Let S be a commutative, cancellative monoid and let s, t be distinct, power-cancellative, non-units of S. Then $S/\rho(s, t)$ is torsion-free if, and only if, s and t are coprime.*

Proof. ($\Leftarrow$) (For this implication, the condition of power-cancellativity is not needed.) Assume that s and t are coprime. Suppose that $u\ \bar{\rho}(s, t)\ v$. Let us suppose that $s^n u = t^n v$ for some $n \in \mathbf{N}$. One can show that s^n and t^n are also coprime, so that $s^n \mid v$ and $t^n \mid u$. Then, using cancellation, one obtains $v = ks^n$ and $u = kt^n$ for some $k \in S$. Thus $u = kt^n\ \rho(s, t)\ ks^n = v$. (We have used here the fact that $s^m\ \rho\ t^m$ for all $m \in \mathbf{N}$.) Part (2) of Proposition 2.5 yields the required result.

($\Rightarrow$) Assume now that S/ρ is torsion-free, so that $\bar{\rho}(s, t) \subseteq \rho = \rho(s, t)$. Suppose that $u \in S$ is such that $s \mid tu$, so that $sq = tu$ for some $q \in S$. Then $(u, q) \in \bar{\rho}(s, t)$ and so $u\ \rho\ q$. If $u = q$, then $s = t$, contrary to the fact that s and t are distinct. So $(u, q) \in \alpha^n$ for some $n \neq 0$. If $n > 0$ then (referring to Proposition 2.1) $u \in sS$ and the result follows. If $n < 0$, on the other hand, $s^m u = t^m q$ for some $m \in \mathbf{N}$, using Corollary 2.2. In this case, multiplication of both terms by st and cancelling yields the equality $s^{m+1} = t^{m+1}$ and so, by assumption, s and t are associates. Let us suppose that $s1 = tz$, where $z \in S$ is a unit. Using again $\bar{\rho}(s, t) \subseteq \rho$, it follows that $z\ \rho\ 1$. But of course $z \neq 1$, since $s \neq t$, and so either s or t is a unit. This is a contradiction. We have therefore shown that $s \mid tu \Rightarrow s \mid u$. Similarly, $t \mid su \Rightarrow t \mid u$, and so the proof is complete.

COROLLARY 5.3. *If S is a power-cancellative, commutative, cancellative monoid, then the following statements are equivalent.*

(1) *Every torsion-free monocyclic S-act satisfies condition (P).*

(2) *No coprime pairs of elements exist in S.*

We conclude this paper by presenting two further examples. The first demonstrates that Corollary 5.3 becomes false if the adjective "power-cancellative" is omitted.

PROPOSITION 5.4. *Let S be the monoid with presentation*

$$\langle a, b \mid a^2 = b^2, ab = ba \rangle^1.$$

Then S is a commutative, cancellative monoid that has no coprime pairs. However, there exists a torsion-free monocyclic S-act that fails to satisfy condition (P).

Proof. Each element of S can be written uniquely in one of the forms $1, a, a^2, \ldots$ or $b, ab, a^2b, \ldots$. It is routine to show that S is cancellative. Recalling that divisors of each other never form a coprime pair, we are left with showing that no pair of the form $a^i b, a^j$ is coprime. If $j \leqslant i$ then $a^j \mid a^i b$.

From the equality $a^k b a^i b = a^{i+k+2}$ we conclude that $a^i b \mid a^j$ whenever $j \geq i+2$. Thus the remaining pairs to consider are those of form $a^i b, a^{i+1}$. If $i>0$, then $a^i b \mid a^{i+1} b$ but $a^i b \nmid b$. If $i=0$, then a, b are not a coprime pair because $a \mid b \cdot b$ but $a \nmid b$.

The monocyclic act $S/\rho(a, b)$ fails to satisfy condition (P), by Proposition 2.5(3), since a and b are distinct non-units of S. It is however torsion-free. To show this, we shall verify that (2) of the same proposition is satisfied. Suppose that x and y are distinct elements of S such that $x\,\bar{\rho}(a, b)\,y$. Let us suppose $a^m x = b^m y$ for some $m \in \mathbf{N}$. If m is even then $x=y$, by cancellation. If $m=2k+1$ for some $k \geq 0$, the supposed equality reduces to $ax=by$, by cancellation of the factor $a^{2k}=b^{2k}$. If $x=a^i$ then $i \geq 1$ and $y = ba^{i-1}\,\rho(a, b)\,aa^{i-1}=x$. If $x=a^i b$ then $y=a^{i+1}\,\rho(a, b)\,a^i b = x$. Thus, in all cases, $x\,\rho(a, b)\,y$, and the proof is complete.

Note. The monoid S above also has the property that, for every $s, t \in S$, there exists a positive integer n such that either $s^n \mid t^n$ or $t^n \mid s^n$, showing that the converse to Corollary 4.2 is false.

Our final example shows that the class of monoids over which all torsion-free *monocyclic* acts satisfy condition (P) is strictly larger than the class of monoids over which all torsion-free cyclic acts satisfy (P).

PROPOSITION 5.5. *Let S be the submonoid $\{a^i b^j : i, j \geq 1\}^1$ of the free commutative monoid on two generators a and b. Then all torsion-free* monocyclic *S-acts satisfy condition (P). However, there exists a torsion-free cyclic S-act that does not satisfy (P).*

Proof. We first show that S is power-cancellative and contains no coprime pairs. Corollary 5.3 will then imply all torsion-free monocyclic S-acts satisfy condition (P). To this end, note that if $(a^i b^j)^m = (a^k b^l)^m$, then $i=k$ and $j=l$ quickly follow.

If $a^i b^j$ and $a^k b^l$ are any two distinct non-units in S, then, without loss of generality, we may assume that $i<k$. But then

$$a^i b^j \mid a^k b^l \cdot a^i b^{j+1}$$

in S because

$$a^k b^l \cdot a^i b^{j+1} = a^{k+i} b^{l+j+1} = a^i b^j \cdot a^k b^{l+1}.$$

However, $a^i b^j$ does not divide $a^i b^{j+1}$ in S. So S contains no coprime pairs of elements.

If all torsion-free, cyclic S-acts satisfied condition (P), then the conclusion of Corollary 4.2 would be satisfied for every pair of elements of S. However, suppose that $(ab^2)^m \mid (a^2 b)^m$ for some $m \in \mathbf{N}$. Then we would have $a^{2m} b^m = a^{m+i} b^{2m+j}$ for some $i, j \geq 1$, which is clearly impossible.

Table 3. Classification of commutative, cancellative monoids by flatness properties of Rees factor acts.

⇒	free = projective = strongly flat	(*P*) = flat = weakly flat = principally weakly flat = torsion-free
(*P*) = flat = weakly flat = principally weakly flat = torsion-free	$\{1\}$	
all	$\{1\}$	groups

§6. *Classification of commutative, cancellative monoids by flatness properties of Rees factor S-acts.* For completeness, we present the table giving the situation when Rees factor acts are considered. It is obtained by straightforward specialization of Table 10 of [1].

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Professor S. Bulman-Fleming,
Department of Mathematics,
Wilfrid Laurier University,
Waterloo,
Ontario N2L 3C5,
Canada.
e-mail: sbulman@wlu.ca

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