ANISOTROPIC SPHEROIDAL MODEL

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Abstract

Exact solutionto the anisotropic Einstein field equations is obtained with a specified form of the anisotropicfactor. The field equations are transformed to a simpler form; the integration of the system is reduced to solving the condition of pressure anisotropy. It is possible to obtain general class of solutions in terms of elementary functions that model the interior of relativistic fluid sphere.

Keywords: Einstein field equations, exact solutions, anisotropic matter

Introduction

The theory of general relativity provides a very satisfactory explanation of the behavior of the gravitational field. The influence of the gravitational field on the matter distribution is expressed by the Einstein field equations which is a nonlinear coupled system of partial differential equations and they are difficult to solve in general. Exact solutions of the Einstein field equations for an anisotropic matter are important in the description of relativistic astrophysical processes. In recent years a number of authors have studied exact solutions to the Einstein field equations corresponding to the anisotropic matter where the radial component of the pressure differs from the angular component. These investigations are contained in the papers (K. Dev and M. Gleiser, 2003), (M. K. Mak and T. Harko, 2003), among others.

The field equations are solved by various restrictions placed by investigators on the geometry of spacetime and the matter content. Mainly two distinct procedures have been adoptedto solve these equations for spherically symmetric static manifolds. Firstly, the coupled differential equations are solved by computations after choosing an equation of state. There exist several reviews of the problem associated with an equation of state. Secondly, the exact Einstein solutions can be obtained by specifying the geometry and the form of the anisotropic factor. In this work we follow the later technique in an attempt to find solutions in terms of elementary that are suitable for description of relativistic stars as pointed out in (K. Komathiraj and S. D. Maharaj, 2010). This approach was recently used by (M. Chaisi and S. D. Maharaj, 2005)that yield a solution in terms of elementary functions. This solution have considered by many authors in the analysis of gravitational behavior of compact objects, and the study of anisotropy under strong gravitational fields.

Our main objective is to obtain simple forms for the solutions to the Einstein field equations with the anisotropic matter that highlights the role of the spheroidal parameter similar to the recent treatment in [3]. In section 2, the Einstein field equations for the static spherically symmetric line element with anisotropic matter is expressed and the condition of pressure anisotropy is written as second order differential equations by specifying one of the gravitational potential. In section 3, we chose particular formfor the anisotropic factor, which enables us to obtain the condition of pressure anisotropy in the remaining gravitational potential with the assistance of a transformation. It is then possible to exhibit exact solutions to the Einstein field equations in a series form. In section 4, we demonstrate that the exact solutions to the Einstein field equations in terms elementary functions are possible and we generate two linearly independent classes of solutions. Finally in section 5, we discuss the physical feature of the solutions.

The anisotropic equations

We take the line element for static spherically symmetric spacetimes to be

$$ds^2 = -e^{2f(r)}dt^2 + e^{2g(r)}dr^2 + r^2(d\theta^2 + \sin\theta d\varphi^2) \tag{1}$$

in Schwarzschild coordinates $(t,\ r,\ \theta,\ \varphi\ )$, where $f(r)$ and $g(r)$are arbitrary functions. For a perfect fluid,
theEinstein field equations can be written in the form

$$\frac{1}{r^2}(1 - e^{-2g}) + \frac{2}{r}\frac{dg}{dr}e^{-2g} = \mu \tag{2}$$

$$-\frac{1}{r^2}(1 - e^{-2g}) + \frac{2}{r}\frac{df}{dr}e^{-2g} = p_r \tag{3}$$

$$e^{-2g}\left[\frac{d^2f}{dr^2} + \left(\frac{df}{dr}\right)^2 + \frac{1}{r}\frac{df}{dr} - \frac{dg}{dr}\frac{df}{dr} - \frac{1}{r}\frac{dg}{dr}\right] = p_t \tag{4}$$

In equation (2) - (4), the quantity μ is the energy density, p_r is the radial pressure and p_t is the tangential pressure.

The Einstein fieldequations (2)-(4)describe the gravitational behaviour foran anisotropic imperfect fluid. For matter distributions with $p_r = p_t$ (isotropicpressures), the Einstein's equations for isotropic fluid may be regained from (2)-(4).To solve the system, it is necessary to specify two of the variables. In our approach we chose e^{2g} and anisotropic factor. In the integration procedure we make the choice

$$e^{2g} = \frac{1 - Kr^2}{1 - r^2} \tag{5}$$

where$K$ is spheroidal parameter. On substituting the choice (5) in (4) - (3), we obtain

$$(1 - Kr^2)^2\Delta = (1 - Kr^2)(1 - r^2)\left(\frac{d^2f}{dr^2} + \left(\frac{df}{dr}\right)^2 - \frac{1}{r}\frac{df}{dr}\right) - r(1 - K)\left(\frac{df}{dr} + \frac{1}{r}\right) + \quad (1 - K)(1 - Kr^2) \tag{9}$$

which is the condition of pressure anisotropy, the quantity $\Delta$is defined as the measure of anisotropy or anisotropy factor. The solution of the system (2) – (4) depends on the integrability of (9). It is necessary to specify the anisotropic factor to integrate (9).

Master equation

The differential equation (9)is difficult to solve. However it can be transformed to a different type of differential equation which can be solved. It is convenient to introduce the following transformation in (9).

$$\omega(x) = \ e^{f(r)}, \qquad x^2 = 1 - r^2 \qquad (10)$$

Substitution of (10) into(9) leads to the equation

$$(1 - K + Kx^2)\frac{d^2\omega}{dx^2} - Kx\frac{d\omega}{dx} + \left(\frac{(1 - K + Kx^2)^2\Delta}{x^2 - 1} + K(K - 1)\right)\omega = 0 \qquad (11)$$

which is the second order linear differential equation in the new variables x and ω . It is necessary to specify the anisotropic factor Δ to solve (11). A variety of choices for Δ is possible but only a few are physically reasonable which generate closed form solutions. The differential equation (11) can be reduced to simpler form if we let

$$\Delta = \frac{aK(x^2 - 1)x}{(1 - K + Kx^2)^2}$$

where$a$ is a real constant. Upon substituting this choice into equation (11) we obtain

$$(1 - K + Kx^2)\frac{d^2\omega}{dx^2} \quad - Kx\frac{d\omega}{dx} + K(a + K - 1)\omega = 0 \qquad (12)$$

As the point $x = 0$ is a regular point of (12), there exists two linearly independent solutions of the form of a power series with centre$x = 0$. Thus, we assume

$$\omega = \sum_{I=0}^{\infty} a_i x^i, \quad a_0 \neq 0 \qquad (13)$$

wherea_i are the coefficients of the series to be determined. For a legitimate solution the coefficients a_i should be determined explicitly. On substituting (13) into (11), we obtain

$$a_{i+2} = \frac{K[a + K - 1 + i(i - 2)]}{(K - 1)(i + 1)(i + 2)} a_i, \qquad i \geq 2 \qquad (14)$$

which is the recurrence formula, or difference equation, governing the structure of the solution. It is possible to express the general even coefficient a_{2i} in terms of the leading coefficient a_0 by establishing a general structure for the coefficient by considering the leading terms. These coefficients generate the pattern

$$a_{2i} = \left(\frac{K}{K - 1}\right)^i \frac{1}{2i!} \prod_{p=1}^{i} [a + K - 1 + 4(p - 1)(p - 2)]\, a_0$$

We can obtain a similar formula for the odd coefficients in terms of the leading coefficient $a_1$as

$$a_{2i+1} = \left(\frac{K}{K - 1}\right)^i \frac{1}{(2i + 1)!} \prod_{p=1}^{i} [a + K - 1 + (2p - 1)(2p - 3)]\, a_1$$

The coefficients are generated from the difference equation (14) and are expressible in terms of the leading coefficients. Now it is possible to establish the general solution to (12) from (13) and these two patterns as

$$\omega = a_0\left[1+\sum_{i=1}^{\infty}\left(\frac{K}{K-1}\right)^i \frac{1}{2i!}\prod_{p=1}^{i}[a+K-1+4(p-1)(p-2)]\,x^{2i}\right] + a_1\left[x\sum_{i=1}^{\infty}\left(\frac{K}{K-1}\right)^i \frac{1}{(2i+1)!}\prod_{p=1}^{i}[a+K-1+(2p-1)(2p-3)]\,x^{2i+1}\right] \qquad (15)$$

Thus we have found the general series solution to differential equation (12). Solution (15) is expressed in terms of a series with real arguments unlike the complex arguments given by software packages.

General solutions with elementary functions

Solution in terms of polynomials

It is well known that the general series solution can be written in terms of polynomials for particular parameter values. This statement is also true for the series in (15). Consequently two sets of solutions in terms of polynomials can be found by restricting the range of values of K and a. We first consider polynomials of even degree. It is convenient to set

$K + a = 2 - (2n-1)^2$, $n > 1$and $i = 2(j-1)$ in (14). This leads to

$$a_{2j} = (-\gamma)^j \frac{(n+j-2)!}{(n-j)!\,(2j)!}x^{2j}, \qquad 0 \le j \le n$$

where we have set $\gamma = 4 - \frac{4}{4n(n-1)+\alpha}$and $a_0 = \frac{1}{n(n-1)}$

With the help of this we can express the polynomial solution to (12) in even powers of x in the form

$$\omega_1 = \sum_{j=0}^{n}(-\gamma)^j \frac{(n+j-2)!}{(n-j)!\,(2j)!}x^{2j}.$$

We now consider polynomial of odd degree. For this case we let

$K + a = 2(1-2n^2)$and$i = 2(j-1)+1$ in (14). We obtain

$$a_{2j+1} = (-\mu)^j \frac{(n+j-1)!}{(n-j)!\,(2j+1)!}x^{2j}, \qquad 0 \le j \le n$$

where we have set $\mu = 4 - \frac{4}{4n^2-1+\alpha}$and $a_1 = \frac{1}{n}$

With the assistance of this pattern we can express the polynomial in odd powers of x as

$$\omega_2 = \sum_{j=0}^{n}(-\mu)^j \frac{(n+j-1)!}{(n-j)!\,(2j+1)!}x^{2j+1}$$

The polynomial solution ω_1 and ω_2 given above comprise the first solution of (12) for appropriate values of the model parameters.

Algebraic functions

We take the second solution of (12) to be of the form

$$\omega = u(x)(1-K+Kx^2)^{3/2}$$

when*u(x)* is an arbitrary polynomial. Particular solution found in the past are special case of this general form; the factor $(1-K+Kx^2)^{3/2}$ helps to simplify the integration process. Now it is possible to write two categories of solutions to (12) in terms of elementary functions (see [3] for details). The first category of solution is given by

$$\omega = \quad A\sum_{j=0}^{n}(-\gamma)^j \frac{(n+j-2)!}{(n-j)!\,(2j)!}x^{2j} + B(1-K+Kx^2)^{3/2}\sum_{j=0}^{n-2}(-\gamma)^j \frac{(n+j)!}{(n-j-2)!\,(2j+1)!}x^{2j+1}$$

For the values $\gamma = 4 - \frac{4}{4n(n-1)+\alpha}$ and $K + \alpha = 2 - (2n-1)^2$.
The second category of solution has the form

$$\omega = \quad A\sum_{j=0}^{n}(-\mu)^j \frac{(n+j-1)!}{(n-j)!\,(2j+1)!}x^{2j+1} + B(1-K+Kx^2)^{3/2}\sum_{j=0}^{n-1}(-\mu)^j \frac{(n+j)!}{(n-j-1)!\,(2j)!}x^{2j}$$

$\mu = 4 - \frac{4}{4n^2-1+\alpha}$and $K + \alpha = 2(1-2n^2)$.
Therefore two categories of solutions in terms of elementary functions can be extracted from the general series in (15). The solutions given above have a simple form and they have been expressed completely as combinations of polynomials and algebraic functions. This has the advantage of simplifying the investigation into the physical properties of a dense anisotropic star.

Discussion

We have found solutions to the Einstein field equations by utilizing the coordinate transformation that do have isotropic analogue. A systematic series analysis produced recurrence relation with coefficients that could be solved in general. This produced exact solutions to the field equations in terms of elementary functions. The anisotropic factor may vanish in the general series solution and we can regain isotropic solutions. It is possible for series to be expressed in terms of polynomials. We used this feature to find two classes of exact solutions to the system in terms of polynomials and product of polynomials and algebraic functions. The simple form of the solutions found facilitates the analysis of the physical features of an anisotropic fluid sphere.

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