# Combinatorial properties of the numbers of tableaux of bounded height

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**Abstract**

We introduce an infinite family of lower triangular matrices $\Gamma^{(s)}$, where $\gamma^s_{n,i}$ counts the standard Young tableaux on $n$ cells and with at most $s$ columns on a suitable subset of shapes. We show that the entries of these matrices satisfy a three-term row recurrence and we deduce recursive and asymptotic properties for the total number $\tau_s(n)$ of tableaux on $n$ cells and with at most $s$ columns.

## 1 Introduction

The first simple expressions of the number of standard Young tableaux of given shape were given by Frobenius and Young (see [6] and [15]) and by Frame-Robinson-Thrall [5]. More recently, the number of standard Young tableaux has been studied according to the height of their shape. Regev [12] gave asymptotic values for the numbers $\tau_s(n)$ of standard Young tableaux whose shape consists of $n$ cells and at most $s$ columns and Stanley (see [14]) discussed the algebraic or differentiably finite nature of the corresponding generating functions. More recently, many authors entered this vein. For example, D.Gouyou-Beauchamps gave both exact forms in [13] and recurrence formulas (see [9]) for the numbers $\tau_s(n)$ when $s \leq 5$ by combinatorial tools, while F.Bergeron and F.Gascon [4] found some recurrence formulas for the numbers $\tau_s(n)$ by analytic argumentations.
In this paper we present a different approach to the subject. Our starting point is the remark that the elements of the Ballot matrix introduced

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by M.Aigner in [1] correspond bijectively to the integers counting standard Young tableaux of a given shape with at most 2 columns. Firstly, we arrange the entries of the Ballot Matrix in a new lower triangular matrix $A$ in such a way that the entries of the $n$-th row count standard Young tableaux with precisely $n$ cells. The integer $\tau_2(n)$ can be therefore recovered by summing the elements of the $n$-th row of $A$.

In the following sections, we extend the results to the general case of standard Young tableaux with at most $s$ columns. For every fixed $s \in \mathbb{N}$ we define an infinite matrix $\Gamma^{(s)}$ whose $(n, i)$-th entry is the total number of standard Young tableaux with at most $s$ columns and such that the difference between the length of the second and of the third column is $i$. The total number $\tau_s(n)$ of tableaux on $n$ cells with at most $s$ columns is the $n$-th row sum of the matrix $\Gamma^{(s)}$. The matrix $\Gamma^{(s)}$ presents a three-term row-recurrence property, which yields a recurrence law satisfied by the integers $\tau_s(n)$. This recurrence allows to get a combinatorial interpretation of the asymptotic behaviour of the ratio

$$\frac{\tau_s(n)}{\tau_s(n-1)}.$$

In Section 3 we treat more extensively the case $s = 3$, in order to exhibit recursive and asymptotic properties for the integers $\tau_3(n) = M_n$, where $M_n$ denotes the $n$-th Motzkin number.

# 2 The two-column case

In the following, we will denote the shape of a standard Young tableau $T$ as a list containing the length of the first, second, ..., last column of $T$.

We first consider the standard Young tableaux whose shape consists of at most two columns. We define an infinite, lower triangular matrix $A = (\alpha_{n,i})$, where $\alpha_{n,i}$ is the number of standard Young tableaux whose shape corresponds to the list $(n - i, i)$, with $i \in \mathbb{N}$.

The matrix $A$ can be recursively constructed as follows:

**Proposition 1** *The matrix $A$ is defined by the following initial conditions:*

*i)* $\alpha_{i,0} = 1$ *for every* $i \geq 0$*;*

*ii)* $\alpha_{1,i} = 0$ *for every* $i > 0$*;*

*iii)* $\alpha_{n,i} = 0$ *for every* $i > \lfloor \frac{n}{2} \rfloor$

*and by the recurrence:*

$$\alpha_{n,i} = \alpha_{n-1,i} + \alpha_{n-1,i-1} \quad \textit{for} \quad n \geq 1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor. \tag{1}$$

**Proof** The initial conditions follow immediately by the definition of the matrix $A$. In order to prove the recurrence, remark that a Young tableau with $n$ cells can be obtained from a tableau with $n-1$ cells by adding a new box containing the symbol $n$ in a corner position. In particular, in a Young tableau of shape $(n-i, i)$, $i > 0$, the symbol $n$ can be placed in at most two corner cells. Hence, the entry $\alpha_{n,i}$ is the sum of the two integers $\alpha_{n-1,i-1}$ and $\alpha_{n-1,i}$, that count the Young tableaux of appropriate shape. □

The recurrence formula (1) has as an immediate fallout the following result:

**Corollary 2** *The entries of the matrix A satisfy the following columnwise recurrence:*

$$\alpha_{n,i} = \sum_{h=2i-1}^{n-1} \alpha_{h,i-1}$$

*for every* $n, i > 0$.

We remark that the elements of the matrix $A$ correspond bijectively to the entries of the Ballot Matrix defined by Aigner in [1]. More precisely, the Ballot Matrix $\widetilde{A} = (\tilde{\alpha}_{i,j})$ can be obtained rearranging the entries of the matrix $A$ as follows:

$$\tilde{\alpha}_{j,k} = \alpha_{2j-k,j-k}.$$

As a consequence, for every $n$, the number of standard Young tableaux with two columns of the same length $n$ is the $n$-th Catalan number $C_n$:

$$\alpha_{2n,n} = C_n \tag{2}$$

.

Denote by $\tau_2(n)$ the number of Young tableaux with $n$ cells and at most two columns. Obviously, we have:

$$\tau_2(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,i}.$$

In the next theorem we deduce a recurrence formula for the sequence $\tau_2(n)_{n\geq 1}$:

**Theorem 3** *The integers $\tau_2(n)$, $n \geq 1$, satisfy the recurrence*

$$\tau_2(n) = 2\tau_2(n-1) - E(n-1) \cdot C_{\frac{n-1}{2}}, \tag{3}$$

*where $C_i$ denotes the $i$-th Catalan number and*

$$E(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** By Proposition 1, we have:

$$\tau_2(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,i} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-1,i-1} + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{n-1,i} =$$

$$= 2\tau_2(n) - E(n-1) \cdot \alpha_{n-1,\frac{n-1}{2}} = 2\tau_2(n) - E(n-1) \cdot C_{\frac{n-1}{2}},$$

as desired. □

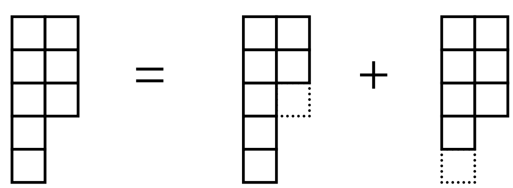

Figure 1: The recurrence formula for the entry $\alpha_{8,3}$.

The preceding result yields an alternative proof of the following theorem, originally proved by Regev in [12]:

**Theorem 4** *The number of standard Young tableaux with exactly $n$ cells and at most two columns is*

$$\tau_2(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

**Proof** It is well known that the central binomial coefficients $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ satisfy the recurrence (3). Remarking that $\tau_2(0) = 1 = \binom{0}{0}$, we have the assertion. □

Moreover, identity (3) yields the following asymptotic property of the integers $\tau_2(n)$:

**Proposition 5** *We have:*

$$\lim_{n\to\infty} \frac{\tau_2(n)}{\tau_2(n-1)} = 2$$

**Proof** By (3) we get:

$$\frac{\tau_2(n)}{\tau_2(n-1)} = \frac{2\tau_2(n-1) - E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_2(n-1)}.$$

This implies that the assertion is proved as soon as we show that

$$\lim_{n\to\infty} \frac{E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_2(n-1)} = 0.$$

If $n$ is even, this identity trivially holds. If $n$ is odd, the described correction term is negligeable with respect to $\tau_2(n-1)$ as $n$ goes to infinity. In fact, we get:

$$\lim_{n\to\infty} \frac{\frac{1}{n}\binom{n-1}{\frac{n-1}{2}}}{\binom{n-1}{\frac{n-1}{2}}} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

as required. □

# 3 The three-column case

We now extend the results of Section 2 to the case of standard Young tableaux with at most three columns. Define a matrix $B = (\beta_{n,i})$ such that the $(n,i)$-th entry of $B$ is the total number of standard Young tableaux of shape $(n-i-2k, i+k, k)$, for every possible value of $k$. In other terms, the integer $\beta_{n,i}$ is the cardinality of the set $Y^3_{n,i}$ containing all standard tableaux on $n$

cells with at most 3 columns such that the difference between the second and third column is $i$.
The entries of the matrix $B$ satisfy a three-term recurrence:

**Proposition 6** *The integers $\beta_{n,0}$ satisfy the identity:*

$$\beta_{n,0} = \beta_{n-1,0} + \beta_{n-1,1}. \tag{4}$$

*Moreover, for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have:*

$$\beta_{n,i} = \beta_{n-1,i-1} + \beta_{n-1,i} + \beta_{n-1,i+1} - r_{n,i}, \tag{5}$$

*where*

$$r_{n,i} = \begin{cases} \text{number of tableaux of shape } (\frac{n+i-2}{3}, \frac{n+i-2}{3}, \frac{n-2i+1}{3}) & \text{if } n-2i \equiv_3 2 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Let $T$ be a standard Young tableau $T$ with 3 columns of shape $(n-i-2k, i+k, k)$, for some $k$, and let $T'$ be the tableau obtained from $T$ by removing the cell containing the symbol $n$. Obviously, the tableau $T'$ has either shape $(n-i-2k-1, i+k, k)$ or $(n-i-2k, i+k-1, k)$ or $(n-i-2k, i+k, k-1)$. This implies that the correspondence $T \mapsto T'$ maps the set $Y^3_{n,i}$ into the union $Y^3_{n-1,i-1} \cup Y^3_{n-1,i} \cup Y^3_{n-1,i+1}$. However, such map is not in general surjective. In fact, it is easy to check that the difference between the set $Y^3_{n-1,i-1} \cup Y^3_{n-1,i} \cup Y^3_{n-1,i+1}$ and the image of the map consists exactly of the tableaux of shape $(\frac{n+i-2}{3}, \frac{n+i-2}{3}, \frac{n-2i+1}{3})$. This proves (5). The first recurrence can be proved by the same arguments, observing that, in this case, the map defined above is surjective. □

The preceding result yields a recurrence formula satisfied by the total number $\tau_3(n)$ of standard Young tableaux with $n$ cells and at most 3 columns:

**Theorem 7** *The numbers $\tau_3(n)$, $n \geq 3$, satisfy the recurrence*

$$\tau_3(n) = 3\tau_3(n-1) - R(n) - E(n-1) \cdot C_{\frac{n-1}{2}} - \beta_{n-1,0}, \tag{6}$$

*where*

$$R(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} r_{n,i}.$$

**Proof** By definition of $\beta_{n,i}$ we have:

$$\tau_3(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{n,i}.$$

Hence,

$$\tau_3(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{n-1,i-1} + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{n-1,i} + \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{n-1,i+1} - \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} r_{n-1,i-1}. \qquad (7)$$

Remark that

$$\tau_3(n-1) = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \beta_{n-1,i}.$$

The proof is an immediate consequence of the following considerations:

- the first summand in the right hand side of Identity (7) is equal to $\tau_3(n-1)$ if $n$ is even. In case of $n$ odd, instead, this term exceeds $\tau_3(n-1)$ by the Catalan number $C_{\frac{n-1}{2}}$,
- the second summand in the right hand side of Identity (7) is exactly $\tau_3(n-1)$,
- the third summand in the right hand side of Identity (7) exceeds the integer $\tau_3(n-1)$ exactly by $\beta_{n-1,0}$,
- the fourth summand in the right hand side of Identity (7) is obviously equal to $R(n-1)$.

$\square$

Remark that an explicit evaluation of the integers $Q(n)$ and $R(n)$ can be obtained via the well known Hook Length Formula.

Previous considerations lead to the following result concerning the asymptotic behaviour of the sequence $\tau_3(n)$:

**Proposition 8** *For every integer $n \geq 1$, we have:*

$$\frac{\tau_3(n)}{\tau_3(n-1)} < 3. \tag{8}$$

*Moreover,*

$$\lim_{n\to\infty} \frac{\tau_3(n)}{\tau_3(n-1)} = 3. \tag{9}$$

**Proof** Inequality (8) follows directly by previous considerations. In order to prove (9) we remark that, by Theorem (7), the ratio $\frac{\tau_(n)}{\tau_(n-1)}$ can be written as:

$$\frac{\tau_3(n)}{\tau_3(n-1)} = 3 - (U_1(n-1) + U_2(n-1) + U_3(n-1)),$$

where

$$U_1(n-1) = \frac{E(n+1) \cdot C_{\frac{n-1}{2}}}{\tau_3(n-1)}$$

$$U_2(n-1) = \frac{\beta_{n-1,0}}{\tau_3(n-1)}$$

$$U_3(n-1) = \frac{R(n-1)}{\tau_3(n-1)}.$$

The statement will be proved as soon as we show that each summand $U_i(n-1)$ goes to 0 as $n$ goes to infinity. The numerator of each $U_i(n-1)$ is a sum of a suitable finite number $f_\lambda$. For every $f_\lambda$ appearing in one of these expression, we can single out an appropriate subset $S_\lambda$ of summands whose cardinality grows as $n$ goes to infinity, such that:

$$\lim_{n\to\infty} \frac{f_\lambda}{\tau_3(n-1)} \leq \lim_{n\to\infty} \frac{f_\lambda}{\sum_{\mu \in S_\lambda} f_\mu} = 0.$$

For instance, consider the rectangular shape $\rho$ over $n-1$ blocks, namely, the shape corresponding to the values $i = 0$, $k = \frac{n-1}{3}$. Obviously, such a shape exists only if $n - 1 \equiv 0 \pmod 3$. In this case, the family $S_\rho$ consists of the shapes over $n-1$ cells and corresponding to the values $i = h$, $k = k = \frac{n-1}{3} - h$, where $h$ ranges from 1 to $k = \frac{n-1}{6}$. By Hook Length Formula, we get:

$$\lim_{n\to\infty} \frac{f_\rho}{\tau_3(n-1)} \leq \lim_{n\to\infty} \frac{f_\rho}{\sum_{\mu \in S_\rho} f_\mu} = \lim_{n\to\infty} \frac{f_\rho}{\sum_{h=1}^{\lfloor \frac{n-1}{6} \rfloor} (h+1)^3 f_\rho} \leq \lim_{n\to\infty} \frac{1}{8 \lfloor \frac{n-1}{6} \rfloor} = 0.$$

The sets $S_\nu$ corresponding to the other summand of each of the numerators of the $U_i(n-1)$ can be described analogously. $\square$

As proved in [12], $\tau_3(n) = M_n$, where $M_n$ is the $n$-th Motzkin number. Hence, these last arguments yield a combinatorial proof of Propositions 4 and 5 in [2].

# 4 The general case

In this section we extend the previous argumentations to the general case of standard Young tableaux with at most $s$ columns. Define a matrix $\Gamma^{(s)} = (\gamma_{n,i}^{(s)})$ such that the $(n,i)$-th entry is the total number of standard Young tableaux with at most $s$ columns and such that the difference between the length of the second and of the third column is $i$.
Also in this case, when $s \geq 4$ and $n \geq s$, the entries of this matrix satisfy a three-term recurrence. In fact, the following proposition can be proved by the same arguments used to prove Proposition 6:

**Proposition 9** *The integers $\gamma_{n,0}^{(s)}$ satisfy the identity:*

$$\gamma_{n,0}^{(s)} = (s-2)\gamma_{n-1,0}^{(s)} + \gamma_{n-1,1}^{(s)} - \sum_{j=3}^{s-1} r_j^{(s)}(n-1,0). \tag{10}$$

*Moreover, for every $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, we have:*

$$\gamma_{n,i}^{(s)} = \gamma_{n-1,i-1}^{(s)} + (s-2)\gamma_{n-1,i}^{(s)} + \gamma_{n-1,i+1}^{(s)} - r_1^{(s)}(n-1,i-1) - \sum_{j=3}^{s-1} r_j^{(s)}(n-1,i), \tag{11}$$

*where $r_j^{(s)}(n-1,i)$ is the number of standard Young tableaux with at most $s$ columns such that the difference between the length of the second and of the third column is $i$ and the $j$-th and the $(j+1)$-th columns have the same length.*

$\square$

Similarly, exploiting the same arguments as in Proposition 8, we get:

**Proposition 10** *The numbers $\tau_s(n)$ satisfy the recurrence:*

$$\tau_s(n) = s\tau_s(n-1) - E(n-1)C_{\frac{n-1}{2}} - \gamma^{(s)}_{n-1,0} - R^{(s)}(n-1), \tag{12}$$

*where $R^{(s)}(n-1)$ is the sum of all the correction terms appearing in Formula* (11)*.*

□

Now we can easily deduce from these identities some information concerning the asymptotic behaviour of the sequence $\tau_s(n)$:

**Proposition 11** *The sequence $\tau_s(n)$ satisfies the following properties:*

$$\frac{\tau_s(n)}{\tau_s(n-1)} < s$$

$$\lim_{n\to\infty} \frac{\tau_s(n)}{\tau_s(n-1)} = s.$$

□

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