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# Multi-frequency chatter analysis using the shift theorem

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**Abstract**

During machining, the use of variable helix tools can potentially improve the system's stability to regenerative chatter. However, this configuration of tool has a distributed time delay, which makes the stability analysis more complex. The analysis is further exacerbated by the time-periodic coefficients that occur during milling. The present contribution demonstrates how the Fourier transform and harmonic transfer function approach can be used to analyse the system stability. This provides new insight into the stability of these tools, based on a mathematically elegant approach that makes extensive use of the shift theorem.

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*Keywords:* Chatter, Self-excited vibration, Milling

## 1. Introduction

Regenerative chatter is a well-known and problematic form of instability that involves delayed dynamics. For standard milling tools, the stability analysis requires consideration of a time-periodic system with a single delay term. For variable pitch tools, multiple delay terms can arise, but variable helix tools are unique in that they lead to a distributed delay[1]. Previous work by the author[2] has shown how this distributed delay can be considered by use of the shift theorem and the Laplace transform. This approach has also been used to investigate the so-called short regenerative effect[3], which can also lead to distributed delays. The advantage of the approach is that it enables a visual interpretation of the system stability, by describing the distributed delays as a filter in the frequency domain. The approach is also computationally efficient. The present contribution demonstrates that the approach can be extended to properly consider the time-periodic nature of milling. Although the resulting computations can be cumbersome, the mathematical formulation relies only on fundamental properties - shift theorems - of the Fourier (or Laplace) transform, thus making the approach an elegant alternative that can provide more insight into the stability of this configuration of tool.

In the well known (e.g.[4]) model of regenerative chatter, cutting forces are assumed to be proportional to the instantaneous chip thickness. Rotation of the cutting tool leads to time-periodic direction coefficients, which map the

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radial and tangential cutting forces into the global coordinates of the tool and workpiece. These forces then induce vibrations of the tool and/or workpiece, which consequently leads to relative vibration and a change in the instantaneous chip thickness. Classical stability models[5] assume time-averaged (or zero-frequency) direction coefficients, which are equivalent to taking the first term of a Fourier series expansion. More recent work[6] has considered multiple terms in this Fourier series. This leads to a stability analysis which is equivalent to the Harmonic Transfer Function concepts that are described in some control engineering literature[7,8]. The present study combines this approach with the author's previous work[2] on variable helix tools. Here, it transpires that careful formulation of the Fourier series expansion of the time-periodic coefficients is needed. Consequently, the manuscript focuses on this aspect of the analytical formulation. Numerical validation is then described by comparing the result with previous studies.

## 2. Theory

Consider a tool with $K$ teeth, where the pitch angle between each tooth is able to vary linearly along the axial depth of the tool. An example (with two teeth) is illustrated in Fig. 1. The function describing the angle for tooth $k$ is:

$$\phi_k(a) = \phi_{k_0} + \beta_k a. \tag{1}$$

Here, $\phi_{k_0}$ is the tooth angle at the end of the tool, and $\beta_k$ is the gradient of the pitch with respect to axial location $a$. So for a tooth of helix angle $\gamma$, and radius $r$,

$$\beta_k = \frac{\tan\gamma}{r}.$$

Note that $\beta_k$ is typically negative (as depicted in Fig. 1) in order to assist with chip evacuation. The spindle speed of the tool is $\Omega$ rad/s, and so at time $t$, the reference rotation of the tool is $\theta_0(t) = \Omega t$ radians. The function describing the position of tooth $k$ is therefore:

$$\theta_k(a,t) = \theta_0(t) + \phi_k(a) \mod 2\pi \tag{2}$$

which is periodic with fundamental period $2\pi$ radians. The function defining the pitch angle difference experienced by tooth $k$ is:

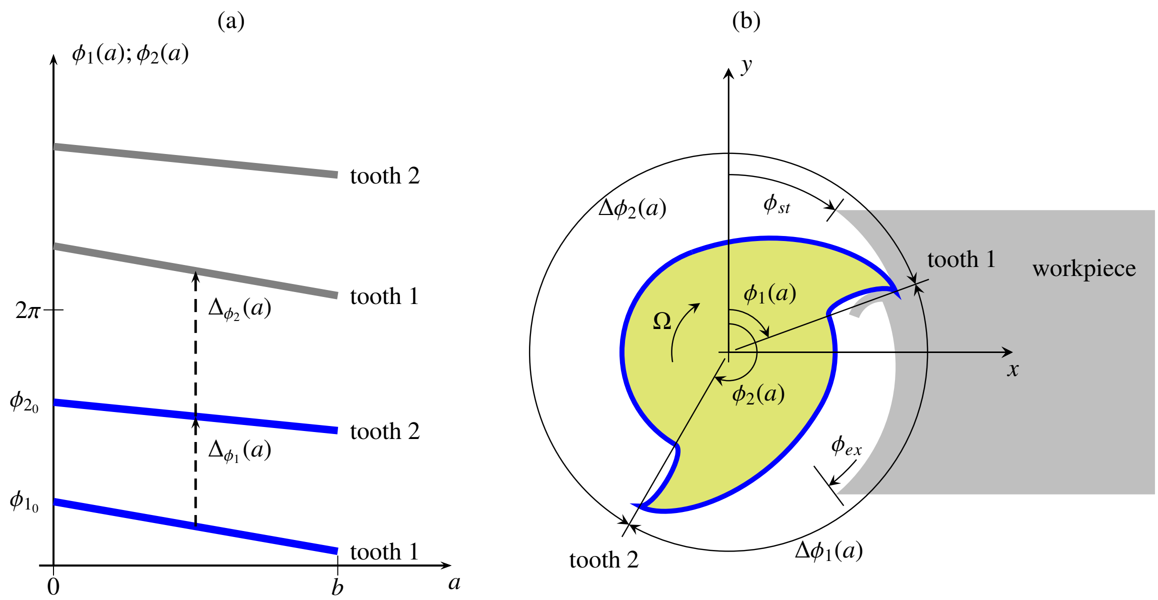


Fig. 1. Schematic representation of a variable helix tool with two teeth, showing the tooth positions at time $t = 0$. (a) tool angles versus axial depth $a$; (b) cross-sectional view at axial location $a$.

$$\Delta_{\phi_k}(a) = \phi_l(a) - \phi_k(a) \qquad \mod 2\pi \tag{3}$$

where

$$l = 1 + (k \mod K).$$

Since the spindle speed is $\Omega$, the corresponding time delay between successive tooth passes is given by the functions:

$$\tau_k(a) = \frac{\Delta_{\phi_k}(a)}{\Omega} \tag{4}$$

Assuming vibrations only occur in the $x$ direction, then the cutting force for each tooth depends on the tooth's instantaneous angle $\theta_k(a,t)$, and the chip thickness $h_k(a,t)$:

$$h_k(a,t) = g(\theta_k(a,t))\sin(\theta_k(a,t))\,(x(t) - x(t - \tau_k(a))) \tag{5}$$

Here, $g$ is a unit step function that defines when the tooth is engaged in the workpiece (as illustrated by the example in Fig. 1):

$$g(\phi) = \begin{cases} 1 & \phi_{st} < (\phi \mod 2\pi) \leq \phi_{ex}, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

For an infinitesimal element at axial location $a$ along the cutter axis, the tangential cutting force is:

$$f_{t_k}(a,t) = K_t h_k(a,t) \tag{7}$$

and the radial cutting force is:

$$f_{r_k}(a,t) = K_r K_t h_k(a,t) \tag{8}$$

where $K_t$ and $K_r$ are empirical cutting force coefficients. The cutting force in the $x$ direction is then

$$f_{x_k}(a,t) = -f_{t_k}(a,t)\cos(\theta_k(a,t)) - f_{r_k}(a,t)\sin(\theta_k(a,t)). \tag{9}$$

Next, define the cutting force direction function $\alpha$ as

$$\alpha(\phi) = g(\phi)\,(-K_t\sin(\phi)(K_r\sin(\phi) + \cos(\phi))) \tag{10}$$

from which the total cutting force becomes:

$$f_x(t) = \sum_{k=1}^{K} \int_{a=0}^{b} \alpha(\Omega t + \phi_{k_0} + \beta_k a)\,(x(t) - x(t - \tau_k(a)))\,da \tag{11}$$

where $b$ is the maximum depth of cut experienced by the tool. Noting that $\alpha(\Omega t + \phi_{k_0} + \beta_k a)$ is periodic allows a Fourier series expansion with respect to time. This gives:

$$\alpha(\Omega t + \phi_{k_0} + \beta_k a) = \sum_{n=-\infty}^{\infty} e^{jn\phi_k(a)} A(n) e^{jn\Omega t} \tag{12}$$

where

$$A(n) = \frac{1}{2\pi} \int_{\phi_{st}}^{\phi_{ex}} \alpha(\theta) e^{-jn\theta} d\theta \tag{13}$$

Here, a key difference emerges compared to classical stability analysis[5], namely the 'phase changing' term $e^{jn\phi_k(a)}$ that is a function of the depth of cut $a$. Note that to keep the terms involving $a$ separated, the phase changing term has been excluded from the function $A(n)$ and included directly in Eq. (12). Substituting the Fourier series expansion from Eq. (12) into Eq. (11) and applying the Fourier transform gives:

$$F_x(j\omega) = \int_{t=0}^{\infty} e^{-j\omega t} \sum_{k=1}^{K} \int_{a=0}^{b} \sum_{n=-\infty}^{\infty} e^{jn\phi_k(a)} A(n) e^{jnt\Omega} \left(x(t) - x(t - \tau_k(a))\right) da\, dt. \tag{14}$$

Note that there is no inter-dependency between any of the limits of integration or summation, which allows rearrangement as follows.

$$F_x(j\omega) = \sum_{n=-\infty}^{\infty} A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ \int_{t=0}^{\infty} e^{-j\omega t} e^{jnt\Omega} x(t) dt - \int_{t=0}^{\infty} e^{-j\omega t} e^{jnt\Omega} x(t - \tau_k(a)) dt \right] da \tag{15}$$

The terms in square brackets can now be simplified by application of the first and second shift theorems:

$$F_x(j\omega) = \sum_{n=-\infty}^{\infty} A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ X(j\omega - jn\Omega) - e^{-\tau_k(a)(j\omega - jn\Omega)} X(j\omega - jn\Omega) \right] da \tag{16}$$

Here, the frequency response $X(j\omega - jn\Omega)$ is independent of $a$ or $k$, giving:

$$F_x(j\omega) = \sum_{n=-\infty}^{\infty} X(j\omega - jn\Omega) A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ 1 - e^{-\tau_k(a)(j\omega - jn\Omega)} \right] da \tag{17}$$

Assuming that the structural dynamics $G(j\omega)$ are linear time invariant gives:

$$X(j\omega) = G(j\omega) \sum_{n=-\infty}^{\infty} X(j\omega - jn\Omega) A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ 1 - e^{-\tau_k(a)(j\omega - jn\Omega)} \right] da \tag{18}$$

This defines a closed-loop relationship between the vibration in the frequency domain, $X(j\omega)$, and itself, modulated by harmonics due to $A(n)$. Consequently the vibration at any one frequency $\omega$ involves components from all of the harmonics $n$. Rewriting with $\omega = \omega + p\Omega$ gives a general expression for the frequency response at any harmonic:

$$X(j\omega + jp\Omega) = G(j\omega + jp\Omega) \sum_{n=-\infty}^{\infty} X(j\omega - jn\Omega + jp\Omega) A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ 1 - e^{-\tau_k(a)(j\omega - jn\Omega + jp\Omega)} \right] da \tag{19}$$

Let $q = p - n$ so that $n = p - q$ giving

$$X(j\omega + jp\Omega) = G(j\omega + jp\Omega) \sum_{p-q=-\infty}^{\infty} X(j\omega + jq\Omega) A(p - q) \sum_{k=1}^{K} \int_{a=0}^{b} e^{j(p-q)\phi_k(a)} \left[ 1 - e^{-\tau_k(a)(j\omega + jq\Omega)} \right] da \tag{20}$$

Defining matrix terms with rows $p$ and columns $q$, $p = -\infty, \ldots, \infty$; $q = -\infty, \ldots, \infty$, and reverting to the format $n = p - q$ for compactness:

$$\hat{x}_p(j\omega) = X(j\omega + jp\Omega) \tag{21}$$

$$\hat{g}_{p,p}(j\omega) = G(j\omega + jp\Omega) \tag{22}$$

$$\hat{h}_{p,q}(j\omega) = A(n) \sum_{k=1}^{K} \int_{a=0}^{b} e^{jn\phi_k(a)} \left[ 1 - e^{-\tau_k(a)(j\omega + jq\Omega)} \right] da \tag{23}$$

$$\hat{x}_p(j\omega) = \hat{g}_{p,p}(j\omega) \sum_{q=-\infty}^{\infty} \hat{h}_{p,q}(j\omega)\hat{x}_q(j\omega) \tag{24}$$

which allows a matrix format where

$$\hat{X}(j\omega) = \hat{\mathbf{GH}}(j\omega)\hat{X}(j\omega). \tag{25}$$

The integration term in Eq. (23) is straightforward to evaluate, because the exponents are linear functions of $a$.

## 3. Stability

Eq. (25) describes a multi-input-multi-output (MIMO) system, with a positive feedback loop defined by the doubly-infinite transfer function $\hat{\mathbf{GH}}(j\omega)$. According to the Generalised Nyquist Stability Criterion[9], the system is stable if $\det(\mathbf{I} - \hat{\mathbf{GH}}(j\omega))$ (where $\mathbf{I}$ is the identity matrix) is non-zero and does not encircle the origin in a clockwise sense. This seems intractible because $\hat{\mathbf{GH}}(j\omega)$ is a doubly-infinite matrix, however, further simplification is possible by exploiting the periodicity of $\hat{\mathbf{GH}}(j\omega)$, and the high frequency behaviour of $G(j\omega)$.

To explore the periodicity of Eq. (24), consider the case where $\omega_1 = \omega + r\Omega$. It can be shown that:

$$\hat{g}_{p,p}(j\omega + jr\Omega) = \hat{g}_{p+r,p+r}(j\omega) \tag{26}$$

$$\hat{h}_{p,q}(j\omega + jr\Omega) = \hat{h}_{p+r,q+r}(j\omega) \tag{27}$$

Consequently, each time the frequency $\omega$ increases by $\Omega$, the elements of the harmonic transfer function $\hat{\mathbf{GH}}(j\omega)$ are offset diagonally by one row and one column.

Meanwhile, it is reasonable to assume that the structural dynamics, described by $G(j\omega)$, will end towards zero at high frequencies. If the maximum frequency considered is $\omega_{max}$, then $\hat{g}_{p,p}(j\omega) = G(j\omega + jp\Omega)$ will be zero unless

$$-\omega_{max} < \omega + p\Omega < \omega_{max} \tag{28}$$

which leads to a range of admissible values of $p$, beyond which $\hat{\mathbf{GH}}(j\omega)$ becomes zero:

$$\frac{-\omega_{max} - \omega}{\Omega} < p < \frac{\omega_{max} - \omega}{\Omega} \tag{29}$$

From Eq. (26) and Eq. (27) it can be seen that the range of $\omega$ need only be $[-\Omega/2, \Omega/2]$, before the harmonic transfer function is duplicated and offset diagonally. Consequently the maximum frequency to be computed gives rise to a maximum required value for $p$:

$$-\frac{\omega_{max}}{\Omega} - \frac{1}{2} < p < \frac{\omega_{max}}{\Omega} + \frac{1}{2} \tag{30}$$

Finally, because other rows will have zero harmonic transfer functions, the number of columns in $\hat{g}$ and $\hat{h}$ can be similarly truncated.

## 4. Validation

In order to validate the new analytical formulation, results are compared to those obtained using the semi-discretisation method[10]. The semi-discretisation approach is an established technique for the stability analysis of delay differential equations[11] that involves eigenvalue analysis of a semi-discretised system model. For conciseness, the scenario depicted in Table 1 and Figure 8 of previous a publication[2] is reconsidered. The results are shown in Fig. 2. It can be seen that there is very close agreement between the newly proposed method and the semi-discretisation approach. The significance of the phase shifting term in Eq. (12) is also demonstrated by the erroneous result shown in cyan.

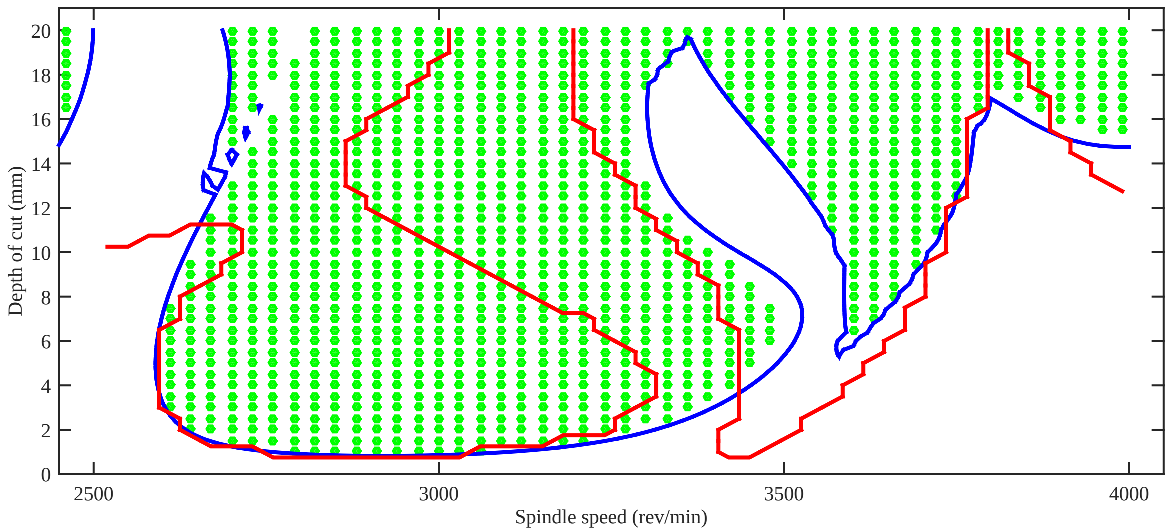


Fig. 2. Comparison with previous work (Fig 8, [2]). semi-discretisation method [2,10]; present study, neglecting the phase-changing term in Eq. (12); × unstable regions including the phase-changing term.

In terms of compuational efficiency, the new approach is typically slightly faster than the semi-discretisation method. However, although each determinant analysis involves a lower order matrix, the determinant must be re-computed for every frequency in the numerical frequency response functions and so the speed is application-specific. Nevertheless it should be noted that the new method has guaranteed convergence which means that no numerical convergence study is required. In comparison of the new approach (or the semi-discretisation method) with the zero order solution [2], the latter does not involve determinant or eigenvalues and so it can be orders of magnitude faster.

## 5. Conclusions

This contribution has demonstrated how the Fourier series and Fourier transform can be used directly to investigate the stability of variable helix milling tools. The approach agrees with alternative methods, but offers the advantage of providing an alternative and elegant mathematical formulation that relies only on the fundamental properties of the Fourier transform. Further work is needed to investigate the promising aspects of the approach, in particular the potential for guaranteed convergence, and the insight into potential designs of tool.

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