Department of Economics

Welfare theorems for random assignments with priorities

Jan Christoph Schlegel[1]
City, University of London

Department of Economics
Discussion Paper Series
No. 17/05

[1] Corresponding author: Jan Christoph Schlegel, Department of Economics, City, University of London, Northampton Square, London EC1V 0HB, UK.
Email: jan-christoph.schlegel@city.ac.uk

Welfare theorems for random assignments with priorities

Jan Christoph Schlegel*

Department of Economics, City, University of London, UK

jan-christoph.schlegel@city.ac.uk

Abstract

Motivated by the application of designing fair and efficient school choice lotteries, we consider constrained efficiency notions for random assignments under priorities. We provide a constrained (priority respecting) version of the ordinal efficiency welfare theorem for random assignments. Moreover, we show that a constrained version of a cardinal second welfare theorem fails to hold.

JEL-classification: C78, D47

Keywords: Matching; Random Assignments; Priority-based Allocation; Constrained Efficiency; Pseudo-Market

1 Introduction

The assignment of students to schools (Abdulkadiroglu and Sönmez, 2003) is one of the major applications of matching theory. A school choice mechanism assigns students to schools taking into account the preferences of students and priorities of the students at the different schools. Thick priorities are a generic problem in school choice. In practice, students are prioritized according to very coarse criteria (based e.g. on proximity, or having a sibling in the school) such that many students have the same priority for a seat at a school. Thus, one can sometimes not avoid to treat students differently ex-post even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This has motivated researchers to study the problem of designing fair school choice lotteries.

A minimal ex-ante fairness requirement for random assignments under priorities is that the lottery should respect the priorities. One way of formalizing this requirement is the following: A student i has *ex-ante justified envy* if there is a school s where

*The author thanks Péter Biró, Battal Doğan, Bettina Klaus, Flip Klijn, Jordi Massó, Panos Protopapas, Marek Pycia and participants of the Sixth World Congress of the Game Theory Society for interesting discussions and comments on a previous version of the paper. The author is particularly grateful to Antonio Miralles for pointing out the connection of the results to the ordinal welfare theorems of McLennan and of Manea. Financial support by the Swiss National Science Foundation (SNSF) under project 100018-150086 is gratefully acknowledged.

a lower priority student j has a positive probability of receiving a seat and i would rather have a seat in s than at some other school at which he has a positive probability of receiving a seat. In this case, it would be natural to eliminate the justified envy, i.e. changing the probability shares such that i has a higher chance of receiving a seat at school s at the expense of the lower ranked student j. *Ex-ante stability* requires that there is no ex-ante justified envy.

Respecting priorities, usually leads to efficiency losses (Abdulkadiroglu and Sönmez, 2003). Naturally, this motivates the question under which conditions efficiency can be reached in a constrained, i.e. priority respecting, way. Our paper contributes to this discussion. We formulate new notions of constrained efficiency properties for random allocations. The notions are motivated by the observation that there can be two sources of efficiency losses in the context of allocating under thick priorities. The first source is the direct effect of choosing higher or lower admission thresholds at the different schools: Already in the context of deterministic assignments with strict priorities, there can be multiple priority respecting assignments that are distinguished by which priority ranks are sufficient to be admitted at a school. By reallocating seat from higher to lower priority applicants, one obtains a priority-respecting assignment that is better from the point of view of applicants. The second source of efficiency loss is a pure mis-allocation effect that can only happen if priority classes are thick. In this case, it can happen that a priority-respecting assignment can be made more efficient just be reallocating (probabilities of obtaining) seats among equal priority applicants without admitting more lower priority student at any school. We focus on the second kind of efficiency losses and call an assignment constrained efficient if is not dominated (in efficiency terms) by a random assignment that does not use a "more lenient" admission policy at any school. Formulating constrained efficiency in this way, lets us clearly distinguish between the effect of harder or softer admission policies and the welfare gains through reallocating school seats within priority classes.

Our main result is a constrained ordinal efficiency welfare theorem for priority respecting lotteries under our constrained efficiency notion. We also consider cardinal welfare theorems (for the difference between cardinal and ordinal welfare theorems see the discussion below) and show by means of a counter example that a cardinal second welfare cannot be established for our constrained efficiency notion in the context of the pseudo markets of He et al. (2017).

1.1 Related Literature

The literature on priority respecting lotteries has been motivated by the application of designing school choice mechanisms. In the school choice set-up, ex-ante stability has been introduced by Kesten and Ünver (2015). For the classical marriage model the condition was first considered by Roth et al. (1993). Kesten and Ünver (2015) consider mechanisms that implement ex-ante stable assignments and satisfy constrained ordinal efficiency properties. He et al. (2017) define an appealing class of mechanisms that implement ex-ante stable lotteries. These mechanisms generalize the pseudo-market mechanisms of Hylland and Zeckhauser (1979) by allowing for priority-specific pricing (agents with different priorities are offered different prices).

Welfare theorems have been studied in the context of random assignments without priorities. It is important here, to distinguish between ordinal efficiency welfare theorems and cardinal welfare theorems. An ordinal efficiency welfare theorem establishes that ordinal efficiency (in the sense of first order stochastic dominance) for a random assignment is equivalent to the existence of a cardinal utility profile consistent with the ordinal preferences under which the random assignment is Pareto efficient when lotteries are evaluated according to expected utility. Pareto efficiency can be strengthened to social welfare efficiency. The original ordinal welfare is due to McLennan (2002). A constructive proof was later provided by Manea (2008). In this paper we generalize the ordinal efficiency welfare theorem to the case of random assignments with priorities.

Cardinal welfare theorems for random assignments are counterparts to the classical welfare theorems for exchange economies. For cardinal welfare theorems, utility profiles are the primitive of the model. In this context, a classical mechanism due to Hylland and Zeckhauser (1979) uses a pseudo market in probability shares to randomly allocate objects. Hylland and Zeckhauser (1979) show that the equilibria of their pseudo markets are Pareto efficient and hence establish a cardinal first welfare theorem. In a recent contribution, Miralles and Pycia (2014) establish a cardinal second welfare theorem that demonstrates that each Pareto efficient random allocation can be decentralized as a pseudo-market equilibrium by appropriately choosing budgets and prices. The main technical difficulty in their proof, in comparison to the classical welfare theorems for exchange economies, is that in the context of lotteries one necessarily deals with satiated preferences (since probabilities have to add up to 1, respectively since each applicants receives at most one school seat). We show that the result does not generalize to the markets with priority-specific pricing of He et al. (2017).

2 Model and Preliminary Results

2.1 Model

There is a set of n agents N and a set of m schools M. A generic agent is denoted by i and a generic school by j. In each school j, there is a finite number of seats $q_j \in \mathbb{N}$. We assume that there are as many school seats as agents, $\sum_{j \in M} q_j = n$.[1] A lottery over schools is a probability distribution over M.

Agents have preferences over lotteries over schools. Preferences of agents can be modeled in two different ways: In the first version, each agent i has **strict preferences** P_i over different schools. We call $P = (P_i)_{i \in N}$ a **preference profile**. We write $j \, R_i \, j'$ if $j \, P_i \, j'$ or $j = j'$. The preferences can be extended to a partial preference order over lotteries using the stochastic dominance criterion: A lottery π' **first-order stochastically dominates** lottery π with respect to preferences P_i, if for each $j \in M$ we have

$$\sum_{j' \in M : j' \, R_i \, j} \pi'_{j'} \geq \sum_{j' \in M : j' \, R_i \, j} \pi_{j'}$$

[1]Our results can be generalized to the case where the number of school seats and agents differ by adding dummy agents and schools. See Aziz and Klaus (2017), for the details of this construction.

and the inequality is strict for at least one school j. We write $\pi' \, P_i^{SD} \, \pi$ if π' first order stochastically dominates π according to P_i and $\pi' \, R_i^{SD} \, \pi$ if either $\pi' \, P_i^{SD} \, \pi$ or $\pi = \pi'$.

In the second version, each agent i has a **von-Neumann-Morgenstern (vNM) utility** vector $U_i = (u_{ij})_{j \in M} \in \mathbb{R}_+^M$. We call $U = (U_i)_{i \in N}$ a **utility profile**. We assume that utilities are **strict**, i.e. for $j \neq j'$ we have $u_{ij} \neq u_{ij'}$. Lotteries are evaluated according to expected utility. Thus agent i prefers lottery π' to lottery π if

$$\sum_{j \in M} u_{ij} \cdot \pi'_j > \sum_{j \in M} u_{ij} \cdot \pi_j.$$

A utility vector contains more information than a preference relation. In addition to ranking the schools, the vNM-utilities express the rates with which agents substitute probabilities of obtaining seats at the different school. Utility vector U_i is **consistent** with preferences P_i, if for each pair of schools $j, j' \in M$ we have $jP_ij' \Leftrightarrow u_{ij} > u_{ij'}$. Each utility vector U_i is consistent with one preference relation P_i that we call the preference relation **induced** by U_i. It is a standard result (see e.g. Proposition 6.D.1 in Mas-Colell et al., 1995), that if lottery π' first order stochastically dominates lottery π according to preferences P_i, then lottery π' yields higher expected utility than π according to any vNM-utilities U_i consistent with P_i.

Each school j has a weak (reflexive, complete and transitive) priority order $\succeq_j$ of the agents. We let $i \sim_j i'$ if and only if $i \succeq_j i'$ and $i' \succeq_j i$. We let $i \succ_j i'$ if and only if $i \succeq_j i'$ but not $i' \succeq_j i$. The priorities $\succeq_j$ of a school j partition N in equivalence classes of equal priority agents, i.e. in equivalence classes with respect to $\sim_j$. We call these equivalence classes **priority classes** and denote them by $N_j^1, N_j^2 \dots, N_j^{\ell_j}$ with indices increasing with priority. Thus for $a > b$, $i \in N_j^a$ and $i' \in N_j^b$ we have $i \succ_j i'$. We use the notation $i \succ_j N_j^k$ to indicate that i has higher priority at j than the agents in the priority class N_j^k.

2.2 Allocations, ex-ante stability, cut-offs and admission policies

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a probability distribution over deterministic assignments. By the Birkhoff-von Neumann Theorem, each random assignment corresponds to a bi-stochastic matrix and, vice versa, each such matrix corresponds to a random assignment (see Kojima and Manea (2010) for a proof in the set-up that we consider). Thus each random assignment is represented by a matrix $(x_{ij}) \in \mathbb{R}^{N \times M}$ such that

$$0 \leq x_{ij} \leq 1, \quad \sum_{j \in M} x_{ij} = 1, \quad \sum_{i \in N} x_{ij} = q_j,$$

where x_{ij} is the probability that agent i is matched to an object of type j. We write $x_i = (x_{ij})_{j \in M}$. The vector x_i is a lottery over M.

A random assignment represented by the matrix (x_{ij}) is **ex-ante blocked** for preference profile P and priorities $\succeq$ by agent i and school j if there is some agent

$i' \neq i$ with $x_{i'j} > 0$ and $i \succ_j i'$ and some school j' with $x_{ij'} > 0$ and $j \, P_i \, j'$. In this case we say that i has **justified envy** at school j. A random assignment is **ex-ante stable** or **ex-ante priority-respecting** if it is not blocked by any agent-school pair.[2] The definition extends to the case where agents have VNM-utilities, by considering the preference profile induced by the utility profile.

We define the **cut-off** $C_j(x)$ for a school j under assignment x to be the lowest priority class containing an agent that has a positive probability of obtaining a set in school j in x. Formally

$$C_j(x) := N_j^{\min\{\ell : \exists i \in N_j^\ell, x_{i,j} > 0\}}.$$

Ex-ante stable assignments can be interpreted as lotteries that ration seats in the cut-offs: An agent is assigned with positive probability to (some) of the schools for which is in the cut-off, with the remaining probability allocated to his most preferred school for which he is above the cut-off. This school can be interpreted as the agent's "safe school" where he gets a seat if he does not get a more-preferred seat in one of the school where he is in the cut-off. A school may have different cut-offs in different ex-ante stable lotteries or it may use the same cut-offs but admit a smaller fraction of agents in the cut-off. We say that school j uses a strictly more **lenient admission policy** under assignment y than under assignment x, if either the school has a strictly lower cut-off in y than in x or it has the same cut-off, but admits a bigger fraction of the students in the cut-off class. Formally

$$C_j(x) \succ_j C_j(y) \text{ or } [C_j(x) = C_j(y) \text{ and } \sum_{i \in C_j(x) = C_j(x)} y_{ij} > \sum_{i \in C_j(x) = C_j(x)} x_{ij}].$$

Example 1. Consider five agents, five schools, each with a single seat ($q_j = 1$ for each j), and the following preferences and priorities.

P_1	P_2	P_3	P_4	P_5	$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$	$\succeq_5$
1	1	1	2	5	$\underline{\underline{1,2,3}}$	5	$\underline{\underline{1,3,5}}$	4, 3, 5	1
4	2	2	5	3	4, 5	1	2, 4	$\underline{\underline{1,2}}$	2
5	5	3	4	2		$\underline{\underline{2,3,4}}$			$\underline{3}$
2	3	5	3	1					$\underline{4,5}$
3	4	4	1	4					

The underlined entries in the priority table correspond to the cut-offs under the following two assignments (rows correspond to agents, columns to schools; schools $j_1, \ldots, j_4$ have the same cut-off under both assignments, school j_5 has a higher cut-off under y than under x):

$$x = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}, y = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{4} & \frac{7}{12} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & 0 \end{pmatrix}$$

[2]For deterministic assignments, ex-ante stability is equivalent to the usual notion of a stable matching. In particular, ex-ante stable assignments always exists, since stable matchings always exist.

Going through all potential blocking pairs, one can check that both x and y are ex-ante stable. We compare x and y in terms of admission policies. School j_1 uses the same admission policy in assignments x and y as

$$\sum_{i \in C_{j_1}(x)} y_{ij_1} = \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1 = \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = \sum_{i \in C_{j_1}(x)} y_{ij_1}.$$

School j_2 uses a more lenient admission policy in x than in y, since the cut-off at j_2 is the same in both assignments, but j_2 admits a bigger fraction of the students in the cut-off class

$$\sum_{i \in C_{j_2}(x)} x_{ij_2} = \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{5}{6} > \tfrac{3}{4} = \tfrac{1}{3} + \tfrac{1}{6} + \tfrac{1}{4} = \sum_{i \in C_{j_2}(x)} y_{ij_2}.$$

School j_3 uses the same admission policy in assignments x and y as

$$\sum_{i \in C_{j_3}(x)} y_{ij_3} = \tfrac{1}{3} + \tfrac{2}{3} = 1 = \tfrac{1}{4} + \tfrac{3}{4} = \sum_{i \in C_{j_3}(x)} y_{ij_3}.$$

School j_4 uses a more lenient admission policy in x than in y, since the cut-off at j_4 is the same in both assignments, but j_4 admits a bigger fraction of the students in the cut-off class

$$\sum_{i \in C_{j_4}(x)} x_{ij_4} = \tfrac{1}{2} > \tfrac{1}{4} = \sum_{i \in C_{j_4}(x)} y_{ij_4}.$$

School j_5 uses a more lenient admission policy in x than in y, since the cut-off at j_5 under x is lower in x than y.

2.3 Constrained Efficiency

The notion of a more lenient admission policy allows us to define constrained efficiency notions by requiring that an allocation is un-dominated among allocations that do not use a more lenient admission policy at any school. We define three different notions of constrained efficiency based on three different and well-known dominance relations. The first notion is defined for ordinal preferences. A lottery y **first order stochastically dominates** (SD-dominates) x if for each $i \in N$ we have $y_i \, R_i^{SD} \, x_i$ and for at least one agent we have $y_i \, P_i^{SD} \, x_i$. Lottery x is **constrained ordinally efficient** if for each lottery y that SD-dominates x there is a school that uses a more lenient admission policy under y than under x.

The other two efficiency notions are defined for cardinal preferences. A lottery y **Pareto dominates** lottery x with respect to VNM utility profile U if for each $i \in N$ we have

$$\sum_{j \in M} u_{ij} \cdot y_{ij} \geq \sum_{j \in M} u_{ij} \cdot x_{ij},$$

and the inequality is strict for at least one agent. A lottery x is **constrained Pareto efficient** with respect to U if for each lottery y that Pareto-dominates x with respect to U there is a school that uses a more lenient admission policy under y than under x.

A lottery y **dominates** lottery x **in social welfare terms** with respect to VNM utility profile U if

$$\sum_{i,j} u_{ij} \cdot y_{ij} > \sum_{i,j} u_{ij} \cdot x_{ij}.$$

A lottery x is **constrained social welfare efficient** if for each lottery y that dominates x in social welfare terms with respect to U there is a school that uses a more lenient admission policy under y than under x.

The efficiency notions have been introduced in increasing order of strength. The following proposition summarizes this fact. The proof is easy and follows along well-known arguments.

Proposition 1. *Let U be a VNM-utility profile, P the ordinal preference profile induced by U, let $\succeq$ be a priority profile and x a random assignment.*

1. *If x is constrained social welfare efficient with respect to U and $\succeq$, then it is constrained Pareto efficient with respect to U and $\succeq$.*
2. *If x is constrained Pareto efficient with respect to U and $\succeq$, then x is constrained ordinally efficient with respect to P and $\succeq$.*

Proof. For the first part, suppose for the sake of contradiction that x is constrained social welfare efficient but Pareto dominated by a lottery y that does not use a more lenient admission policy at any school. By the definition of Pareto dominance, we have for each $i \in N$ that

$$\sum_{j\in M} u_{ij} \cdot y_{ij} \geq \sum_{j\in M} u_{ij} \cdot x_{ij},$$

where the inequality is strict for at least one agent. Summing up the inequalities for different agents we obtain:

$$\sum_{i\in N}\sum_{j\in M} u_{ij} \cdot y_{ij} > \sum_{i\in N}\sum_{j\in M} u_{ij} \cdot x_{ij}.$$

But this contradict the assumption that x is social welfare maximizing among lotteries that do not use a more lenient admission policy at any school than lottery x.

For the second part, suppose for the sake of contradiction that x is constrained Pareto efficient but first order stochastically dominated by a lottery y that does not use a more lenient admission policy at any school. As noted in Section 2, if for i we have $y_i \, P_i^{SD} \, x_i$ then, as U_i is consistent with P_i $\sum_{j\in M} u_{ij} \cdot y_{ij} > \sum_{j\in M} u_{ij} \cdot x_{ij}$. Thus if y first order stochastically dominates x, for each $i \in N$ we have

$$\sum_{j\in M} u_{ij} \cdot y_{ij} \geq \sum_{j\in M} u_{ij} \cdot x_{ij},$$

where the inequality is strict for at least one agent. But this contradict the assumption that x is Pareto optimal among lotteries that do not use a more lenient admission policy at any school than lottery x. □

2.3.1 Constrained ordinal efficiency and strong stable improvement cycles

If a priority-respecting random assignment is not constrained ordinally efficiency, then one can generate a more efficient and priority-respecting random assignment by reallocating probability shares between agents of equal priority or by reallocating probability shares from higher to lower priority agents. It turns out that it is sufficient to look at the reallocation of probability shares within priority classes. Moreover, it is sufficient to look at efficiency improvements through pairwise trade within priority classes. This leads us to the notion of a strong stable improvement cycle that we will introduce next. We will later prove that the absence of these kind of cycles is equivalent to constrained ordinal efficiency. A **strong (ex-ante) stable improvement cycle** for an ex-ante stable assignment x is a sequence of agents $i_1, i_2, \ldots, i_k$ and schools $j_1, j_2, \ldots, j_k$ such that for each $1 \leq \ell \leq k$ (taking indices modulo k) the following holds:

1. $x_{i_\ell, j_\ell} > 0$
2. Agent i_ℓ prefers $j_{\ell+1}$ to j_ℓ.
3. Agents i_ℓ and $i_{\ell-1}$ have the same priority at j_ℓ. [3]

Strong stable improvement cycles can be used to increase ex-ante efficiency without violating ex-ante stability, by reallocating probability shares through pairwise trade within priority classes. This can be achieved by transferring probability shares in j_1 from i_2 to i_1, probability shares in j_2 from i_3 to i_2, and so on. Changing probability shares in this way will not change the shares of school seats allocated to the different indifference classes. It only reallocates shares within priority classes. Each ex-ante stable random assignment can be transformed into a constrained ordinally efficient ex-ante stable random assignment, by successively reallocating probability shares along a strong stable improvement cycles, until no such cycle is left.

Example 2. Consider a market with two agents and two schools with one seat each. Both agents have the same priority at both schools and agent rank the schools in opposite order:

P_1	P_2	$\succeq_1$	$\succeq_2$
2	1	1, 2	1, 2
1	2		

Consider the assignment:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

[3] Usually a weaker notion of a stable improvement cycle is considered that has originally been defined for deterministic stable matchings (Erdil and Ergin, 2008) and generalized to the probabilistic set-up by Kesten and Ünver (2015). In comparison to a strong stable improvement cycles, in the definition of a (weak) stable improvement cycle the third item in the definition is replaced by the weaker condition that i_ℓ is one of the highest priority agents at school $j_{\ell+1}$ that prefers $j_{\ell+}$ to some of the object types that he is matched to under Π. Strong improvement cycles are the most obvious sources for an ex-ante, stability-preserving efficiency gain, in the sense that their resolution only reallocates probability shares within priority classes, whereas other stable improvement cycles might redistribute probability shares across priority classes.

Since both agents have the same priority at both schools, the assignment is ex-ante stable. The assignment has however a strong stable improvement cycle consisting of agents 1 and 2 and schools 1 and 2. Reallocating probability shares for school 1 from 1 to 2 and reallocating probability shares for school 2 from 2 to 1, would lead to an efficiency gain. □

A consequence of the absence of strong stable improvement cycles is that it allows us to order schools in a way that reflects a common component of the agents' preferences. The following lemma describes the ordering and will be crucial in much of the following discussion.

Lemma 2. *Let x be an ex-ante stable random assignment with respect to P and $\succeq$. If x has no strong stable improvement cycle with respect to P and $\succeq$, then there is an ordering $\rhd$ of the schools such that the following holds: If $i \in C_j(x) \cap C_{j'}(x)$ and $x_{ij} > 0$ then*

$$j' \rhd j \Rightarrow j' \, P_i \, j.$$

Proof. The lemma is a consequence of the combinatorial result that directed graphs are acyclic if and only if they can be topologically ordered. This means that vertices can be ordered such that if there is an arc from vertex v to u then u comes before v in the ordering.[4] To apply this result to our context, we construct a directed graph $G(x)$ on the vertex set consisting of all schools $V(G(x)) = M$ and draw a directed edge from j to j' if and only if there is an agent i with $x_{ij} > 0$ who prefers j' to j, i.e. $j' \, P_i \, j$. The absence of a strong stable improvement cycle then implies that the graph $G(x)$ is acyclic. Thus we can topologically order the graph $G(x)$. The ordering is the desired ordering of schools (see Figure 1 for an illustration). □

3 Results

3.1 Pseudo-Market Mechanisms

In this section, we consider the pseudo-markets with priority-specific pricing of He et al. (2017). A random assignment is generated by a pseudo-market of probability shares. Each agent has a budget of tokens and can "buy" probability shares at the different object types. Agents face different prices depending on their priority. More precisely, for each object type there is a cut-off priority class.[5] All agents ranked below the cut-off of an object type cannot buy shares at the object type, i.e. they face an infinite price, all agents in the cut-off class face the same finite price and all agents ranked above the cut-off class can get shares at the object type for free. By construction, the mechanisms always implement an ex-ante stable random assignment. Moreover a first welfare theorem holds and pseudo-market equilibrium assignments are constrained Pareto efficient (Theorem 5 in He et al., 2017).[6]

[4] See e.g. West (2001) for a proof that acyclicity is equivalent to the existence of a topological ordering.

[5] The cut-offs can always be chosen such that in equilibrium they correspond to the cut-offs, as defined in Section 2.2, of the equilibrium assignment. Thus we denote both concepts by the same name.

[6] He et al. (2017) use a slightly different efficiency notion that however is equivalent for ex-ante stable assignments: They require that an assignment is not Pareto dominated by any assignment which also first

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$

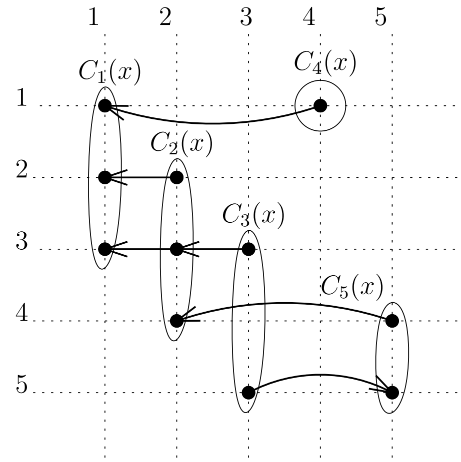

Figure 1: We consider the priorities and preferences from Example 1 and the assignment x. The graph on the right represents the preference information as follows: An arc is drawn from the column corresponding to school j to the column corresponding to school j' in the row corresponding to agent i if i is in the cut-off for both j and j', if $x_{ij} > 0$ and i prefers j' to j. For example, agent i_3 is in the cut-off for schools j_1 and j_2 and is assigned with positive probability to j_2. Moreover, agent i_2 prefers j_1 to j_2. Therefore we draw an arc from j_2 to j_1. Arcs implied by transitivity are not represented in the picture. A topological of the schools is $j_1 \rhd j_4 \rhd j_2 \rhd j_5 \rhd j_3$. Another such ordering is $j_1 \rhd j_2 \rhd j_4 \rhd j_5 \rhd j_3$.

Formally, a pseudo-market of He et al. (2017) is a triple $(U, b, \succeq)$ consisting of a VNM-utility profile U, priorities $\succeq$ and a vector of budgets $b \in \mathbb{R}^M_+$. A **constrained equilibrium** for the pseudo market $(U, b, \succeq)$ is a triple (x, C, p) consisting of a random assignment x, priority classes according to $\succeq$ for the different schools $C = (C_j)_{j \in M}$ called the priority cut-offs, and cut-off prices $p = (p_j)_{j \in M} \in \mathbb{R}^M_+$ such that the following holds: Defining for each $i \in N$ a price vector $p_i = (p_{ij})_{j \in M} \in \mathbb{R}^M_+$ where

$$p_{ij} = \begin{cases} \infty, & \text{if } C_j \succ_j i, \\ p_j, & \text{if } i \sim_j C_j, \\ 0, & \text{if } i \succ_j C_j. \end{cases} \tag{1}$$

for each $i \in N$ the random assignment x is an optimum for the problem

$$\begin{aligned} \max_{\pi \in \mathbb{R}^M} \quad & \sum_{j \in M} u_{ij} \cdot \pi_j \\ \text{subject to} \quad & \sum_{j \in M} p_{ij} \cdot \pi_j \leq b_i \\ & \sum_{j \in M} \pi_j \leq 1 \\ & \pi_j \geq 0 \quad \forall j \in M. \end{aligned}$$

order stochastically dominates the other assignment with respect to school priorities.

Equivalently we can formulate the maximization problem as follows:

$$\begin{aligned} \max_{\pi \in \mathbb{R}^M} \quad & \sum_{j: i \succeq_j C_j} u_{ij} \cdot \pi_j \\ \text{subject to} \quad & \sum_{j: i \in C_j} p_j \cdot \pi_j \leq b_i \\ & \sum_{j: i \succeq_j C_j} \pi_j \leq 1 \\ & \pi_j \geq 0 \quad \forall j : i \succeq_j C_j. \end{aligned}$$

It can be shown (He et al., 2017) that for each market $(U, b, \succeq)$ there exists a constrained equilibrium. We denote the set of constrained equilibria for $(U, b, \succeq)$ by $\mathcal{E}(U, b, \succeq)$. By construction, each equilibrium assignment is an ex-ante stable random assignment.

Proposition 3 (He et al., 2017)**.** *For each utility profile, budget vector and priorities , each constrained equilibrium assignment is ex-ante stable with respect to the induced preference profile.*

Proof. Let $(x, C, p) \in \mathcal{E}(U, b, \succeq)$. Suppose x is ex-ante blocked by i and j. Then $i \succ_j C_j$ and therefore $p_{ij} = 0$. Moreover, there is a j' with $j \, P_i \, j'$ and $x_{ij'} > 0$. Thus i would obtain a higher expected utility by obtaining shares in j which i can get for free rather than shares in j'. We have a contradiction. □

We show that constrained equilibrium assignments are constrained Pareto efficient. The argument is very similar to the standard argument that establishes the first welfare theorem in exchange economies. The proposition is a reformulation of a theorem by He et al. (2017) with our constrained efficiency notion.

Proposition 4 (Constrained First Welfare Theorem, He et al., 2017)**.** *For each utility profile U, budget vector b and priorities $\succeq$, each constrained equilibrium allocation is constrained Pareto efficient with respect to U and $\succeq$.*

Proof. Let $(x, C, p) \in \mathcal{E}(U, b, \succeq)$. Suppose y Pareto dominates x and does not use a more lenient admission policy than x at any school. By the definition of Pareto dominance, for each $i \in N$,

$$\sum_{j \in M} u_{ij} \cdot y_{ij} \geq \sum_{j \in M} u_{ij} \cdot x_{ij},$$

where the inequality is strict for at least one agent. For an agent i, for which the inequality is strict, we have by revealed preferences

$$\sum_{j: i \in C_j} p_j \cdot y_{ij} > b_i \geq \sum_{j: i \in C_j} p_j \cdot x_{ij}.$$

Next we show that for an agent i, for which equality holds, i.e.

$$\sum_{j \in M} u_{ij} \cdot y_{ij} = \sum_{j \in M} u_{ij} \cdot x_{ij},$$

we have

$$\sum_{j:i\in C_j} p_j \cdot y_{ij} \geq \sum_{j:i\in C_j} p_j \cdot x_{ij}.$$

Suppose for the sake of contradiction that

$$\sum_{j:i\in C_j} p_j \cdot y_{ij} < \sum_{j:i\in C_j} p_j \cdot x_{ij} \leq b_i.$$

Since $\sum_{j\in M} u_{ij} \cdot y_{ij} = \sum_{j\in M} u_{ij} \cdot x_{ij}$, lottery y_j is a solution to the utility maximization problem. Let $\bar{j}$ be the most preferred school with a finite price for agent i, i.e.

$$u_{i\bar{j}} = \max_{\{j:p_{ij}<\infty\}} u_{ij}.$$

We show that y_i is a degenerate lottery that chooses $\bar{j}$ for sure, $y_{i\bar{j}} = 1$. Indeed, suppose not. Then there is a school $j \neq \bar{j}$ with $y_{ij} > 0$. Since $\sum_{j:i\in C_j} p_j \cdot y_{ij} < b_i$ and $u_{i\bar{j}} > u_{ij}$ (utilities are strict), increasing probability shares in $\bar{j}$ by a small positive amount and proportionally decreasing probability shares in j would lead to another feasible lottery π' with $\sum_{j\in M} u_{ij} \cdot \pi'_j > \sum_{j\in M} u_{ij} \cdot y_{ij}$ and $\sum_{j:i\in C_j} p_j \cdot \pi'_j \leq b_i$ contradicting the observation that y_i is a solution to the utility maximization problem. Thus y_i is the degenerate lottery that chooses the best school with finite price for agent i. But if this lottery is affordable for i, then it is the unique solution to the utility maximization problem. Thus $x_i = y_i$. But this contradicts the assumption that

$$\sum_{j:i\in C_j} p_j \cdot y_{ij} < \sum_{j:i\in C_j} p_j \cdot x_{ij}.$$

We have established that for $i \in N$ we have

$$\sum_{j:i\in C_j} p_j \cdot y_{ij} \geq \sum_{j:i\in C_j} p_j \cdot x_{ij},$$

with strict inequality for at least one agent. Summing the inequalities over all agents, we obtain

$$\sum_{i\in N} \sum_{j:i\in C_j} p_j \cdot y_{ij} > \sum_{i\in N} \sum_{j:i\in C_j} p_j \cdot x_{ij}.$$

Rearranging the terms we obtain

$$\sum_{j\in M} p_j \cdot \left(\sum_{i\in C_j} y_{ij} \right) > \sum_{j\in M} p_j \cdot \left(\sum_{i\in C_j} x_{ij} \right).$$

This implies that there is an object type j such that $p_j > 0$ and

$$\sum_{i\in C_j} y_{ij} > \sum_{i\in C_j} x_{ij}.$$

Since agents below the cut-off C_j face infinite prices, we have $C_j(x) \succeq_j C_j$. Since y is not more lenient than x at j, we have $C_j(y) \succeq_j C_j(x) \succeq_j C_j$. Since $\sum_{i\in C_j} y_{ij} > 0$, we have in fact equality: $C_j(y) = C_j(x) = C_j$. But then, as $\sum_{i\in C_j} y_{ij} > \sum_{i\in C_j} x_{ij}$, j uses a more lenient admission policy at y than at x, a contradiction. $\square$

It is a natural question whether the constrained efficiency notion in Proposition 4 can be further strengthened. It particular, it seems to be natural to require only that the constrained equilibrium assignment is un-dominated by any random assignment that uses the same cut-offs but does not necessarily assign the same probability mass to each cut-off. Formally, call an assignment x **strongly constrained efficient** if for each y that Pareto-dominates x there is a school j such that $C_j(x) \succ_j C_j(y)$. The following example shows that constrained equilibrium assignments in general fail to satisfy strong constrained efficiency:

Example 3. Consider four agents, four schools, each with a single seat ($q_j = 1$ for each j), and the following utilities and priorities.

U_1	U_2	U_3	U_4		$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$
4	2	2	4		$2,3$	4	$\underline{2,3}$	1
3	3	3	1		$\underline{1}$	$\underline{2,3}$	$1,4$	$\underline{4}$
2	4	4	2		4	1		$2,3$
1	1	1	3					

and budgets

$$b_1 = 1, \quad b_2 = 7, \quad , b_3 = 7, \quad b_4 = 1.$$

The underlined entries in the priorities are cut-offs C such that with prices

$$p_1 = 3, \quad p_2 = 6, \quad p_3 = 12, \quad p_4 = 3,$$

the assignment

$$x = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

is a constrained equilibrium assignment, i.e. $(x, C, p) \in \mathcal{E}(U, b, \succeq)$. The assignment fails to be strongly constrained efficient however, since it is dominated by the (ex-ante stable) random assignment

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and we have $C_j(y) = C_j(x)$ for each $j \in M$.

3.2 The Constrained Ordinal Efficiency Welfare Theorem

In this section, we prove the main theorem of our paper.

Theorem 5 (Constrained Ordinal Efficiency Welfare Theorem)**.** *Let x be ex-ante stable according to P and $\succeq$. The following statements are equivalent:*

1. *x has no strong stable improvement cycle with respect to P and $\succeq$,*
2. *x is constrained ordinally efficient with respect to P and $\succeq$,*

3. *there exists a VNM-utility profile U consistent with P such that x is constrained Pareto-efficient with respect to U and $\succeq$,*

4. *there exists a VNM-utility profile U consistent with P, such that x is constrained social welfare maximizing with respect to U and $\succeq$,*

5. *there exists a VNM-utility profile U consistent with P, a budget vector b, prices p and cut-offs C such that $(x, C, p) \in \mathcal{E}(U, b, \succeq)$.*

The utility profile U can be chosen in the same way in (3),(4) and (5).

Proof. Let x be ex-ante stable. By Propositions 1, we have that (4) $\Rightarrow$ (3) $\Rightarrow$ (2). It follows immediately from the definitions that (2) $\Rightarrow$ (1). By Proposition 4, we have (5) $\Rightarrow$ (3). We will show that (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) which will complete the proof. After the proof, we present an example that will illustrate the proof. The reader may want to consult the example, as a running example during the proof.

Suppose x is constrained ordinally efficient. First we define the utility profile U that will be the same in the proof that (1) $\Rightarrow$ (4) and in the proof that (1) $\Rightarrow$ (5). By Lemma 2, there is an ordering $\rhd$ of the schools such that if an agent prefers a school j' for which he is in the cut-off under x to another school j for which he is in the cut-off and $x_{ij'} > 0$ and $j' \rhd j$, then i prefers j' to j. We choose positive numbers $(u_j)_{j \in M}$ such that for each $j, j' \in M$ we have

$$j' \rhd j \Rightarrow u_{j'} > u_j.$$

We will define utilities in such way that for each school j all agents in the cut-off of j under x that are matched to j with positive probability have the same utility u_j for j. Moreover, this utility will be the maximal utility that any agent who is in the cut-off for j has for receiving a seat in j. Thus we require for each i and j that

$$x_{ij} > 0, i \in C_j(x) \Rightarrow u_{ij} = u_j, \tag{2}$$

$$x_{ij} = 0, i \in C_j(x) \Rightarrow u_{ij} \leq u_j. \tag{3}$$

Furthermore we choose a number $\bar{u} > 0$ with $\bar{u} < u_j$ for each $j \in M$ and require for each agent $i \in N$ that for schools for which he is strictly above the cut-off should yield him a utility of at most $\bar{u}$, with utility of exactly $\bar{u}$ if $x_{ij} > 0$. Thus we require for each i and j that i.e.

$$x_{ij} > 0, i \succ_j C_j(x) \Rightarrow u_{ij} = \bar{u}, \tag{4}$$

$$x_{ij} = 0, i \succ_j C_j(x) \Rightarrow u_{ij} \leq \bar{u}. \tag{5}$$

We show that we can construct U consistent with P such that Conditions (2)-(5) hold. For each agent $i \in N$ we distinguish between four types of schools: The set M_1 consists of schools for which i is above the cut-off, the set M_2 of schools for which he is in the cut-off and receives a seat with positive probability under x_i, the set M_3 consists of schools for which i is in the cut-off but does not receive a seat under x_i, and the set M_4 consists of schools for which he is below the cut-off. We first choose utilities for the first three types of schools. Afterwards, for schools in M_4 utilities can be chosen such that U_i is consistent with P_i but otherwise arbitrarily.

If $M_1 \neq \emptyset$ we consider the unique school $j = \max_{P_i} M_1$ and let $u_{ij} = \bar{u}$. For the other schools in M_1 we choose $u_{ij} \leq \bar{u}$ and consistent with $P_i|_{M_1}$ but otherwise arbitrarily. Note that, by ex-ante stability of x, if there is a $j \in M_1$ with $x_{ij} > 0$, then $j = \max_{P_i} M_1$. Thus Conditions (4) and (5) hold. For $j \in M_2$, we let $u_{ij} = u_j$. Note that, by ex-ante stability of x, i prefers all schools in M_2 to all schools in M_1 and that by Lemma 2 for $j, j' \in M_2$ with $j' \, P_i \, j$ we have $u_{j'} > u_j$. Thus the choice of utilities is consistent with $P_i|_{M_1 \cup M_2}$. Moreover, by construction, Condition (2) holds. Finally for schools in M_3 we have to choose $u_{ij} \leq u_j$ and consistent with $P_i|_{M_1 \cup M_2 \cup M_3}$ to obtain Condition (3). This is possible by Lemma 2: If i prefers $j \in M_2$ to some of the schools in M_1, then choosing $u_{ij} = u_j$ is consistent with $P_i|_{M_1 \cup M_2 \cup M_3}$ by Lemma 2 and $u_{ij} \leq u_j$ holds with equality. If i prefers each school in M_1 to school $j \in M_2$, then we can choose any $u_{ij} < \min_{j'} u_{j'} \leq u_j$ that makes U_i consistent with $P_i|_{M_1 \cup M_2 \cup M_3}$.

Now that we have defined U, we can show that x is constrained social welfare efficient with respect to U and $\succeq$. Suppose for the sake of contradiction that there is a random assignment y that does not use a more lenient admission policy than x at any school and such that

$$\sum_{j \in M} \sum_{i \in N} u_{ij} \cdot y_{ij} > \sum_{j \in M} \sum_{i \in N} u_{ij} \cdot x_{ij}. \tag{6}$$

First we rearrange the right-hand side of the inequality. Note that by the definition of a cut-off we have

$$\sum_{j \in M} \sum_{i \in N} u_{ij} \cdot x_{ij} = \sum_{j \in M} \sum_{i \in C_j(x)} u_{ij} \cdot x_{ij} + \sum_{j \in M} \sum_{i \succ_j C_j(x)} u_{ij} \cdot x_{ij}.$$

By Condition (2) we have

$$\sum_{j \in M} \sum_{i \in C_j(x)} u_{ij} \cdot x_{ij} = \sum_{j \in M} u_j \cdot \left(\sum_{i \in C_j(x)} x_{ij} \right).$$

By Condition (4) we have

$$\begin{aligned} \sum_{j \in M} \sum_{i \succ_j C_j(x)} u_{ij} \cdot x_{ij} &= \sum_{j \in M} \bar{u} \cdot \left(\sum_{i \succ_j C_j(x)} x_{ij} \right) = \sum_{j \in M} \bar{u} \cdot \left(q_j - \sum_{i \in C_j(x)} x_{ij} \right) \\ &= n \cdot \bar{u} - \sum_{j \in M} \bar{u} \cdot \left(\sum_{i \in C_j(x)} x_{ij} \right). \end{aligned}$$

In conclusion, we have

$$\sum_{i \in N} \sum_{j \in M} u_{ij} \cdot x_{ij} = \sum_{j \in M} (u_j - \bar{u}) \cdot \left(\sum_{i \in C_j(x)} x_{ij} \right) + n \cdot \bar{u}. \tag{7}$$

Next we derive an upper for the left-hand side of Inequality (6) as follows: Since y does not use a more lenient admission policy than x at any school, we have:

$$\sum_{i\in N}\sum_{j\in M} u_{ij}\cdot y_{ij} = \sum_{j\in M}\sum_{i\in C_j(x)} u_{ij}\cdot y_{ij} + \sum_{j\in M}\sum_{i\succ_j C_j(x)} u_{ij}\cdot y_{ij}.$$

By Condition (3) we have

$$\sum_{j\in M}\sum_{i\in C_j(x)} u_{ij}\cdot y_{ij} \le \sum_{j\in M} u_j\cdot\left(\sum_{i\in C_j(x)} y_{ij}\right).$$

By Condition (5) we have

$$\sum_{j\in M}\sum_{i\succ_j C_j(x)} u_{ij}\cdot y_{ij} \le \sum_{j\in M}\bar{u}\cdot\left(\sum_{i\succ_j C_j(x)} y_{ij}\right) = \sum_{j\in M}\bar{u}\cdot\left(q_j - \sum_{i\in C_j(x)} y_{ij}\right)$$
$$\le \bar{u}\cdot n - \sum_{j\in M}\bar{u}\cdot\left(\sum_{i\in C_j(x)} y_{ij}\right).$$

In conclusion, we have

$$\sum_{j\in M}\sum_{i\in N} u_{ij}\cdot y_{ij} \le \sum_{j\in M}(u_j-\bar{u})\cdot\left(\sum_{i\in C_j(x)} y_{ij}\right) + n\cdot\bar{u}.$$

Combining this inequality with Inequality (6) and Equality (7), we obtain the inequality

$$\sum_{j\in M}(u_j-\bar{u})\cdot\left(\sum_{i\in C_j(x)} y_{ij}\right) > \sum_{j\in M}(u_j-\bar{u})\cdot\left(\sum_{i\in C_j(x)} x_{ij}\right).$$

Since $u_j-\bar{u}>0$ for each $j\in M$, there exists in particular a $j\in M$ with

$$\sum_{i\in C_j(x)} y_{ij} > \sum_{i\in C_j(x)} x_{ij} \ge 0.$$

But this implies that j has a more lenient admission policy under y than x, a contradiction. This concludes the proof that (1) $\Rightarrow$ (5).

Next we show (1) $\Rightarrow$ (4). We have to define prices p, cut-offs C and budgets b such that $(x,C,p)\in\mathcal{E}(U,b,\succeq)$. We define the cut-off classes to be the cut-offs under x, i.e. for each $j\in M$ we let $C_j := C_j(x)$. For each $j\in M$ we let

$$p_j = u_j - \bar{u}.$$

Defining prices in this way guarantees that the relative prices between two schools that an agent obtains with positive probability equals the marginal rate of substitution of the two schools where utility is measured in excess of the utility of the "fall-back school":

$$\frac{p_j}{p_{j'}} = \frac{u_j - \bar{u}}{u_{j'} - \bar{u}}.$$

We define budgets $b = (b_i)_{i \in N}$ by

$$b_i = \sum_{j : i \in C_j} p_j \cdot x_{ij}.$$

We show that for each $i \in N$ assignment x_i is optimal given prices, cut-offs and his budget, i.e. we want to show that x_i is an optimum for the problem:

$$\begin{aligned} \max_{\pi} \quad & \sum_{j : i \succeq_j C_j} u_{ij} \cdot \pi_j \\ \text{subject to} \quad & \sum_{j : i \in C_j} p_j \cdot \pi_j \leq b_i, \\ & \sum_{j : i \succeq_j C_j} \pi_j \leq 1, \\ & \pi_j \geq 0, \quad \forall j : i \succeq_j C_j. \end{aligned}$$

The dual problem is

$$\begin{aligned} \min_{\lambda, \mu} \quad & \lambda \cdot b_i + \mu \\ \text{subject to} \quad & p_j \cdot \lambda + \mu \geq u_{ij}, \quad \forall j : i \in C_j, \\ & \mu \geq u_{ij}, \quad \forall j : i \succ_j C_j, \\ & \lambda, \mu \geq 0. \end{aligned}$$

The choice of $\lambda = 1$ and $\mu = \bar{u}$ is feasible for the dual, since, by Conditions (2) and (3), for each j with $i \in C_j$ we have

$$p_j = u_j - \bar{u} \geq u_{ij} - \bar{u},$$

and since, by Conditions (4) and (5), for each j with $i \succ_j C_j$ we have

$$\bar{u} \geq u_{ij}.$$

Now by Conditions (2) and (4) we have

$$\begin{aligned} \sum_{j : i \succeq_j C_j} u_{ij} \cdot x_{ij} &= \sum_{j : i \in C_j} u_j \cdot x_{ij} + \bar{u} \cdot (1 - \sum_{j : i \in C_j} x_{ij}) = \sum_{j : i \in C_j} (u_j - \bar{u}) \cdot x_{ij} + \bar{u} \\ &= \sum_{j : i \in C_j} p_j \cdot x_{ij} + \bar{u} = b_i + \bar{u} = b_i + \bar{u} = \lambda \cdot b_i + \mu. \end{aligned}$$

By linear programming duality, this shows that x_i is an optimal solution to the agent's maximization problem (and $(\lambda = 1, \mu = \bar{u})$ is optimal for the dual).

□

Example 1 (cont.). We reconsider the previous Example 1 with the same preferences and priorities as before. Consider the assignment x. One can check that the x is constrained ordinally efficient with respect to P and $\succeq$. To construct a utility profile U such that x is constrained social welfare efficient with respect to U and $\succeq$, we proceed as follows: We can order the schools $j_1 \rhd j_4 \rhd j_2 \rhd j_5 \rhd j_3$ (see Figure 1). We let

$$u_1 = 6 > u_4 = 5 > u_2 = 4 > u_5 = 3 > u_3 = 2 > \bar{u} = 1.$$

For agent 1, school 5 is the most preferred school for which he is above the cut-off. He prefers schools 1 and 4 to school 5 and is in the cut-off for both of them and receives both of them with positive probability under x. Thus we let

$$u_{11} = u_1 = 6 > u_{14} = u_4 = 5 > u_{15} = \bar{u} = 1.$$

The other utilities may be chosen consistent with P_1 but otherwise arbitrarily, e.g.

$$u_{12} = \frac{1}{2} > u_{13} = \frac{1}{3}.$$

Following the same construction for the other agents, we obtain:

U_1	U_2	U_3	U_4	U_5
6	6	6	$\frac{1}{3}$	$\frac{1}{2}$
$\frac{1}{2}$	4	4	4	1
$\frac{1}{3}$	$\frac{1}{2}$	2	$\frac{1}{2}$	2
5	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
1	1	1	3	3

We choose the cut-offs of x as cut-offs to decentralize the assignment.

$$C_1 = C_1(x) = \{1, 2, 3\}, C_2 = C_2(x) = \{2, 3, 4\}, C_3 = C_3(x) = \{1, 3, 5\},$$

$$C_4 = C_4(x) = \{1, 2\}, C_5 = C_5(x) = \{4, 5\}$$

We choose prices such that for each school j we have $p_j = u_j - \bar{u}$. Thus

$$p_1 = 6 - 1 = 5,\ p_2 = 4 - 1 = 3,\ p_3 = 2 - 1 = 1,\ p_4 = 5 - 1 = 4,\ p_5 = 3 - 1 = 2.$$

Budgets are

$$b_1 = p_1 \cdot x_{11} + p_4 \cdot x_{14} = 5 \cdot \frac{1}{3} + 4\frac{1}{2} = 3\tfrac{2}{3},$$
$$b_2 = p_1 \cdot x_1 + p_2 \cdot x_2 = 5 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2\tfrac{2}{3},$$
$$b_3 = p_1 \cdot x_{31} + p_2 \cdot x_{32} + p_5 \cdot x_{35} = 5 \cdot \frac{1}{3} + 3 \cdot \frac{1}{6} + \frac{1}{3} = 2\tfrac{1}{2},$$
$$b_4 = p_2 \cdot x_{42} + p_5 \cdot x_{45} = 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} = 1\tfrac{1}{3},$$
$$b_5 = p_3 \cdot x_{53} + p_5 \cdot x_{55} = \frac{2}{3} + 2 \cdot \frac{1}{6} = 1.$$

One can show that for each agent i the lottery x_i is a solution to his utility maximization problem. As an example consider agent 1. His optimization problem and the dual of this problem are given by

$$\begin{aligned}
\max_{\pi=(\pi_1,\dots,\pi_5)} \quad & 6\cdot\pi_1 + \tfrac{1}{2}\cdot\pi_2 + \tfrac{1}{3}\cdot\pi_3 + 5\cdot\pi_4 + \pi_5 \\
\text{subject to} \quad & 5\cdot\pi_1 + \pi_3 + 4\pi_4 \leq 3\tfrac{2}{3}, \\
& \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 \leq 1, \\
& \pi_1, \pi_2, \pi_3, \pi_4, \pi_5 \geq 0.
\end{aligned}$$

$$\begin{aligned}
\min_{\lambda,\mu} \quad & 3\tfrac{2}{3}\cdot\lambda + \mu \\
\text{subject to} \quad & 5\cdot\lambda + \mu \geq 6 \\
& \mu \geq \frac{1}{2}, \\
& \lambda + \mu \geq \frac{1}{3}, \\
& 4\cdot\lambda + \mu \geq 5, \\
& \mu \geq 1 \\
& \lambda, \mu \geq 0.
\end{aligned}$$

Choosing $\lambda = 1, \mu = 1$ is feasible for the dual. Moreover

$$6\cdot x_{11} + \tfrac{1}{2}\cdot x_{12} + \tfrac{1}{3}\cdot x_{13} + 5\cdot x_{14} + x_{15} = 2 + 0 + 0 + 2\tfrac{1}{2} + \tfrac{1}{6} = 4\tfrac{2}{3} = 3\tfrac{2}{3}\cdot\lambda + \mu.$$

By linear programming duality this demonstrates that x_1 is optimal for agent 1. □

3.3 The Impossibility of a Cardinal Second Welfare Theorem

In our ordinal efficiency welfare theorem we started with ordinal preferences and constructed vNM-utilities, budgets, cut-offs and prices to decentralize an ex-ante stable and constrained ordinally efficient assignment. It it a natural question, whether the result can be strengthened in the following way: Start with a profile of vNM-utilities and show that each constrained Pareto efficient assignment can be decentralized as a price equilibrium. In this final section, we demonstrate by means of a counter-example that this is not possible and a cardinal second welfare theorem does not hold under our constrained efficiency notion. Since the validity of the first welfare theorem (Proposition 4) seems to be very dependent on the particular constrained efficiency notion that we employ (compare Example 3), this is a strong indication that the first and second welfare theorem cannot be obtained jointly for the same constrained efficiency notion.

Example 4. Consider four agents, four schools, each with a single seat ($q_j = 1$ for each j), and the following utilities and priorities.

U_1	U_2	U_3	U_4	$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$
4	2	2	4	$2,3$	4	$\underline{2,3}$	1
3	3	3	1	$\underline{1}$	$\underline{2,3}$	$1,4$	$\underline{4}$
2	4	5	2	4	1		$2,3$
1	1	1	3				

The underlined entries in the priorities are the cut-offs for the following assignment

$$x = \begin{pmatrix} \frac{3}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}.$$

By going through all potential blocking pairs, one can check that x is ex-ante stable. Next we show that x is constrained Pareto efficient: Suppose y Pareto dominates x and does not use a more lenient admission policy at any school. Since y does not use a more lenient admission policy at any school, agent 1 cannot receive a seat in school 2 or 3 and cannot receive a higher probability for receiving a seat in school 1. Thus $y_1 = x_1$. Since y does not use a more lenient admission policy at any school, agent 4 cannot receive a seat in school 2 or 3 and cannot receive a higher probability for receiving a seat in school 4. Thus $y_4 = x_4$. Thus, agents 1 and 4 receive the same assignment in x and y and we can focus on agents 2 and 3. Moreover, the sum of probability shares allocated to agents 2 and 3 is the same in both assignments $y_2 + y_3 = x_2 + x_3 = (\frac{1}{4}, \frac{3}{4}, 1, 0)$. Since $u_{21} = u_{31} = 2$, $u_{22} = u_{32} = 3$ and $u_{23} = 5 > 4 = u_{33}$ the only way that y could Pareto improve upon x is that $y_{23} > x_{23}$ and $y_{33} < x_{33}$. But in this case agent 3 can have an expected utility from y_3 that is higher than his expected utility from x_3, only if $y_{31} > x_{31}$ and $y_{32} > x_{32}$. However, $y_{31} > x_{31}$ implies that $x_{21} > y_{21}$, contradicting the fact that $x_{21} = 0$.

Next we show that x cannot be decentralized as an equilibrium. Suppose there are budgets $b \in \mathbb{R}_+^N$, cut-offs C and prices $p \in \mathbb{R}_+^M$ such that $(x, C, p) \in \mathcal{E}(U, b, \succeq)$. Since $x_{11} > 0, x_{21} > 0$ and $2 \succ_1 1$, we have $2 \succ_1 C_1$ and therefore $p_{21} = 0$. Since $x_{21} > 0, x_{22} > 0, x_{23} > 0$ and $u_{23} > u_{22} > u_{21}$ we have $0 = p_{21} < p_2 = p_{22} < p_3 = p_{32} < \infty$. Thus agent 2's optimization problem is

$$\begin{aligned} \max_{\pi} \quad & 2\pi_1 + 3\pi_2 + 4\pi_3 \\ \text{subject to } & \pi_1 + \pi_2 + \pi_3 \leq 1 \\ & p_2 \cdot \pi_2 + p_3 \cdot \pi_3 \leq b_2 \\ & \pi_1, \pi_2, \pi_3 \geq 0 \end{aligned}$$

for $p_2, p_3 > 0$ (since $u_{24} < u_{21}$ and $p_{21} = 0$, we can ignore school 4 in the above problem). Since $x_{21}, x_{22}, x_{23} > 0$ and x_2 is an optimal solution to the problem, both constraints bind for $\pi = x_2$. But this implies that

$$\frac{p_2}{p_3} = \frac{u_{22} - u_{21}}{u_{23} - u_{21}} = \frac{3 - 2}{4 - 2} = \frac{1}{2}.$$

Agent 3 is in the same priority classes as 2 at all schools. Therefore he faces the same prices as agent 2. But this implies that agent 3 can afford the lottery $(\frac{1}{4}, 0, \frac{3}{4}, 0)$, as

$$b_3 \geq p_2 \cdot x_{32} + p_3 \cdot x_{33} = p_2 \frac{1}{2} + p_3 \cdot \frac{1}{2} \geq p_2 \cdot 0 + p_3 \cdot \frac{3}{4}.$$

But this lottery yields a higher expected utility as x_2 since

$$2 \cdot \tfrac{1}{4} + 5 \cdot \tfrac{3}{4} = \tfrac{17}{4} > 4 = 3 \cdot \tfrac{1}{2} + 5 \cdot \tfrac{1}{2}.$$

This contradicts the assumption that (x, C, p) is an equilibrium.

References

Abdulkadiroglu, A. and Sönmez, T. (2003): "School Choice: A Mechanism Design Approach." *American Economic Review*, 93(3): 729–747.

Aziz, H. and Klaus, B. (2017): "Random Matching under Priorities: Stability and No Envy Concepts." Technical report.

Erdil, A. and Ergin, H. (2008): "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice." *The American Economic Review*, 98(3): 669–689.

He, Y., Miralles, A., Pycia, M., and Yan, J. (2017): "A Pseudo-Market Approach to Allocation with Priorities." *American Economic Journal: Microeconomics*, forthcoming.

Hylland, A. and Zeckhauser, R. (1979): "The Efficient Allocation of Individuals to Positions." *The Journal of Political Economy*, 87(2): 293–314.

Kesten, O. and Ünver, M. U. (2015): "A Theory of School-choice Lotteries." *Theoretical Economics*, 10(2): 543–595.

Kojima, F. and Manea, M. (2010): "Incentives in the Probabilistic Serial Mechanism." *Journal of Economic Theory*, 145(1): 106–123.

Manea, M. (2008): "A Constructive Proof of the Ordinal Efficiency Welfare Theorem." *Journal of Economic Theory*, 141(1): 276–281.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995): *Microeconomic Theory.* Oxford University Press, New York.

McLennan, A. (2002): "Ordinal Efficiency and the Polyhedral Separating Hyperplane Theorem." *Journal of Economic Theory*, 105(2): 435–449.

Miralles, A. and Pycia, M. (2014): "Prices and Efficient Assignments without Transfers." Technical report.

Roth, A. E., Rothblum, U. G., and Vande Vate, J. H. (1993): "Stable Matchings, Optimal Assignments, and Linear Programming." *Mathematics of Operations Research*, 18(4): 803–828.

West, D. B. (2001): *Introduction to Graph Theory.* Sattle River: Prantice Hall.