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Countable products of probabilistic normed spaces

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ABSTRACT

Countable products of probabilistic normed spaces are introduced and studied. In particular, a comparison is made with the analogous constructions for probabilistic metric spaces.

1. Introduction

Probabilistic Normed (briefly PN) spaces were introduced by Šerstnev in a series of papers [1–4]. Then a new definition was proposed by Alsina, Schweizer and Sklar [5]. This new definition revived the study of PN spaces. The properties of these spaces were studied by several authors; here we shall mention [6–9] (but see also the survey paper [10]). We recall below for the reader's convenience the definition of a PN space; the notation is essentially that fixed by the classic book by Schweizer and Sklar [11].

Definition 1.1. A *probabilistic normed* space is a quadruple (V, ν, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions and the mapping $\nu : V \to \Delta^+$ satisfies, for all p and q in V, the conditions

(N1) $\nu_p = \epsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);
(N2) $\forall p \in V \quad \nu_{-p} = \nu_p$;
(N3) $\nu_{p+q} \geq \tau\left(\nu_p, \nu_q\right)$;
(N4) $\forall \alpha \in [0, 1] \quad \nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$.

The function ν is called the *probabilistic norm*. If (V, ν, τ, τ^*) satisfies the condition, weaker than (N1),

$$\nu_\theta = \epsilon_0,$$

then it is called a *Probabilistic Pseudo-Normed* space (briefly, a PPN space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^* then $(V, \nu, \tau_T, \tau_{T^*})$ is denoted by (V, ν, T) and is a *Menger* PN space.

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A PN space is called a *Šerstnev space* if it satisfies conditions (N1), (N3) and following condition

(Š) $\forall \alpha \in \mathbb{R} \setminus \{0\}\ \forall x \in \overline{\mathbb{R}}_+ \qquad \nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$,

which implies (N2) and (N4) in the strengthened form

$$\forall \alpha \in [0, 1] \quad \forall p \in V \qquad \nu_{\alpha p} = \tau_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right).$$

The definition of a t-norm may be found in [11–13].

Some examples of t-norms T and its t-conorms T^* are: $M(x, y) = \min\{x, y\}$, $\Pi(x, y) = x \cdot y$ and $M^*(x, y) = max\{x, y\}$, $\Pi^*(x, y) = x + y - x \cdot y$. We recall the following sequence of inequalities:

$$T \leq M < M^* \leq T^*,$$

and in particular

$$W < \Pi < M < M^* < \Pi^* < W^*.$$

Definition 1.2. A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,

(1) $\tau(F, \epsilon_0) = F$,
(2) $\tau(F, G) = \tau(G, F)$,
(3) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H, G \leq K$,
(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Relevant triangle functions are the functions τ_T, τ_{T^*} and those of the form Π_T which, for any continuous t-norm T, and any $x > 0$, are given by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\},$$
$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(u), G(v)) \mid u + v = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

See [14,15] for a recent study of triangle functions.

The purpose of this note is to study the countable product of PN spaces. The case of the finite product of PN spaces was studied by the first author in [16]. The countable product of probabilistic metric spaces was the object of the two important papers by Alsina [17] and Alsina and Schweizer [18]; the differences between these works and the present one will be noted; in the first place, while a single triangle function appears in the definition of a probabilistic metric space, two triangle functions, τ and τ^*, have to be considered in a PN space.

2. The dominance relation

Definition 2.1. Let $(S, \leq)$ be a partially ordered set and let f and g be commutative and associative binary operations on S with common identity e. Then f *dominates* g, and one writes $f \gg g$, if, for all x_1, x_2, y_1, y_2 in S,

$$f(g(x_1, y_1), g(x_2, y_2)) \geq g(f(x_1, x_2), f(y_1, y_2)).$$

Setting $y_1 = x_2 = e$ in this inequality, one has $f(x_1, y_2) \geq g(x_1, y_2)$; thus $f \gg g$ implies $f \geq g$, which in turn implies that the dominance relation is antisymmetric. It is easily shown that the dominance relation is also reflexive. However, although it is known that the dominance relation is transitive on certain subsets of the set of continuous t-norms (see [19–21]), it is not, in general, transitive, as examples due to Sherwood [13] and Sarkoci [22] show.

We are interested in the dominance relation as it applies to t-norms, s-norms[1] and triangle functions. Here the following are known:

Lemma 2.1. *The following statements hold:*

(a) *For every t-norm T, $M \gg T$;*
(b) *For every s-norm S, $S \gg M^*$;*
(c) *For every continuous t-norm T, $\Pi_T \gg \tau_T$;*
(d) *For every triangle function τ, $\Pi_M \gg \tau$.*

[1] s-norms are often called t-conorms in the literature.

Proof. (a) For all x_1, x_2, y_1 and y_2 in I, one has

$$x_j \geq M(x_1, x_2), \qquad y_j \geq M(y_1, y_2) \quad (j = 1, 2);$$

thus

$$T(x_1, y_1) \geq T\left(M(x_1, x_2), M(y_1, y_2)\right),$$
$$T(x_2, y_2) \geq T\left(M(x_1, x_2), M(y_1, y_2)\right),$$

whence

$$M\left(T(x_1, y_1), T(x_2, y_2)\right) \geq T\left(M(x_1, x_2), M(y_1, y_2)\right),$$

i.e. $M \gg T$. Corresponding arguments prove (b), (c) and (d).

Next, a straightforward calculation yields

Lemma 2.2. *If T_1 and T_2 are t-norms such that $T_1 \gg T_2$, then $T_2^* \gg T_1^*$, and conversely.*

The following theorem is due to Tardiff [23].

Theorem 2.1. *The following statements are equivalent for all continuous t-norms T_1 and T_2:*

(1) $T_1 \gg T_2$;
(2) $\Pi_{T_1} \gg \Pi_{T_2}$;
(3) $\tau_{T_1} \gg \tau_{T_2}$;
(4) $\Pi_{T_1} \gg \tau_{T_2}$;
(5) $\tau_{T_2^*} \gg \tau_{T_1^*}$.

3. Main results: (I)—The Σ-product

In a given PN space (V, ν, τ, τ^*) the value of the probabilistic norm of $p \in V$ at the point t will be denoted either by $\nu(p)(t)$ or by $\nu_p(t)$.

Definition 3.1. Given a sequence $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$ of PN spaces and a sequence $\left(\beta_j\right)_{j\in\mathbf{N}}$ of strictly positive real numbers such that $\sum_{j\in\mathbf{N}} \beta_j = 1$, their *$\Sigma$-product* is the pair (V, ν) where V is the countable cartesian product of the spaces V_j, $V := \prod_{j\in\mathbf{N}} V_j$, and where, for $p = (p_j) \in V$,

$$\nu_p := \sum_{j\in\mathbf{N}} \beta_j \, \nu_j(p_j).$$

Our first result is the following.

Theorem 3.1. *Let $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$ be a sequence of PN spaces such that for, every $j \in \mathbf{N}$, $\tau_j \geq \tau_W$ and $\tau_j^* \leq \tau_{W^*}$. Then their Σ-product (V, ν) is a Menger PN space under W.*

Proof. Let $\theta := (\theta_1, \theta_2, \ldots, \theta_j, \ldots)$, where θ_j is the origin of the vector space V_j ($j \in \mathbf{N}$). Then

$$\nu_\theta = \sum_{j\in\mathbf{N}} \beta_j \, \nu_j(\theta_j) = \sum_{j\in\mathbf{N}} \beta_j \, \epsilon_0 = \epsilon_0.$$

In the other direction, if $\nu_p = \epsilon_0$, then one has $\nu_p(t) = 1$ for every $t > 0$; thus

$$1 = \sum_{j\in\mathbf{N}} \beta_j \, \nu_j(p_j)(t) \leq \sum_{j\in\mathbf{N}} \beta_j = 1,$$

so that one has, for every $j \in \mathbf{N}$ and every $t > 0$, $\nu_j(p_j)(t) = 1$, namely $p_j = \theta_j$ and, therefore, $p = \theta$. This proves (N1). The proof of (N2) is trivial. As for (N3), one has, for every $t > 0$,

$$\begin{aligned}
\nu_{p+q}(t) &= \sum_{j\in\mathbf{N}} \beta_j \, \nu_j(p_j + q_j)(t) \geq \sum_{j\in\mathbf{N}} \beta_j \, \tau_j\left(\nu_j(p_j), \nu_j(q_j)\right)(t) \\
&\geq \sum_{j\in\mathbf{N}} \beta_j \, \tau_W\left(\nu_j(p_j), \nu_j(q_j)\right)(t) \\
&= \sum_{j\in\mathbf{N}} \beta_j \sup_{u\in[0,t]} W\left(\nu_j(p_j)(u), \nu_j(q_j)(t-u)\right);
\end{aligned}$$

thus, for every u in $[0, t]$,

$$\begin{aligned}\nu_{p+q}(t) &\geq \sum_{j\in\mathbf{N}} \beta_j \max\left\{\nu_j(p_j)(u) + \nu_j(q_j)(t-u) - 1, 0\right\}\\ &\geq \sum_{j\in\mathbf{N}} \beta_j \left\{\nu_j(p_j)(u) + \nu_j(q_j)(t-u) - 1\right\}\\ &= \nu_p(u) + \nu_q(t-u) - 1,\end{aligned}$$

and, hence

$$\nu_{p+q}(t) \geq \sup_{u\in[0,t]} \left\{\nu_p(u) + \nu_q(t-u) - 1\right\};$$

since $\nu_{p+q} \geq 0$, one has $\nu_{p+q} \geq \tau_W(\nu_p, \nu_q)$, which proves (N3).

Similarly, for every $\alpha \in [0, 1]$ for every $p \in V$ and for every $t > 0$, one has

$$\begin{aligned}\nu_p(t) &= \sum_{j\in\mathbf{N}} \beta_j\, \nu_j(p_j)(t) \leq \sum_{j\in\mathbf{N}} \beta_j\, \tau_j^*\left(\nu_j(\alpha p_j), \nu_j((1-\alpha)p_j)\right)(t)\\ &\leq \sum_{j\in\mathbf{N}} \beta_j\, \tau_{W^*}\left(\nu_j(\alpha p_j), \nu_j((1-\alpha)p_j)\right)(t)\\ &= \sum_{j\in\mathbf{N}} \beta_j \inf_{u\in[0,t]} W^*\left(\nu_j(\alpha p_j)(u), \nu_j((1-\alpha)p_j)(t-u)\right).\end{aligned}$$

Recalling that $W^*(s, t) = \min\{s + t, 1\}$, one has, for every u in $[0, t]$,

$$\begin{aligned}\nu_p(t) &\leq \sum_{j\in\mathbf{N}} \beta_j\, W^*\left(\nu_j(\alpha p_j)(u), \nu_j((1-\alpha)p_j)(t-u)\right)\\ &\leq \sum_{j\in\mathbf{N}} \beta_j \left(\nu_j(\alpha p_j)(u) + \nu_j((1-\alpha)p_j)(t-u)\right)\\ &= \nu_{\alpha p}(u) + \nu_{(1-\alpha)p}(t-u),\end{aligned}$$

so that

$$\nu_p(t) \leq \inf_{u\in[0,t]} \left\{\nu_{\alpha p}(u) + \nu_{(1-\alpha)p}(t-u)\right\};$$

since $\nu_p \leq 1$, one has $\nu_p \leq \tau_{W^*}\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$, which establishes (N4) and concludes the proof. □

Corollary 3.1. *If $\left(V_j, \nu_j\right)_{j\in\mathbf{N}}$ is a sequence of Menger PN spaces under W, then their Σ-product is again a Menger PN space under W.*

However, even if $\left(V_j, \nu_j\right)_{j\in\mathbf{N}}$ is a sequence of PN spaces with respect to the *same* triangle functions τ and τ^*, their Σ-product need not be a PN space with respect to the same triangle functions. In order to see this it is sufficient to modify Example 1.3 in Alsina's paper [17].

Example 3.1. Consider the simple space $(\mathbb{R}, |\cdot|, U)$ where U is the d.f. of a random variable uniformly distributed on the interval $(0, 1)$ and the sequences p and q in $\prod_{n\in\mathbf{N}} \mathbb{R}$ given by

$$p := \{cn : n \in \mathbf{N}\} \quad \text{and} \quad q := \{n : n \in \mathbf{N}\},$$

Assume that $\beta_1 := \max\{\beta_j : j \in \mathbf{N}\}$ and, moreover, assume that the constant $c \in]0, 1/2[$ is small enough to satisfy the inequality

$$\beta_1 > \frac{1}{c+1}. \tag{3.1}$$

Thus, for every $t > 0$,

$$\nu_{p+q}(t) = \sum_{j\in\mathbf{N}} \beta_j\, U\left(\frac{t}{(c+1)j}\right) < U\left(\frac{t}{c+1}\right).$$

In particular,

$$\nu_{p+q}(1) < \frac{1}{c+1}$$

On the other hand

$$\nu_p(1/2) = \sum_{j\in\mathbf{N}} \beta_j U\left(\frac{1}{2cj}\right) > \beta_1 U\left(\frac{1}{2c}\right) = \beta_1$$

since $1/(2c) > 1$, and

$$\nu_q(1/2) = \sum_{j\in\mathbf{N}} \beta_j U\left(\frac{1}{2j}\right) > \beta_1 U\left(\frac{1}{c}\right) = \beta_1,$$

so that

$$\tau_M\left(\nu_p, \nu_q\right)(1) \geq \nu_p(1/2) \wedge \nu_q(1/2) \geq \beta_1.$$

Therefore one has, because of Eq. (3.1),

$$\nu_{p+q}(1) < \frac{1}{c+1} < \beta_1 \leq \tau_M\left(\nu_p, \nu_q\right)(1),$$

viz. the Σ-product of the sequence of simple spaces we have considered is not a simple space.

The previous example also exhibits one drawback of the construction of the Σ-product of a sequence of PN spaces, namely that even when $\left(V_j, \nu_j, \tau, \tau^*\right)_{j\in\mathbf{N}}$ is a sequence of PN spaces under the *same* triangle functions τ and τ^*, their Σ-product need not be a PN space under the same two triangle functions. This drawback is evident also in another class of examples, which we now proceed to construct.

Definition 3.2. The pair (V, ν) is said to be a *Probabilistic Semi-Normed space* (briefly a PSN space) when $\nu : V \to \Delta^+$ satisfies (N1) and (N2). A PSN space (V, ν) is said to be a *Wald space*, when $\tau = *$, namely when the triangle function τ is the convolution,

$$\nu_{p+q} \geq \nu_p * \nu_q. \tag{3.2}$$

In a PN space with $\tau_T = \tau_\Pi$, where Π is the t-norm $\Pi(a, b) = a \cdot b$, one has

$$\nu_{p+q}(x+y) \geq \nu_p(x) \cdot \nu_q(y).$$

Theorem 3.2. *A Wald space $(V, \nu, *)$ can be made in a PN space $(V, \nu, *, \Pi_{T^*})$, for every t-norm T.*

Proof. The inequality $\nu_p \leq \nu_{\beta p}$, which holds for every $p \in V$ and for every $\beta \in [0, 1]$, yields, when applied to $\beta = \alpha$ and $\beta = 1 - \alpha$ ($\alpha \in [0, 1]$),

$$\nu_p \leq \min\left\{\nu_{\alpha p}, \nu_{(1-\alpha)p}\right\} = \Pi_M\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right) \leq \Pi_{T^*}\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right),$$

which concludes the proof. □

Example 3.2. The quadruple $(V, \nu, *, \Pi_M)$ where V is a normed linear space and the probabilistic norm ν is defined via

$$\nu_p(x) := \frac{x}{x+1} \mathrm{e}^{-\|p\|},$$

and $*$ is convolution, is a PN space; it is neither a Šerstnev space nor a topological vector space. In fact, (N1) and (N2) are obvious.

(N3) For every $x > 0$ and for all p and q in V, one has explicitly, setting $h := x + 1 > 1$,

$$\begin{aligned}
\int_0^x \nu_p(x-t)\,\mathrm{d}\nu_q(t) &= \int_0^x \frac{x-t}{x-t+1} \mathrm{e}^{-\|p\|} \frac{1}{(t+1)^2} \mathrm{e}^{-\|q\|}\,\mathrm{d}t \\
&= \mathrm{e}^{-(\|p\|+\|q\|)} \left[\int_0^x \frac{\mathrm{d}t}{(t+1)^2} - \int_0^x \frac{\mathrm{d}t}{(x+1-t)(t+1)^2}\right] \\
&= \mathrm{e}^{-(\|p\|+\|q\|)} \left(1 - \frac{1}{h} - \frac{2\ln h}{(1+h)^2} - \frac{x}{(1+h)h}\right).
\end{aligned}$$

Property (N3) is equivalent, in turn to each of the following inequalities

$$\frac{x}{h}\,\mathrm{e}^{-\|p+q\|} \geq \mathrm{e}^{-(\|p\|+\|q\|)}\left(1-\frac{1}{h}-\frac{2\ln h}{(1+h)^2}-\frac{x}{(1+h)h}\right),$$

$$\frac{x}{h} \geq \mathrm{e}^{\|p+q\|-(\|p\|+\|q\|)}\left(1-\frac{1}{h}-\frac{2\ln h}{(1+h)^2}-\frac{x}{(1+h)h}\right),$$

$$x \geq \frac{1}{\mathrm{e}^{\|p\|+\|q\|-\|p+q\|}}\left(x-\frac{2h\,\ln h}{(1+h)^2}-\frac{x}{(1+h)h}\right),$$

$$x \geq \frac{1}{\mathrm{e}^{\|p\|+\|q\|-\|p+q\|}}\left(x\frac{h}{1+h}-\frac{2h\,\ln h}{(1+h)^2}\right).$$

But

$$\frac{1}{\mathrm{e}^{\|p\|+\|q\|-\|p+q\|}} \leq 1, \qquad -\frac{2h\,\ln h}{(1+h)^2} < 0 \quad \text{and} \quad \frac{h}{1+h} < 1,$$

so that

$$x \geq \frac{1}{\mathrm{e}^{\|p\|+\|q\|-\|p+q\|}}\,x\,\frac{h}{1+h} \geq \frac{1}{\mathrm{e}^{\|p\|+\|q\|-\|p+q\|}}\left(x\frac{h}{1+h}-\frac{2h\,\ln h}{(1+h)^2}\right),$$

which proves (N3).

(N4) is immediate since for every $p \in V$ and for every $\alpha \in (0, 1)$, one has $\|p\| \geq \alpha\,\|p\|$ and $\|p\| \geq (1-\alpha)\,\|p\|$.

Moreover, $(V, \nu, *, \Pi_M)$ is not a Šerstnev PN space because in this case the equation (Š) is equivalent to the relationship

$$\frac{x+1}{x+|\alpha|} = \mathrm{e}^{(1-\alpha)\,\|p\|},$$

which, for $x \neq -|\alpha|$ is an equation that admits the unique solution

$$x = \frac{|\alpha|\,\mathrm{e}^{(1-\alpha)\,\|p\|}-1}{1-\mathrm{e}^{(1-\alpha)\,\|p\|}};$$

this means that equation (Š) cannot hold for all $x > 0$ and $p \in V$.

Example 3.3. The Σ-product of Wald PN spaces is not necessarily a Wald PN space.

Consider the normed space $(\mathbb{R}, |\cdot|)$ as a Wald space $(\mathbb{R}, \nu, *)$, where

$$\nu_p = \epsilon_{|p|} = \epsilon_0(x-|p|),$$

for all $p \in \mathbb{R}$. Let $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$ be a sequence of Wald spaces where $V_j = \mathbb{R}$ for every $j \in \mathbf{N}$. Then their Σ-product is the pair (V, ν), where

$$V := \Pi_{j\in\mathbf{N}} V_j = \Pi_{j\in\mathbf{N}}\mathbb{R},$$

and where, for every $p = (p_j) \in V$ one has

$$\nu_p := \sum_{j\in\mathbf{N}} \frac{1}{2^j}\,\nu_j(p_j).$$

Choose $p = (1, 1, 0, \ldots, 0, \ldots)$, $q = (0, 0, 1, \ldots, 1, \ldots)$, and $r = (1, 1, 1, \ldots, 1, \ldots)$; then $p + q = r$,

$$\nu_p = \frac{1}{4}\,\epsilon_0 + \frac{3}{4}\,\epsilon_1, \qquad \nu_q = \frac{3}{4}\,\epsilon_0 + \frac{1}{4}\,\epsilon_1, \quad \text{and} \quad \nu_r = \epsilon_1,$$

whence one concludes that

$$\nu_p * \nu_q = \frac{1}{16}\,(3\epsilon_2 + 10\epsilon_1 + 3\epsilon_0),$$

and, for $0 < x < 1$,

$$\left(\nu_p * \nu_q\right)(x) = \frac{3}{16} > 0 = \nu_r(x),$$

and the proof is complete.

From the point of view of topology, the following result is essential.

Theorem 3.3. *The strong topology of the Σ-product $(V, \nu, \tau_W, \tau_{W^*})$ coincides with the product topology of the sequence of spaces $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$.*

Proof. The neighbourhoods of the type

$$\mathcal{N}_\theta(t) := \left\{q \in V : \nu_q(t) > 1 - t\right\}$$

form a basis for the topology of the Σ-product $(V, \nu, \tau_W, \tau_{W^*})$, while the cylinders

$$\prod_{j=1}^{k} \mathcal{N}_{\theta_j}(t_j) \times \prod_{j>k} V_j$$

are a basis for the product topology.

Let a neighbourhood $\mathcal{N}_\theta(t)$ be given with $t \in]0, 1[$ (if $t \geq 1$, then $\mathcal{N}_\theta(t) = V$). Since there exists $n \in \mathbf{N}$ such that $\sum_{j=1}^{n} \beta_j > 1 - t$, it is possible to define

$$t' := 1 - \frac{1-t}{\sum_{j=1}^{n} \beta_j} > 0$$

and let $t_0 := t \wedge t'$. If a point q belongs to the cylinder

$$U_n(t_0) := \prod_{j=1}^{n} \mathcal{N}_{\theta_j}(t_0) \times \prod_{j>n} V_j,$$

then one has $\nu_j(q_j)(t_0) > 1 - t_0$ for every $j = 1, 2, \ldots, n$. Now

$$\nu_q(t) \geq \nu_q(t_0) \geq \sum_{j=1}^{n} \beta_j \, \nu_j(q_j)(t_0) > \sum_{j=1}^{n} \beta_j \, (1 - t_0) \geq \sum_{j=1}^{n} \beta_j \, (1 - t') = 1 - t;$$

so that $U_n(t_0) \subset \mathcal{N}_\theta(t)$.

In the other direction, consider the cylinder

$$U_n(t_1, t_2, \ldots, t_n) := \prod_{j=1}^{n} \mathcal{N}_{\theta_j}(t_j) \times \prod_{j>n} V_j, \tag{3.3}$$

and set

$$t := \min\{t_1, t_2, \ldots, t_n\} > 0$$

$$t' := 1 - \max\left\{\beta_j \, (1 - t_j) + \sum_{\substack{k \in \mathbf{N} \\ k \neq j}} \beta_k : j = 1, 2, \ldots, n\right\}$$

$$t_0 := t \wedge t'.$$

Then, if q belongs to the neighbourhood $\mathcal{N}_\theta(t_0)$, one has, for every $j = 1, 2, \ldots, n$,

$$\nu_q(t_j) \geq \nu_q(t_0) > 1 - t_0 \geq 1 - t' = \max\left\{\beta_j \, (1 - t_j) + \sum_{\substack{k \in \mathbf{N} \\ k \neq j}} \beta_k : j = 1, 2, \ldots, n\right\}$$

$$\geq \beta_j \, (1 - t_j) + \sum_{\substack{k\in\mathbf{N} \\ k\neq j}} \beta_k \geq \beta_j \, (1 - t_j) + \sum_{\substack{k\in\mathbf{N} \\ k\neq j}} \beta_k \, \nu_k(q_k)(t_j);$$

this inequality implies

$$\beta_j \, \nu_j(q_j)(t_j) > \beta_j \, (1 - t_j),$$

namely $\nu_j(q_j)(t_j) > 1 - t_j$; therefore

$$\mathcal{N}_\theta(t_0) \subset U_n(t_1, t_2, \ldots, t_n);$$

which concludes the proof. □

4. (II)—The τ-product

By extending Alsina's construction [17] a different definition is possible for the countable product of PN spaces. Before proceeding, it is necessary to recall the definition of the $\tau^{(\infty)}$-product of a sequence $(\tau_n)_{n\in\mathbf{N}}$ of triangle functions. One proceeds inductively by setting, for any sequence $(F_n)_{n\in\mathbf{N}}$ of distribution functions in Δ^+, $\tau^{(1)} := \tau_1$, and, for $n \geq 2$,

$$\tau^{(n)}(F_1, \ldots, F_n, F_{n+1}) := \tau_n\left(\tau^{(n-1)}(F_1, \ldots, F_n), F_{n+1}\right).$$

The sequence $\left\{\tau^{(n)}(F_1, \ldots, F_{n+1})\right\}$ is decreasing so that its weak limit always exists, although it may be identically be equal to zero; it is denoted by $\tau^{(\infty)}\{F_n\}$,

$$\tau^{(\infty)}\{F_n\} := \mathrm{w} - \lim_{n\to+\infty} \tau^{(n)}(F_1, \ldots, F_{n+1}).$$

Definition 4.1. Given a sequence $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$ of PN spaces and a continuous triangle function τ, their *τ-product* is the pair (V, ν), where $V = \prod_{j\in\mathbf{N}} V_j$ and $\nu : V \to \Delta^+$ is defined, if $p = (p_n)$ with $p_n \in V_n$ for every $n \in \mathbf{N}$, by

$$\nu_p := \tau^{(\infty)}\{\nu_n(p_n)\}. \tag{4.1}$$

Theorem 4.1. *Let $\left(V_n, \nu_n, \tau_n, \tau_n^*\right)_{n\in\mathbf{N}}$ be a sequence of PN spaces. Assume that there exist two continuous triangle functions τ and τ^* such that*

(a) *for every $n \in \mathbf{N}$, $\tau \leq \tau_n \leq \tau_n^* \leq \tau^*$;*
(b) *τ^* dominates τ, $\tau^* \gg \tau$.*

Then their τ-product (V, ν) is a PN space under τ and τ^*.

Proof. One has $\nu_\theta = \tau^{(\infty)}\left\{\nu_{\theta_n}\right\} = \tau^{(\infty)}\{\epsilon_0, \ldots, \epsilon_0, \ldots\} = \epsilon_0$. Notice that this also proves that $\tau^{(\infty)}$ is not identically equal to zero. Now assume that $\nu_p = \epsilon_0$; then one has $\nu_n(p_n) = \epsilon_0$, and hence $p_n = \theta_n$ for every n; therefore $p = \theta$ and (N1) is proved, property (N2) is trivial.

Let $p = (p_n)$ and $q = (q_n)$ be elements of V. Then, on account of the continuity of τ, one has

$$\begin{aligned}
\nu_{p+q} &= \tau^{(\infty)}\{\nu_n(p_n + q_n)\} \geq \tau^{(\infty)}\{\tau_n(\nu_n(p_n), \nu_n(q_n))\} \\
&\geq \tau^{(\infty)}\{\tau(\nu_n(p_n), \nu_n(q_n))\} = \tau\left(\tau^{(\infty)}(\nu_n(p_n)), \tau^{(\infty)}(\nu_n(q_n))\right) \\
&= \tau\left(\nu_p, \nu_q\right),
\end{aligned}$$

which proves (N3).

For every $\alpha \in [0, 1]$, one has, since τ^* is continuous and τ^* dominates τ,

$$\begin{aligned}
\nu_p &= \tau^{(\infty)}\{\nu_n(p_n)\} \leq \tau^{(\infty)}\left\{\tau_n^*(\nu_n(\alpha p_n), \nu_n((1-\alpha)p_n))\right\} \\
&\leq \tau^{(\infty)}\left\{\tau^*(\nu_n(\alpha p_n), \nu_n((1-\alpha)p_n))\right\} \\
&\leq \tau^*\left(\tau^{(\infty)}\{\nu_n(\alpha p_n)\}, \tau^{(\infty)}\{\nu_n((1-\alpha)p_n)\}\right) \\
&= \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right),
\end{aligned}$$

which proves (N4) and concludes the proof. □

In the case of Menger spaces we have a stronger result as a special case of the previous theorem.

Example 4.1. In the previous theorem, if for every $n \in \mathbf{N}$, $\tau_n = \tau_M$, $\tau_n^* = \tau_{M^*}$ and $\tau^* = \tau_M$, namely, if the spaces $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$ are all PN spaces of Menger, then their τ_M-product (V, ν) is also a PN space of Menger denoted by (V, ν, M).

Example 4.2. Under the same hypotheses of Theorem 4.1, assume that for every $n \in \mathbf{N}$, $\tau_n = \tau_T$, $\tau_n^* = \tau_{M^*}$; then the τ_T-product (V, ν) is a PN space under τ_T and Π_M. Notice that one has $\tau_{M^*}^* \leq \Pi_M$: in fact, for all F and G in Δ^+ and $x > 0$, one has

$$\tau_{M^*}(F, G)(x) = \tau_M(F, G)(x) = \sup\{M(F(u), G(v)) \mid u + v = x\} \leq M(F(x), G(x)),$$

since for all u and v in $\mathbb{R}$ such that $u + v = x$, $M(F(u), G(v)) \leq M(F(x), G(x))$ holds. On the other hand $\Pi_M \gg \tau$ for every triangle function τ, which establishes the assertion.

However, even if $\left(V_j, \nu_j\right)_{j\in\mathbf{N}}$ is a sequence of PN spaces of Menger with respect to the *same* triangle functions τ_T and τ_{T^*}, their τ_T-product need not be a PN space of Menger with respect to the same triangle functions because $\tau_{T^*} \gg \tau_T$ is not true.

In a way similar to that of Theorem 4.1 one proves the following

Theorem 4.2. *Under the same assumptions of the previous Theorem, if the mapping $\nu' : V \to \Delta^+$ is defined, for every $p = (p_n)$ in V, by*

$$\nu'_p := \tau^{*(\infty)}\{\nu_n(p_n)\},$$

then the τ^*-product (V, ν', τ, τ^*) of the given sequence of PN spaces is a PN space, called the τ^*-product of $\left(V_j, \nu_j, \tau_j, \tau_j^*\right)_{j\in\mathbf{N}}$.

Notice that for the probabilistic norms ν and ν' of the τ- and τ^*-products, respectively, one has $\nu \leq \nu'$.

Next, we give the example of a τ-product of a sequence of PN spaces. To this end, recall that an equilateral PN space (V, ν, F) (see [7]) is not a Šerstnev space unless the d.f. F is constant on $\mathbb{R}$.

Example 4.3. Let $(\beta_n)_{n\in\mathbf{N}}$ be a sequence of strictly positive numbers such that $\sum_{n\in\mathbf{N}} \beta_n < +\infty$ and let $(V_n, \nu_n, \Pi_M)_{n\in\mathbf{N}}$ be a sequence of equilateral PN spaces, where, for all $n \in \mathbf{N}$, $\nu_n = \epsilon_{\beta_n}$. The convergence of the series $\sum_{n\in\mathbf{N}} \beta_n$ ensures that $\Pi_M^{(\infty)}$ is not identically zero. The Π_M-product is again an equilateral PN space. Notice also that, if τ is a continuous triangle function such that $\tau(\epsilon_a, \epsilon_b) = \epsilon_{a+b}$ for all real numbers $a, b \in [0, +\infty]$, then the τ-product (V, ν) is a PN space under τ and Π_M.

It suffices to recall that $\Pi_M \gg \tau$ for every triangle function τ.

Theorem 3.3 has only a partial analogue in the case of the τ-product of PN spaces.

Theorem 4.3. *The product topology of the sequence $\left(V_n, \nu_n, \tau_n, \tau_n^*\right)_{n\in\mathbf{N}}$ of PN spaces is weaker than the strong topology of their τ-product.*

Proof. As in the proof of Theorem 3.3 let $U_n(t_1, t_2, \ldots, t_n)$ be the same neighbourhood as in (3.3) and define $t := \min\{t_1, , \ldots, t_n\} > 0$. Let $q = (q_n)$ belong to $N_\theta(t)$; then, one has, for all $j \in \{1, 2, \ldots, n\}$,

$$1 - t_j \leq 1 - t < \nu_q(t) \leq \nu_j(q_j)(t) \leq \nu_j(q_j)(t_j),$$

so that $N_\theta(t) \subset \prod_{j\in\mathbf{N}} N_{\theta_j}(t_j) \subset U_n(t_1, t_2, \ldots, t_n)$. □

Considerations analogous to those presented in [17] could be made in order to explain (i) why the two topologies that are the object of Theorem 4.3 in general cannot coincide and (ii) the different probabilistic meaning, in the topologies of σ- and τ-products, of the statement "a point $p = (p_n)$ belongs to the neighbourhood $\mathbf{N}_\theta(t)$". Since they would just be a rephrasing of those in Alsina's paper, they will not be repeated here.

5. Final considerations

In [18] Alsina and Schweizer were successful in constructing the countable product of Probabilistic Metric(= PM) spaces in such a way that the following natural, and, hence, desirable, properties of the product space hold

(a) if all the PM spaces in the sequence $(S_n, \mathcal{F}_n, \tau)_{n\in\mathbf{N}}$ share the same triangle function τ, then the product space $(S, \mathcal{F})$ is a PM under the same triangle function τ;
(b) the strong topology of the product space coincides with the product topology.

One naturally wonders whether the construction given in [18] may be modified so as to extend it to the case of the countable product of PN, rather than PM, spaces. It will be shown below that the answer to this question must be negative. This is a consequence of the fact that *two* triangle functions, τ and τ^*, rather than only one, appear in the definition of a PN space.

Let $\left(V_j, \nu_j, \tau, \tau^*\right)_{j\in\mathbf{N}}$ be a sequence of PN spaces under the same two continuous triangle functions τ and τ^*.

The same concepts will be used as in [18]. As in that paper, let $\left(\beta_j\right)_{j\in\mathbf{N}}$ be a sequence of strictly positive numbers such that the series $\sum_{j\in\mathbf{N}} \beta_j$ converges. For every $j \in \mathbf{N}$, let φ_j belong to M_{β_j}, the set of strictly increasing functions from $[0, \beta_j]$ onto $[0, +\infty]$. The functions φ_j's are requested to be τ-superadditive, viz., for all F and G in Δ^+ and for every $j \in \mathbf{N}$,

$$\tau(F, G) \circ \varphi_j \geq \tau\left(F \circ \varphi_j, G \circ \varphi_j\right).$$

Define, for every $j \in \mathbf{N}$, $\widetilde{\nu}(p_j) := \nu(p_j) \circ \varphi_j$ and, for $p = \{p_j : j \in \mathbf{N}\}$ in $\prod_{j\in\mathbf{N}} V_j$,

$$\varphi_p := \tau^\infty\{\widetilde{\nu}(p_j)\}.$$

Then again, with the same proof as in [18], one shows that ν_p belongs to $\mathcal{D}^+$ and that the probabilistic norm thus defined satisfies property (N3) of a PN space. The problems arise with property (N4). In fact, in order to satisfy this latter property one has to require that each φ_j be τ^*-*subadditive*, namely, for all F and G in Δ^+ and for every $j \in \mathbf{N}$,

$$\tau^*(F, G) \circ \varphi_j \leq \tau^*\left(F \circ \varphi_j, G \circ \varphi_j\right).$$

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