A study of boundedness in probabilistic normed spaces

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ABSTRACT

It was shown in Lafuerza-Guillén, Rodríguez-Lallena and Sempi (1999) [8] that uniform boundedness in a Šerstnev PN space (V, ν, τ, τ^*), (named *boundedness* in the present setting) of a subset $A \subset V$ with respect to the strong topology is equivalent to the fact that the probabilistic radius R_A of A is an element of $\mathcal{D}^+$. Here we extend the equivalence just mentioned to a larger class of PN spaces, namely those PN spaces that are topological vector spaces (briefly TV spaces), but are not Šerstnev PN spaces.

We present a characterization of those PN spaces, whether they are TV spaces or not, in which the equivalence holds. Then, a characterization of the Archimedeanity of triangle functions τ^* of type $\tau_{T.L}$ is given. This work is a partial solution to a problem of comparing the concepts of distributional boundedness ($\mathcal{D}$-bounded in short) and that of boundedness in the sense of associated strong topology.

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1. Introduction

Probabilistic Normed (briefly PN) spaces were introduced by Šerstnev in a series of papers [1–4]. Then a new definition was proposed by Alsina, Schweizer and Sklar [5]. This new definition revived the study of PN spaces. The properties of these spaces were studied by several authors; here we shall mention [6–9] (but see also the survey papers [10,11]).

We shall consider the space Δ of distribution functions (=d.f.'s); a d.f. is an increasing (in the weak sense) function $F : \overline{\mathbb{R}} := [-\infty, +\infty] \to [0, 1]$ that is left-continuous and such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The d.f.'s belonging to the subset $\Delta^+ := \{F \in \Delta \mid F(0) = 0\}$ will be called *distance distribution functions*. Also the subset $\mathcal{D}^+ \subset \Delta^+$ of the *proper* distance distribution functions, i.e., those $F \in \Delta^+$ for which $\lim_{x \to +\infty} F(x) = 1$ will be needed. The space Δ can be metrized in several equivalent ways [12–14] in such a manner that the metric topology coincides with the topology of weak convergence for distribution functions. Here, we assume that Δ is metrized by the *Sibley metric* d_S, which is the metric denoted by d_L in [15]. The d.f.'s of constant random variables will be needed: for every $a \in \overline{\mathbb{R}}$,

$$\varepsilon_a(t) := \begin{cases} 0, & t \le a, \\ 1, & t > a. \end{cases}$$

For the reader's convenience, now we recall the definition of a PN space; the notation is essentially that fixed by the classical book by Schweizer and Sklar [15].

Definition 1.1. A *probabilistic normed* space is a quadruple (V, ν, τ, τ^*), where V is a real linear space, τ and τ^* are continuous triangle functions and the mapping $\nu : V \to \Delta^+$ satisfies, for all p and q in V, the conditions

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(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$ (θ is the null vector in V);

(N2) $\forall\, p \in V \quad \nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau\left(\nu_p, \nu_q\right)$;

(N4) $\forall\, \alpha \in [0, 1] \quad \nu_p \leq \tau^*\left(\nu_{\alpha p}, \nu_{(1-\alpha)p}\right)$.

The function ν is called the *probabilistic norm*. If (V, ν, τ, τ^*) satisfies the condition, weaker than (N1),

$$\nu_\theta = \varepsilon_0,$$

then it is called a *Probabilistic Pseudo-Normed* space (briefly, a PPN space). If $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^* then $(V, \nu, \tau_T, \tau_{T^*})$ is denoted by (V, ν, T) and is called a *Menger* PN space. A PN space is called a *Šerstnev space* if it satisfies (N1), (N3) and the following condition,

(Š) $\nu_{\alpha p}(x) = \nu_p\left(\frac{x}{|\alpha|}\right)$ for every $\alpha \in \overline{\mathbb{R}} \setminus \{0\}$ and for every $x > 0$.

    From the point of view of topological vector spaces the most interesting PN spaces are those that are not Šerstnev spaces. In these cases vector addition is still continuous (provided the triangle function is determined by a continuous t-norm), while scalar multiplication, in general, is not continuous with respect to the strong topology (see [16]).

    A set A in the PN space (V, ν, τ, τ^*) is said to be $\mathcal{D}$-*bounded* if its *probabilistic radius*, R_A belongs to $\mathcal{D}^+$. The probabilistic radius of A is defined by

$$R_A(x) := \begin{cases} \ell^- \inf\{\nu_p(x) : p \in A\}, & x \in [0, +\infty[, \\ 1, & x = +\infty. \end{cases}$$

    Of course, if V is a normed space under the norm $\|\cdot\|$, then the set A may be bounded when regarded as a subset of the normed space $(V, \|\cdot\|)$, but the two notions need not coincide (see [8]). The aim of this paper is to investigate the cases in which the notions of $\mathcal{D}$-boundedness and boundedness coincide. The papers [17–19] on the relationship between the two types of boundedness ought also to be kept in mind.

    The notion of t-norm will be used in the sequel; its definition may be found in [20,21,15]. Briefly a t-norm is any binary operation T on $[0, 1]$ that is commutative, associative, increasing in each place and such that $T(a, 1) = a$ for every $a \in [0, 1]$. A t-conorm T^* is associated with every t-norm T; it is defined by $T^*(x, y) := 1 - T(1 - x, 1 - y)$.

    Important examples of t-norms and of their t-conorms are: $M(x, y) = \min\{x, y\}$, $\Pi(x, y) = xy$ and $M^*(x, y) = \max\{x, y\}$, $\Pi^*(x, y) = x + y - xy$. We recall the following inequalities

$$T \leq M < M^* \leq T^*,$$

which hold for every t-norm T; in particular

$$W < \Pi < M < M^* < \Pi^* < W^*.$$

Definition 1.2. A PN space (V, ν, τ, τ^*) will be said to be *strict* if $\nu(V) \subseteq \mathcal{D}^+$, or, equivalently, if ν_p belongs to $\mathcal{D}^+$ for every $p \in V$.

Definition 1.3. A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,

(1) $\tau(F, \varepsilon_0) = F$,

(2) $\tau(F, G) = \tau(G, F)$,

(3) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H, G \leq K$,

(4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

    Particular and relevant triangle functions are the functions τ_T, τ_{T^*} and those of the form Π_T which, for every continuous t-norm T, and every $x > 0$, are defined by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\},$$
$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(u), G(v)) \mid u + v = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

See [22,23] for a recent study of triangle functions.

Definition 1.4. A subset A of a PN space which is a TV space is said to be *bounded* if for every $m \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that

$$A \subset k\mathcal{N}_\theta(1/m).$$

Definition 1.5. The subset A is said to be *totally bounded* if for every $m \in \mathbf{N}$ there is a finite set $A_1 \subseteq A$ such that

$$A \subseteq \bigcup_{p \in A_1} (p + \mathcal{N}_\theta (1/m)^{[k]}).$$

Furthermore, whenever a metric space has the structure of a TV space, another type of boundedness is relevant. A subset A of a TV space is *topologically bounded* if for every sequence $(\alpha_n) \subset \mathbb{R}$ with $\lim_{n \to +\infty}(\alpha_n) = 0$ and for every sequence $(p_n) \subset A$, one has $\lim_{n \to +\infty}(\alpha_n p_n) = \theta$.

Definition 1.6. Let $\mathcal{L}$ be the set of all binary operations L on $\mathbb{R}^+$ satisfying the following conditions:

(i) $\operatorname{Ran} L = \mathbb{R}^+$
(ii) L is non-decreasing in each place;
(iii) L is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ except, at most, in $(0, \infty)$ and $(\infty, 0)$.

Theorem 1.1. *Let T be a left-continuous t-norm, and let L belong to $\mathcal{L}$ and satisfy the following conditions:*

(a) *L is commutative;*
(b) *L is associative;*
(c) *$u_1 < u_2$ and $v_1 < v_2$ imply $L(u_1, v_1) < L(u_2, v_2)$;*
(d) *$L(x, 0) = x$;*

then the function $\tau_{T,L}$ is a triangle function.

Lemma 1.1. *Let T be a continuous t-norm and let L satisfy the condition of Theorem 1.1 (a); then $\tau_{T,L}(\varepsilon_a, \varepsilon_b) = \varepsilon_{L(a,b)}$ for all a and b in $[0, +\infty[$.*

Definition 1.7. A copula is a function $C : [0, 1]^2 \to [0, 1]$ that satisfies the following conditions:

(C1) for every $t \in [0, 1]$, $C(0, t) = C(t, 0) = 0$ and $C(1, t) = C(t, 1) = t$;
(C2) C is 2-increasing, i.e., for all s, s', t and t' in $[0, 1]$, with $s \le s'$ and $t \le t'$,

$$C(s', t') - C(s', t) - C(s, t') + C(s, t) \ge 0.$$

It follows from Definition 1.7 that every copula C is increasing in each place. Moreover for any copula C one has $W \le C \le M$.

Definition 1.8. Let φ be a continuous, strictly decreasing function from $\mathbb{I} = [0, 1]$ to $[0, +\infty]$ such that $\varphi(1) = 0$. The *pseudo-inverse* of φ is the function $\varphi^{[-1]}$ with $\operatorname{Dom} \varphi^{[-1]} = [0, +\infty]$ and $\operatorname{Ran} \varphi^{[-1]} = \mathbb{I}$ defined by

$$\varphi^{[-1]}(t) := \begin{cases} \varphi^{-1}(t), & 0 \le t \le \varphi(0), \\ 0, & \varphi(0) \le t \le +\infty. \end{cases} \tag{1.1}$$

The following results are well known and can be found in [15] or in [24].

Lemma 1.2. *Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, +\infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined by (1.1). Let C be the function from $\mathbb{I}^2$ to $\mathbb{I}$ defined by*

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \tag{1.2}$$

Then C satisfies the boundary conditions (C1). Moreover C is 2-increasing if, and only if, for all $v \in \mathbb{I}$,

$$C(u_2, v) - C(u_1, v) \le u_2 - u_1, \tag{1.3}$$

whenever $u_1 \le u_2$.

Theorem 1.2. *Let φ be a continuous, strictly decreasing function from $\mathbb{I}$ to $[0, +\infty]$ such that $\varphi(1) = 0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined by (1.1). Then the function C from $\mathbb{I}^2$ to $\mathbb{I}$ defined by (1.2) is a copula if, and only if, φ is convex.*

Copulas of the form (1.2) are called *Archimedean*. An Archimedean copula is associative and a *t*-norm. One owes the following result to Alsina, Schweizer and Sklar [16].

Theorem 1.3. *Every* PN *space* (V, ν, τ, τ^*), *when it is endowed with the strong topology induced by the probabilistic norm* ν, *is a topological vector space if, and only if, for every* $p \in V$ *the map from* $\mathbb{R}$ *into* V *defined by*

$$\lambda \mapsto \lambda p \tag{1.4}$$

is continuous, i.e., for every $\eta > 0$, *there exists a number* $\delta > 0$ *such that*

$$d_S(\nu_{\alpha' p - \alpha p}, \varepsilon_0) < \eta$$

whenever $|\alpha' - \alpha| < \delta$, *or, equivalently, such that* $d_S(\nu_{\beta p}, \varepsilon_0) < \eta$ *whenever* $|\beta| < \delta$.

A triangle function τ is said to be *Archimedean* if ε_0 and $\varepsilon_{+\infty}$ are the only solutions in Δ^+ of the functional equation $\tau(F, F) = F$. Any such a solution is said to be an *idempotent* of the triangle function τ; thus a triangle function τ is Archimedean if, and only if, ε_0 and $\varepsilon_{+\infty}$ are the only idempotents of τ. Thus, for instance, the triangle functions τ_W and τ_Π are Archimedean.

It was proved in [16, Theorem 4], that, if the triangle function τ^* is Archimedean, then the mapping (1.4) is continuous and, as a consequence of Theorem 1.3, the PN space (V, ν, τ, τ^*) is a TV space. It was shown in [8] that uniform boundedness in a Šerstnev PN space (V, ν, τ, τ^*), (named *boundedness* in the present setting) of a subset $A \subset V$ with respect to the strong topology is equivalent to the fact that the probabilistic radius R_A of A is an element of $\mathcal{D}^+$. Here we extend this equivalence to a larger class of PN spaces, namely those PN spaces that are topological vector spaces (briefly TV spaces), but are not Šerstnev PN spaces.

Section 2 presents a characterization of those PN spaces, whether they are TV spaces or not, in which the equivalence holds. In Section 3, a characterization of the Archimedeanity of triangle functions τ^* of the type $\tau_{T,L}$ is given. This work is a partial solution to a problem of comparing the concepts of distributional boundedness ($\mathcal{D}$-bounded in short) and that of boundedness in the sense of associated strong topology.

2. Main results (I)—A particular class of PN spaces

Definition 2.1. The PN space (V, ν, τ, τ^*) is said to satisfy the *DI-condition*[1] if the probabilistic norm ν is such that, for all $\alpha \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R}$ and $p \in V$,

$$\nu_{\alpha p}(x) = \nu_p(\varphi(\alpha, x)),$$

where $\varphi : \mathbb{R} \times [0, +\infty[\to [0, +\infty[$ satisfies

$$\lim_{x \to +\infty} \varphi(\alpha, x) = +\infty \quad \text{and} \quad \lim_{\alpha \to 0} \varphi(\alpha, x) = +\infty.$$

Example 2.1. Let $(V, \| \cdot \|)$ be a normed space. For $\beta \in]0, 1[$, define $\nu : V \to \Delta^+$ by

$$\nu_p(x) := \begin{cases} 0, & x \leq 0, \\ (1 - \beta\varepsilon_0(\|p\|)) \dfrac{\ln(1 + x)}{\ln(1 + x) + \|p\|}, & x \in]0, +\infty[, \\ 1, & x = +\infty. \end{cases}$$

Below, we shall prove that $(V, \nu, \tau_\Pi, \tau_M)$

(1) is a PN space;
(2) is neither a Šerstnev space nor a TV space nor a strict PN space;
(3) satisfies the *DI*-condition, with

$$\varphi(\alpha, x) = (x + 1)^{1/|\alpha|} - 1.$$

(1) $(V, \nu, \tau_\Pi, \tau_M)$ *is a* PN *space.* It is easily verified that (N1) and (N2) hold.
For all p and q in V, for all s and t in R^+, one has

$$\frac{\|p + q\|}{\ln(1 + s + t)} \leq \frac{\|p\| + \|q\|}{\ln(1 + s + t)} \leq \frac{\|p\|}{\ln(1 + s)} + \frac{\|q\|}{\ln(1 + t)}.$$

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[1] The acronym *DI* stands for the expression *Double Infinity*.

Notice that the factor $(1 - \beta \varepsilon_0(\|p\|))$ equals 1 if, and only if, $p = \theta$; otherwise, it equals $1 - \beta$. Then, if both p and q differ from θ,

$$\nu_{p+q}(s+t) = (1-\beta)\frac{\ln(1+s+t)}{\ln(1+s+t)+\|p+q\|} = (1-\beta)\frac{1}{1+\frac{\|p+q\|}{\ln(1+s+t)}}$$

$$\geq (1-\beta)\frac{1}{1+\frac{\|p\|}{\ln(1+s)}+\frac{\|q\|}{\ln(1+t)}}$$

$$\geq (1-\beta)^2 \frac{1}{1+\frac{\|p\|}{\ln(1+s)}+\frac{\|q\|}{\ln(1+t)}+\frac{\|p\|\|q\|}{\ln(1+s)\ln(1+t)}}$$

$$= \frac{(1-\beta)}{1+\frac{\|p\|}{\ln(1+s)}} \cdot \frac{(1-\beta)}{1+\frac{\|q\|}{\ln(1+t)}}$$

$$= (1-\beta)\frac{\ln(1+s)}{\ln(1+s)+\|p\|} \cdot (1-\beta)\frac{\ln(1+t)}{\ln(1+t)+\|q\|} = \nu_p(s) \cdot \nu_q(t).$$

As a consequence (N3) holds, i.e., for all p and q in V, and for every $x \in R^+$,

$$\nu_{p+q}(x) \geq \tau_\pi(\nu_p, \nu_q)(x).$$

This latter inequality is trivially true if at least one of p and q equals θ.
For all s and t in R^+, one has

$$1 - \frac{\ln(1+s)}{\ln(1+s+t)} = \frac{\ln(1+s+t)-\ln(1+s)}{\ln(1+s+t)} = \frac{\ln\frac{s+t+1}{s+1}}{\ln(1+s+t)}$$

$$= \frac{\ln(1+\frac{t}{s+1})}{\ln(1+s+t)} \leq \frac{\ln(1+t)}{\ln(1+s+t)}.$$

Therefore, because of Lemma 2 in [7],

$$\nu_{\left(1-\frac{\ln(1+s)}{\ln(1+s+t)}\right)p} \geq \nu_{\frac{\ln(1+t)}{\ln(1+s+t)}p}.$$

For all $\alpha \in]0, 1[$ and $x \in R^+$, there exist s_0, t_0 in R^+ such that $s_0 + t_0 = x$ and

$$\alpha = \frac{\ln(1+s_0)}{\ln(1+s_0+t_0)};$$

in fact $s_0 = (1+x)^\alpha - 1$. Whence, if $p \neq \theta$,

$$\tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})(x) = \sup_{s+t=x} \min(\nu_{\alpha p}(s), \nu_{(1-\alpha)p}(t))$$

$$= \sup_{s+t=x} \min\left(\nu_{\left(\frac{\ln(1+s_0)}{\ln(1+s_0+t_0)}\right)p}(s), \nu_{\left(1-\frac{\ln(1+s_0)}{\ln(1+s_0+t_0)}\right)p}(t)\right)$$

$$\geq \min\left(\nu_{\left(\frac{\ln(1+s_0)}{\ln(1+s_0+t_0)}\right)p}(s_0), \nu_{\left(1-\frac{\ln(1+s_0)}{\ln(1+s_0+t_0)}\right)p}(t_0)\right)$$

$$\geq \min\left(\nu_{\left(\frac{\ln(1+s_0)}{\ln(1+s_0+t_0)}\right)p}(s_0), \nu_{\left(\frac{\ln(t_0+1)}{\ln(1+s_0+t_0)}\right)p}(t_0)\right)$$

$$= \min\left(\frac{(1-\beta)\ln(1+s_0+t_0)}{\ln(1+s_0+t_0)+\|p\|}, \frac{(1-\beta)\ln(1+s_0+t_0)}{\ln(1+s_0+t_0)+\|p\|}\right)$$

$$= \min(\nu_p(x), \nu_p(x)) = \nu_p(x).$$

Therefore (N4) holds for $p \neq \theta$; since (N4) is obvious for $p = \theta$, it holds for every $p \in V$ and for every $\alpha \in]0, 1[$. For $\alpha = 0$ or $\alpha = 1$, (N4) obviously holds, i.e.,

$$\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}).$$

(2) $(V, \nu, \tau_\Pi, \tau_M)$ is neither a Šerstnev space nor a TV space nor a strict PN space. A straightforward calculation shows that $\nu_{\alpha p}(x) \neq \nu_p(x/|\alpha|)$ so that $(V, \nu, \tau_\pi, \tau_M)$ is not a Šerstnev space.

We recall that a sufficient condition for a PN space (V, ν, τ, τ^*) to be a TV space is that τ^* be an Archimedean triangle function. Now, τ_M is not Archimedean so that all one needs to prove is that the scalar multiplication $\lambda \mapsto \lambda p$ is not

continuous. For every sequence (λ_n) of real numbers that converges to 0 as n tends to $+\infty$, and for every p in V one has $\lambda_n p \not\to \theta$ in the strong topology of V. Now

$$\lim_{n\to\infty} \nu_{\lambda_n p}(x) = 1 - \beta < 1,$$

or equivalently,

$$\lim_{n\to\infty} \nu_{\lambda_n p} \neq \varepsilon_0.$$

Also since $\lim_{x\to\infty} \nu_p(x) = 1 - \beta < 1$, the distance d.f. ν_p is in $\Delta^+ \setminus \mathcal{D}^+$, so that $(V, \nu, \tau_\pi, \tau_M)$ is not a strict PN space.

(3) $(V, \nu, \tau_\Pi, \tau_M)$ satisfies the DI-condition. For all $x \in \mathbb{R}^+$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in V$, one has, for $p \neq \theta$,

$$\nu_{\lambda p}(x) = \frac{(1-\beta)\ln(1+x)}{\ln(1+x) + |\lambda|\|p\|} = \frac{(1-\beta)\frac{1}{|\lambda|}\ln(1+x)}{\frac{1}{|\lambda|}\ln(1+x) + \|p\|}$$

$$= \frac{(1-\beta)\ln(1+x)^{\frac{1}{|\lambda|}}}{\ln(1+x)^{\frac{1}{|\lambda|}} + \|p\|} = \nu_p((1+x)^{1/|\lambda|} - 1).$$

Let

$$\varphi(\lambda, x) = (1+x)^{1/|\lambda|} - 1.$$

It is easily checked that $(\lambda, x) \mapsto \varphi(\lambda, x)$ satisfies

$$\lim_{x\to+\infty} \varphi(\lambda, x) = +\infty \quad \text{and} \quad \lim_{\lambda\to 0} \varphi(\lambda, x) = +\infty.$$

Example 2.2. Let (V, ν, τ, τ^*) be an α-Šerstnev space (see [6]). It is easy to show that (V, ν, τ, τ^*) satisfies the DI-condition, where

$$\varphi(\lambda, x) = \frac{x}{|\lambda|^\alpha}.$$

Example 2.3. The quadruple (V, ν, Π_Π, Π_M) where V is a normed linear space and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ defined via $\nu_p(x) := e^{-\|p\|/x}$ for $x > 0$, is a PN space as that of Example 2.2 with $\alpha = 1$.

Notice that this probabilistic norm is an increasing function of x and a decreasing function of $\|p\|$.

Properties (N1) and (N2) are obvious.

Property (N3) is obviously true since

$$e^{-\|p+q\|/x} \geq e^{-\|p\|/x} \cdot e^{-\|q\|/x}.$$

As for property (N4), since $\|p\| \geq \lambda\|p\|$ for every $p \in V$ and for every $\lambda \in [0, 1]$, one has for every $\alpha \in]0, 1[$

$$e^{-\|p\|/x} \leq \min\{e^{-\|\alpha p\|/x}, e^{-\|(1-\alpha)p\|/x}\}.$$

Moreover, this is a Šerstnev PN space, as is immediately seen.

It is also a topological vector space. As a matter of fact, for every $p \neq \theta$, and for every arbitrary sequence (α_n) with $\alpha_n \neq 0$, $(n \in \mathbf{N})$ such that $\alpha_n \to 0$ as n tends to $+\infty$, one has

$$\nu_{\alpha_n p}(x) = e^{-|\alpha_n|\|p\|/x} \xrightarrow[n\to+\infty]{} 1.$$

Moreover there is a neighbourhood of the origin θ that is topologically bounded: For every sequence (α_n) with $\alpha_n \neq 0$, $(n \in \mathbf{N})$ such that $\alpha_n \to 0$ as n tends to $+\infty$, and for every sequence (p_n) of elements of $\mathcal{N}_\theta(t)$ one has

$$\nu_{\alpha_n p_n}(t) = e^{-|\alpha_n|\|p_n\|/t} \xrightarrow[n\to+\infty]{} 1.$$

Theorem 2.1. Let (V, ν, τ, τ^*) be a PN space that satisfies the DI-condition. Then for a subset $A \subset V$ the following statements are equivalent:

(a) A is $\mathcal{D}$-bounded.
(b) A is bounded, namely, for every $n \in \mathbf{N}$ and for every $p \in A$, there is $k \in \mathbf{N}$ such that $\nu_{p/k}(1/n) > 1 - 1/n$.
(c) A is topologically bounded.

Proof. (a) $\Longrightarrow$ (b). Let A any $\mathcal{D}$-bounded subset of V. By definition of $\mathcal{D}$-boundedness, the probabilistic radius R_A of A (see [8]) is a distance d.f. such that $\lim_{x \to +\infty} R_A(x) = 1$, Therefore, for every $n \in \mathbf{N}$, there exists $x_n > 0$ such that $R_A(x_n) > 1 - 1/n$. Thus, for every $p \in A$,

$$\nu_p(x_n) \geq R_A(x_n) > 1 - 1/n.$$

Since, in view of the *DI*-condition,

$$\lim_{\lambda \to 0} \varphi(\lambda, 1/n) = +\infty,$$

for every $n \in \mathbf{N}$, there exists $\lambda' \in \mathbb{R}$ such that $\varphi(\lambda', 1/n) \geq x_n$. Therefore,

$$\nu_{\lambda'p}(1/n) = \nu_p(\varphi(\lambda', 1/n)) \geq \nu_p(x_n) \geq R_A(x_n) > 1 - 1/n.$$

As a consequence letting $k = [1/\lambda']$, where $[t]$ denotes the integral part of t, one has

$$\nu_{p/k}(1/n) > 1 - 1/n,$$

namely, A is bounded.

(b) $\Longrightarrow$ (a) Let A be a bounded subset of V and consider the neighbourhood of θ, $N_\theta(1/n)$. Then there exists $\lambda_0 \in \mathbb{R}$ such that for every $p \in V$, $p = \lambda_0 q$ for some $q \in N_\theta(1/n)$.

Because of the *DI*-condition, $\lim_{x \to +\infty} \varphi(\lambda_0, x) = +\infty$, for every $n \in \mathbf{N}$; then, there exists $x_0 > 0$ such that, $\varphi(\lambda_0, x_0) \geq 1$. Then, for $x \geq x_0$,

$$\nu_p(x) = \nu_{\lambda_0 q}(x) \geq \nu_{\lambda_0 q}(x_0) = \nu_q(\varphi(\lambda_0, x_0)) \geq \nu_q(1) \geq \nu_q(1/n) > 1 - 1/n,$$

so that

$$R_A(x) \geq 1 - \frac{1}{n},$$

i.e., R_A is in $\mathcal{D}^+$.

(a) $\Longrightarrow$ (c). Let A any $\mathcal{D}$-bounded subset of V. One has, as above,

$$\nu_{\alpha_n p_n}(x_n) = \nu_{p_n}(\varphi(\alpha_n, x_n)) \geq R_A(x_n) > 1 - \frac{1}{n},$$

which implies

$$\alpha_n p_n \xrightarrow[n \to +\infty]{} \theta.$$

(c) $\Longrightarrow$ (a). Let A be a subset of V which is not $\mathcal{D}$-bounded. Then

$$\lim_{x \to +\infty} R_A(x) = \gamma < 1.$$

By definition of R_A, for every $n \in \mathbf{N}$ there is $p_n \in A$ such that, for every $x > 0$,

$$\nu_{p_n}(x) < \frac{1 + \gamma}{2} < 1.$$

Then for every $x > 0$,

$$\nu_{\alpha_n p_n}(x) = \nu_{p_n}(\varphi(\alpha_n, x)) < \frac{1 + \gamma}{2} < 1,$$

which shows that $\nu_{\alpha_n p_n}$ does not tend to ε_0, even if it has a weak limit, viz., $(\alpha_n p_n)$ does not tend to θ in the strong topology; in other words, A is not topologically bounded. $\square$

The next example exhibits a PN space (V, ν, τ, τ^*) that is a TV space but in which the triangle function τ^* is not Archimedean.

Example 2.4. Let (V, ν, τ, τ^*) be a PN space in which the triangle function τ^* is not Archimedean. Let f and g be functions satisfying the following conditions:

(i) $f : [0, +\infty] \to \mathbb{I}$ is continuous, decreasing and $f(0) = 1$;
(ii) $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{I}$ is continuous in either place, increasing in the first place and decreasing in the second place with $g(x, 0) = 1$.

Then the quadruple (V, ν, τ, τ^*) where the probabilistic norm ν is defined via

$$\nu_p(x) := f(\|p\|) \cdot g(x, \|p\|),$$

is a TV space.

Notice that every PN space (V, ν, τ, τ^*) in which $\tau^* = \tau_M$ and such that $\nu(V) \subseteq \mathcal{D}^+$ is a TV space since τ_M is Archimedean on the subset $\mathcal{D}^+$ of Δ^+.

The condition $\nu(V) \subseteq \mathcal{D}^+$ is not necessary to obtain a TV space as shown in [25, Theorem 9] and in the following example.

Example 2.5. With the same assumptions of the Example 2.4 one has that the quadruple (V, ν, Π_Π, Π_M) where

$$\nu_p(x) = \frac{1}{1 + \|p\|} \cdot \frac{x}{x + \|p\|}$$

is a PN space that is a TV space, which is not Šerstnev and is not strict.

Example 2.6. The quadruple (V, ν, Π_Π, Π_M) where the probabilistic norm ν is defined via

$$\nu_p(x) := \exp\left(-\frac{\|p\|x}{\|p\|x + 1}\right),$$

is a PN space that is neither Šerstnev nor strict, but it is a TV space.

Example 2.7 ([16]). For $p \in \mathbb{R}$ define ν by setting $\nu_0 = \varepsilon_0$, and

$$\nu_p := \frac{1}{\|p\| + 2}\varepsilon_0 + \frac{\|p\| + 1}{\|p\| + 2}\varepsilon_\infty,$$

for $p \neq 0$, it is straightforward to show that $(\mathbb{R}, \nu, \tau_W, \tau_M)$ is a PN space, and that, for any $p \neq 0$ and any sequence (α_n) converging to 0, the sequence $(\nu_{\alpha_n p})$ converges weakly, not to ε_0, but to $(\varepsilon_0 + \varepsilon_\infty)/2$. Thus $\nu_{\alpha p}$ is not continuous in its first place, namely, scalar multiplication is not a continuous mapping from $\mathbb{R}$ into V. Thus $(\mathbb{R}, \nu, \tau_W, \tau_M)$ is not a TV space.

Example 2.8. Let $(V, \|\cdot\|)$ be a normed space and, for $\alpha \in]0, 1[$, let $\nu : V \to \Delta^+$ be given by

$$\nu_p(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{\ln(1+x)}{\ln(1+x) + \|p\|}, & x \in]0, +\infty[, \ \|p\| < 1 \\ \dfrac{a\ln(1+x)}{\ln(1+x) + \|p\|}, & x \in]0, +\infty[, \ \|p\| \geq 1, \\ 1, & x = +\infty. \end{cases}$$

Then

(1) $(V, \nu, \tau_\pi, \tau_M)$ is a PN space satisfying the *DI*-condition with

$$\varphi(\lambda, x) = (1 + x)^{\frac{1}{\|\lambda\|}} - 1;$$

(2) $(V, \nu, \tau_\pi, \tau_M)$ is a TV PN space;
(3) the subset $A = \{p : \|p\| < 1\}$ is both $\mathcal{D}$-bounded and bounded.

Only property (2) needs to be checked. For every sequence (λ_n) of real numbers that converges to 0 as n tends to $+\infty$, and for every $p \in V$, one has $\lambda_n p \to \theta$ in the strong topology of V; in fact, for every $x \in \mathbb{R}^+$, $\lim_{n\to\infty} \nu_{\lambda_n p}(x) = 1$, namely, $\lim_{n\to\infty} \nu_{\lambda_n p} = \varepsilon_0$.

3. II—A class of PN spaces that are TV spaces

Theorem 1.1 implies that $L \geq \max$; one also knows that $\tau_{T,L} \leq \Pi_T$.

Lemma 3.1. *Let $L \in \mathcal{L}$ satisfy the assumptions of Theorem 1.1; then $L = \max$, if, and only if, $L(a, a) = a$ for every $a \in]0, +\infty[$.*

The following theorem studies whether certain classes of PN spaces are TV spaces.

Theorem 3.1. *Let T be an Archimedean t-norm and assume that $L \in \mathcal{L}$ satisfies the assumptions of Theorem 1.1, then the following statements are equivalent:*

(a) *$\tau_{T,L}$ is a triangle function having no nontrivial idempotents in Δ^+;*
(b) *L satisfies the condition $L(x, x) > x$ for every $x \in]0, +\infty[$.*

Proof. Recall that $\tau_{T,L}$ is a continuous triangle function (see [15, Theorems 7.2.4 and 7.2.8]).

(a) $\Longrightarrow$ (b) A continuous triangle function with no nontrivial idempotents in Δ^+ is Archimedean (see [15, Lemma 14.1.3]). Then $\tau_{T,L}$ is an Archimedean triangle function, and, for every function $F \notin \{\varepsilon_0, \varepsilon_\infty\}$, one has $\tau_{T,L}(F, F) \neq F$. By Lemma 1.1, $\varepsilon_x \neq \tau_{T,L}(\varepsilon_x, \varepsilon_x) = \varepsilon_{L(x,x)}$ for every $x \in]0, +\infty[$. Therefore $L(x, x) \neq x$ and from Lemma 3.1 one has $L(x, x) > x$ for every $x \in]0, +\infty[$, which is the assertion.

(b) $\Longrightarrow$ (a) Let $F \in \Delta^+$ be different from ε_a for every $a \in [0, +\infty]$. Then there exists $x_0 \in]0, +\infty[$ such that $F(x_0)$ is in $]0, 1[$. Since T is Archimedean, the following holds

$$\tau_{T,L}(F, F)(x_0) = \sup\{T(F(u), F(v)) : L(u, v) = x_0\} \leq T(F(x_0), F(x_0)) < F(x_0).$$

Therefore F is not an idempotent element of $\tau_{T,L}$. Now let x be in $]0, +\infty[$. By Lemma 1.1,

$$\tau_{T,L}(\varepsilon_x, \varepsilon_x) = \varepsilon_{L(x,x)} \neq \varepsilon_x,$$

whence $\tau_{T,L}$ has no nontrivial idempotents elements. This concludes the proof. $\square$

Example 3.1. If C is an Archimedean copula, then the triangle function $\tau_{C,L}$ where $L \in \mathcal{L}$ satisfies the assumptions of Theorem 1.1 is Archimedean.

Theorem 3.2. *Let T_1 and T_2 be t-norms such that $T_1 \leq T_2$; then $\tau_{T_1,L}$ and $\tau_{T_2,L}$ have the same idempotents.*

Proof. If F is in Δ^+ and $\tau_{T_1,L}$ admits F as idempotent, then

$$F = \tau_{T_1,L}(F, F) \leq \tau_{T_2,L}(F, F) \leq F,$$

so that $\tau_{T_2,L}(F, F) = F$. $\square$

Corollary 3.1. *Let T be a t-norm such that $T \leq \Pi$, then the family of all PN spaces of the type $(V, \nu, \tau_{T,L}, \tau_{\Pi,L})$ are TV spaces.*

Corollary 3.2. *Let T be a t-norm such that $T \leq W$, then the family of all PN spaces of the type $(V, \nu, \tau_{T,L}, \tau_{W,L})$ are TV spaces.*

Theorem 3.3. *Let T and C be a continuous t-norm and an Archimedean copula, respectively; if, for every $(u, v) \in [0, 1]^2$, $T(u, v) \leq C(u, v)$, then every PN space of the form $(V, \nu, \tau_{T,L}, \tau_{C,L})$ is a TV space.*

The above result provides many examples of Archimedean triangle functions. It suffices to take

$$L = K_\alpha(x, y) = (x^\alpha + y^\alpha)^{1/\alpha} \quad (x, y \in \mathbb{R}^+; \alpha \geq 1).$$

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