Sobolev orthogonal polynomials: balance and asymptotics

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Abstract

Let μ_0 and μ_1 be measures supported on an unbounded interval and S_{n,λ_n} the extremal varying Sobolev polynomial which minimizes

$$\langle P,P\rangle_{\lambda_n} = \int P^2\,d\mu_0 + \lambda_n \int P'^2\,d\mu_1, \quad \lambda_n > 0$$

in the class of all monic polynomials of degree n. The goal of this paper is twofold. On one hand, we discuss how to balance both terms of this inner product, that is, how to choose a sequence (λ_n) such that both measures μ_0 and μ_1 play a role in the asymptotics of (S_{n,λ_n}). On the other, we apply such ideas to the case when both μ_0 and μ_1 are Freud weights. Asymptotics for the corresponding S_{n,λ_n} are computed, illustrating the accuracy of the choice of λ_n.

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1 Introduction

One of the central problems in the analytic theory of orthogonal polynomials is the study of their asymptotic behavior. In this paper we are concerned with the asymptotic properties of Sobolev orthogonal polynomials, that is polynomials orthogonal with respect to an inner product involving derivatives. In this sense, given μ_0 and μ_1 finite Borel measures supported on an interval $I \subset \mathbb{R}$ and $\lambda > 0$ we consider the Sobolev inner product

$$\langle P, Q\rangle_\lambda = \int P\,Q\,d\mu_0 + \lambda \int P'\,Q'\,d\mu_1 \tag{1}$$

in the space of all polynomials with real coefficients.

We denote by P_{n,μ_0}, P_{n,μ_1} and $S_{n,\lambda}$ the corresponding monic polynomials orthogonal with respect to μ_0, μ_1 and $\langle\cdot,\cdot\rangle_\lambda$, respectively.

Let μ_0 and μ_1 be measures compactly supported on $\mathbb{R}$. Whether (μ_0, μ_1) is a coherent pair, which means that there exist nonzero constants σ_n such that the corresponding monic polynomials satisfy for each n

$$P_{n,\mu_1} = \frac{P'_{n+1,\mu_0}}{n+1} + \sigma_n \frac{P'_{n,\mu_0}}{n}$$

or, if μ_0 and μ_1 fulfill much milder conditions, i.e., they belong to the well known Szegő class, it has been established (see [9] and [8]) that the ratio asymptotics

$$\lim_{n\to\infty} \frac{S_{n,\lambda}(z)}{P_{n,\mu_1}(z)} = \frac{2}{\varphi'(z)}$$

holds uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$, where $\varphi(z) = z + \sqrt{z^2-1}$ with $\sqrt{z^2-1} > 0$ when $z > 1$. In other words, the measure μ_0 does not appear explicitly within the asymptotic expression.

Nevertheless, a closer look at the inner product (1) explains the "dominance" of the measure μ_1 in the asymptotics: the derivative makes the leading coefficient of the polynomials in the second integral of (1) be multiplied by the degree of the polynomial. Thus, if we want both measures to have an impact on the behavior of the polynomials for $n \to \infty$, it seems natural to "balance" the inner product, that is, to compensate both integrals by introducing a varying parameter λ_n.

In a general framework, we consider the varying Sobolev inner product $\langle P, Q\rangle_{\lambda_n}$. We denote by S_{n,λ_n} the monic polynomial which minimizes the expression $\langle Q_n, Q_n\rangle_{\lambda_n}$ in the class of all monic polynomials Q_n of degree n.

Concerning the choice of the varying parameter λ_n, it is interesting to write the expression of the Sobolev inner product in terms of monic polynomials, that is

$$\langle Q_n, Q_n\rangle_{\lambda_n} = \int (Q_n)^2\, d\mu_0 + \lambda_n\, n^2 \int \left(\frac{Q_n'}{n}\right)^2 d\mu_1. \tag{2}$$

In this expression each integral in the right hand side is bounded from below by $\int P_{n,\mu_0}^2\, d\mu_0$ and $\int P_{n-1,\mu_1}^2\, d\mu_1$, respectively, as long as Q_n is a monic polynomial of degree n.

If the measures μ_0 and μ_1 are supported on the same bounded interval where they satisfy the Szegő condition, then $\int P_{n,\mu_0}^2\, d\mu_0$ behaves as $\int P_{n-1,\mu_1}^2\, d\mu_1$, when $n \to \infty$. More precisely, the ratio $\dfrac{\int P_{n,\mu_0}^2\, d\mu_0}{\int P_{n-1,\mu_1}^2\, d\mu_1}$ has a limit. Therefore, in order to balance both terms in (2) it is natural to keep $\lambda_n\, n^2$ bounded.

In fact, it was proved in [1] that if (λ_n) is a decreasing sequence of positive real numbers such that $\lim_n \lambda_n\, n^2 \in (0, +\infty)$ then

$$\lim_{n\to\infty} \frac{S_{n,\lambda_n}(z)}{R_n(z)} = 1$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$, where (R_n) is the sequence of monic polynomials orthogonal with respect to a measure constructed as a certain combination of the measures μ_0 and μ_1.

Let us consider now that the measures μ_0 and μ_1 are supported on an unbounded interval. There are many asymptotic results (strong asymptotics) for the monic polynomials $S_{n,\lambda}$ orthogonal with respect to the inner product (1) for a fixed λ; see for instance [2] and [11] for coherent pairs, [3] and [4] for Freud weights and, more recently, the survey [7]. But as far as we know, nothing has been said about asymptotics in the balanced case. In this sense, the first question that should be answered is: what is the appropriate choice for the sequence (λ_n)? We understand by this a sequence of parameters for which polynomials S_{n,λ_n} exhibit a nontrivial asymptotic behavior, depending on both measures μ_0 and μ_1. One of the goals of this paper is to raise that $\lambda_n = n^{-2}$ is not, in general, the right choice when the support of μ_0 and μ_1 is unbounded.

The structure of the paper is as follows. In Section 2, we use heuristic arguments, based on potential theory, about the "size" of λ_n in order to achieve an appropriate "balancing". In this sense, the Mhaskar–Rakhmanov–Saff numbers turn out to be a powerful tool. On account of the above results, in Section 3 we obtain asymptotics for Sobolev polynomials and their norms for a particular case of Freud weights, which illustrates that the choice of λ_n is accurate.

2 Selection of the parameters

We point out some heuristic reasonings about the asymptotic behavior of the parameters λ_n in order to balance both terms in the varying Sobolev inner product $\langle P, Q\rangle_{\lambda_n}$.

Firstly, we recall some basic tools from the classical potential theory with external field which will be used later on.

Let μ be a probability measure with support in a closed set Σ of the complex plane. Recall that, the logarithmic potential V^μ associated with μ is defined by $V^\mu(z) = -\int \log|z-t|\, d\mu(t)$. Let us assume that $w(z) = e^{-Q(z)}$ is an admissible and continuous weight function in Σ. It is well known that there exists a unique probability measure μ_w, called extremal or equilibrium measure associated with w, minimizing the weighted energy:

$$I_w(\mu) = \int_\Sigma (V^\mu(z) + 2Q(z))\, d\mu(z)$$

for all probability measures with support in Σ. This measure μ_w is compactly supported and there exists a constant F_w (the modified Robin constant of Σ) such that $V^{\mu_w}(z) + Q(z) = F_w$ quasi–everywhere on $\mathrm{supp}(\mu_w)$, see [14, Theorem 1.3, p. 27]. Moreover, if Q is an even function with some additional properties it can be deduced that

$$\|w^n\, Q_n\|_{L_\infty(\Sigma)} = \|w^n\, Q_n\|_{L_\infty(\mathrm{supp}(\mu_w))}$$

for every polynomial Q_n of degree $\leq n$, see [14, p. 203]. As a straightforward application of these results, we can obtain for weighted polynomials a symmetric compact interval on which its supremum norm lives, more precisely, we have

$$\|w\, Q_n\|_{L_\infty(\Sigma)} = \|w\, Q_n\|_{L_\infty([-a_n, a_n])}$$

for every polynomial Q_n of degree $\leq n$. The number a_n $(n \geq 1)$ is the so–called n–th Mhaskar–Rakhmanov–Saff number for Q, that is, the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n\, t\, Q'(a_n\, t)}{\sqrt{1-t^2}}\, dt\,.$$

The link between the equilibrium measure and the asymptotics of orthogonal polynomials is given by the following observation: for a polynomial $Q_n(z) = (z-c_1)(z-c_2)\dots(z-c_n)$ we can write $\log|Q_n(z)| = -n\,V^{\nu_n}(z)$ where ν_n is the normalized counting measure on the zeros of Q_n, that is, $\nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{c_i}$. Then $|w^n(z)\,Q_n(z)|^{1/n} = e^{-(V^{\nu_n}(z)+Q(z))}$.

If we denote by $T_{n,w}$ the $n-$th weighted monic Chebyshev polynomial corresponding to w, that is, the solution of the extremal problem

$$\inf\{\|w^n\,Q_n\|_{L_\infty(\Sigma)};\, Q_n(z) = z^n + \dots\}$$

then

$$\lim_{n\to\infty} \|w^n\,T_{n,w}\|_{L_\infty(\Sigma)}^{1/n} = e^{-F_w},$$

see [14, Theorem 3.1, p. 163].

Keeping in mind our balance problem, we are interested in the asymptotic behavior of the L_2–norm in $[-1,1]$ with varying weights. Since

$$\lim_{n\to\infty}\left(\frac{\|w^n\,Q_n\|_{L_\infty([-1,1])}}{\|w^n\,Q_n\|_{L_2([-1,1])}}\right)^{1/n} = 1$$

for every polynomial of degree n, (see [15, Theorem 3.2.1, p. 65]), the asymptotic extremality of $\|w^n\,Q_n\|_{L_2([-1,1])}^{1/n}$ can be thought as the corresponding one of $\|w^n\,Q_n\|_{L_\infty([-1,1])}^{1/n}$. In fact, if we denote by P_{n,w^n} the solution of the extremal problem

$$\inf\{\|w^n\,Q_n\|_{L_2([-1,1])};\, Q_n(z) = z^n + \dots\}$$

it can be deduced (see [15, Theorem 3.3.3, p. 78]) that there exists

$$\lim_{n\to\infty} \|w^n\,P_{n,w^n}\|_{L_2([-1,1])}^{1/n}. \tag{3}$$

¿From now on, $f_n(x) \sim g_n(x)$ in a domain D will denote that there are positive constants C_1, C_2 such that $C_1\,g_n(x) \le f_n(x) \le C_2\,g_n(x)$, for all $x \in D$ and n large enough.

In relation with our problem, we consider the varying Sobolev inner product $\langle\cdot,\cdot\rangle_{\lambda_n}$ where $d\,\mu_i = W^2(x)\,dx$, $i = 0,1$. Here, we assume that $W(x) = e^{-Q(x)}$ is a weight function where $Q : I = (-c,c) \to [0,+\infty)$ is a convex, smooth, and even function with $Q(c^-) = +\infty = Q((-c)^+)$ and $Q(x) = 0$ only for $x = 0$ (we take Q an even function for simplicity). For these weights W, see [5, Theorem 4.1, p. 95], the L_2–norm on I for weighted polynomials is asymptotically equivalent to the L_2–norm on a compact interval. More precisely,

$$\|W\,Q_n\|_{L_2([-a_{n+1},a_{n+1}])} \le \|W\,Q_n\|_{L_2(I)} \le \sqrt{2}\,\|W\,Q_n\|_{L_2([-a_{n+1},a_{n+1}])} \tag{4}$$

holds for every n and every polynomial Q_n with degree $\le n$, where a_n are the Mhaskar–Rakhmanov–Saff numbers associated with Q.

¿From (4), we deduce that for every polynomial $Q_n(x) = x^n + \dots$

$$\begin{aligned}
&\langle Q_n, Q_n\rangle_{\lambda_n} \qquad (5)\\
&\sim \int_{-a_{n+1}}^{a_{n+1}} Q_n^2(x)\,W^2(x)\,dx + \lambda_n \int_{-a_{n+1}}^{a_{n+1}} \left(Q_n'(x)\right)^2\,W^2(x)\,dx\\
&= a_{n+1}\left[\int_{-1}^{1} Q_n^2(a_{n+1}t)W^2(a_{n+1}t)dt + \lambda_n \int_{-1}^{1} \left(Q_n'(a_{n+1}t)\right)^2 W^2(a_{n+1}t)dt\right]\\
&= a_{n+1}^{2n+1}\left[\int_{-1}^{1} U_n^2(t)\,W^2(a_{n+1}t)\,dt + \frac{\lambda_n\,n^2}{a_{n+1}^2}\int_{-1}^{1} V_{n-1}^2(t)\,W^2(a_{n+1}t)\,dt\right],
\end{aligned}$$

where U_n and V_{n-1} are monic polynomials of degree n and $n-1$, respectively.

Observe that (5) remains true if we take $d\mu_i = L_i\,W^2(x)\,dx,\ i = 0,1$, where L_0 and L_1 are any positive constants. At first sight, the presence of the constants L_i could seem irrelevant but in the next section it will allow us to give an alternative reading to explain why our selection of λ_n is accurate.

Therefore, in order to balance both terms in (5) it is reasonable to require the following:

i) $\lambda_n\,n^2 \sim a_{n+1}^2$.

ii) the asymptotic extremality of the $L_2(W^2(a_{n+1}t),\,[-1,1])$–norm for monic polynomials of degree n behaves as the corresponding one of degree $n-1$.

The previous results about potential theory lead us to think that a sufficient condition to get ii) is

$$W^{1/n}(a_{n+1}t) \sim w(t), \quad \forall t \in (-1,1) \tag{6}$$

where w is an admissible and continuous weight function.

Concerning the choice of the parameters λ_n observe that, when the support of the measures μ_0 and μ_1 is unbounded, the size of λ_n as n^{-2} is not the right one, in general. If the weight satisfies (6), the choice of the parameters depends on the distribution of the measure $W^2(t)\,dt$, that is, on the corresponding Mhaskar–Rakhmanov–Saff numbers.

We would like to point out that these ideas can be also applied in a more general framework. Indeed, consider a Sobolev inner product with two different weights, W_0^2 and W_1^2, which are linked in such a way so that $\langle\cdot,\cdot\rangle_{\lambda_n}$ can be expressed in terms of only one weight (either W_0^2 or W_1^2) satisfying condition (6). Actually, important examples in this situation are the Hermite coherent pairs. Notice that if the pair of measures (W_0^2, W_1^2) constitutes a Hermite symmetrically coherent pair (see [2] and [11]), then either

$$\text{I:}\quad W_0^2(x) = (x^2+a^2)\,e^{-x^2} \quad \text{and} \quad W_1^2(x) = e^{-x^2},\ a \in \mathbb{R}, \quad \text{or}$$

$$\text{II:}\quad W_0^2(x) = e^{-x^2} \quad \text{and} \quad W_1^2(x) = \frac{e^{-x^2}}{x^2+a^2}, \quad a \in \mathbb{R}\setminus\{0\}.$$

In both cases we have

$$\langle Q_n, Q_n\rangle_{\lambda_n} = \int_{\mathbb{R}} \left[Q_n^2(x)(x^2+a^2)\right] W_1^2(x)\,dx + \lambda_n \int_{\mathbb{R}} \left(Q_n'(x)\right)^2 W_1^2(x)\,dx,$$

and it is not difficult to check that

$$\frac{\langle Q_n, Q_n\rangle_{\lambda_n}}{a_{n+2}^{2n+3}} \sim \int_{-1}^{1} U_{n+1}^2(t) W_1^2(a_{n+2}t)\,dt + \frac{\lambda_n n^2}{a_{n+2}^4} \int_{-1}^{1} V_{n-1}^2(t)\, W_1^2(a_{n+2}t)\,dt,$$

where in each case a_n are the Mhaskar–Rakhmanov–Saff numbers for the corresponding weight W_1, and U_{n+1} and V_{n-1} are monic polynomials of degree $n+1$ and $n-1$, respectively.

Since $\dfrac{a_n}{\sqrt{n}} \to \sqrt{2}$, observe that

$$\lim_{n\to\infty} W_1^{1/n}(a_{n+2}t) = e^{-t^2}, \quad \forall t \in (-1,1)$$

and therefore, according to the theory stated above, the adequate choice of λ_n should be $\lambda_n \sim a_{n+2}^4\, n^{-2}$. In other words, $\lambda_n \sim$ constant. Hence, it can be said that the Hermite–Sobolev coherent inner products are self–balanced.

3 Freud–Sobolev orthogonal polynomials

We are going to test the arguments developed in the previous section for the case of a Sobolev inner product related to Freud weights. The simplest example corresponds to $W_0^2(x) = W_1^2(x) = e^{-x^2}$, but this is a trivial case since for any choice of λ_n the Sobolev orthogonal polynomial S_{n,λ_n} is the n–th monic Hermite polynomial. In this section, we show asymptotics for the Sobolev orthogonal polynomials with $W^2(x) = W_0^2(x) = W_1^2(x) = \exp(-x^4)$.

Throughout the section, $(P_n)_{n\geq 0}$ denotes the sequence of monic polynomials orthogonal with respect to the weight W^2, $\|\cdot\|$ stands for the $L^2(W^2)$-norm, and S_{n,λ_n} is the monic polynomial which minimizes

$$\langle Q_n \, , Q_n\rangle_{\lambda_n} = \int_{\mathbb{R}} Q_n^2(x)\, W^2(x)\, dx + \lambda_n \int_{\mathbb{R}} (Q_n')^2(x)\, W^2(x)\, dx$$

in the class of all monic polynomials of degree n.

The Mhaskar–Rakhmanov–Saff numbers for $W(x) = \exp(-x^4/2)$ satisfy $a_n \sim n^{1/4}$ and therefore condition (6) holds for W. As we have explained in Section 2, to balance this Sobolev inner product we must take $\lambda_n\, n^2 \sim a_{n+1}^2$, that is, λ_n like $n^{-3/2}$ when $n \to \infty$.

Next, we study the asymptotic behavior of the ratio $\dfrac{S_{n,\lambda_n}}{P_n}$ showing that the choice of λ_n provides the reasonable one in a sense we will explain later. For technical reasons some additional constraints should be imposed on parameters λ_n, so we deal with a decreasing sequence (λ_n) of positive real numbers such that

$$\lim_{n\to\infty} n^{3/2}\lambda_n = L \in [0, +\infty]\, , \tag{7}$$

and

$$\lim_{n\to\infty} n^{7/4}(\lambda_{n-2} - \lambda_n) = 0 = \lim_{n\to\infty} n^{1/4}\left(\frac{\lambda_{n-2}}{\lambda_n} - 1\right) . \tag{8}$$

Notice that the sequence $\lambda_n = n^{-3/2}$ satisfies (7) and (8).

Proposition 1 *Let (λ_n) be a decreasing sequence of positive real numbers which satisfies $\frac{\lambda_{n-2}}{\lambda_n} \to 1$ and $n^{3/2}\lambda_n \to L \in [0, +\infty]$. Then*

$$\kappa(L) := \lim_{n\to\infty} \frac{\langle S_{n,\lambda_n}, S_{n,\lambda_n}\rangle_{\lambda_n}}{\|P_n\|^2} = \begin{cases} 1 & \text{if } L = 0 \\ \frac{2L}{\sqrt{3}}\, \varphi\left(\frac{20\, L + 3\sqrt{3}}{12\, L}\right) & \text{if } 0 < L < +\infty \\ +\infty & \text{if } L = +\infty\, , \end{cases} \tag{9}$$

where $\varphi(x) = x + \sqrt{x^2 - 1}$.

Proof. We consider the Fourier expansion of the polynomial P_n in terms of the basis $(S_{m,\lambda_n})_{m\geq 0}$. Because of the weight e^{-x^4} is a symmetric function we have

$$P_n(z) = S_{n,\lambda_n}(z) + \sum_{j=0}^{n-2} \alpha_j(\lambda_n) S_{j,\lambda_n}(z)\,,$$

where

$$\alpha_j(\lambda_n) = \frac{\langle P_n, S_{j,\lambda_n}\rangle_{\lambda_n}}{\langle S_{j,\lambda_n}, S_{j,\lambda_n}\rangle_{\lambda_n}} = \frac{\lambda_n \int_{\mathbb{R}} P_n'(x) S_{j,\lambda_n}'(x) e^{-x^4} dx}{\langle S_{j,\lambda_n}, S_{j,\lambda_n}\rangle_{\lambda_n}}, \quad 0 \le j \le n-2.$$

Since the orthogonal polynomials P_n satisfy the following structure relation, (see [12]),

$$P_n'(z) = nP_{n-1}(z) + \frac{4\|P_n\|^2}{\|P_{n-3}\|^2} P_{n-3}(z), \tag{10}$$

the coefficients $\alpha_j(\lambda_n)$ vanish for $0 \le j < n-2$.. For $j = n-2$ we get

$$\alpha_{n-2}(\lambda_n) = \frac{4\,(n-2)\,\lambda_n\,\|P_n\|^2}{\langle S_{n-2,\lambda_n}, S_{n-2,\lambda_n}\rangle_{\lambda_n}}\,, \tag{11}$$

and therefore

$$P_n(z) = S_{n,\lambda_n}(z) + \alpha_{n-2}(\lambda_n) S_{n-2,\lambda_n}(z), \quad n \ge 3. \tag{12}$$

¿From now on, we will write $\kappa_m(\lambda_n) = \langle S_{m,\lambda_n}, S_{m,\lambda_n}\rangle_{\lambda_n}$, $n, m \ge 0$.

Now, observe that (12) leads to

$$\begin{aligned}\kappa_n(\lambda_n) &= \langle P_n - \alpha_{n-2}(\lambda_n) S_{n-2,\lambda_n}, P_n - \alpha_{n-2}(\lambda_n) S_{n-2,\lambda_n}\rangle_{\lambda_n} \\ &= \int_{\mathbb{R}} \left[(P_n - \alpha_{n-2}(\lambda_n) S_{n-2,\lambda_n})^2 + \lambda_n \left(P_n' - \alpha_{n-2}(\lambda_n) S_{n-2,\lambda_n}'\right)^2 \right] e^{-x^4}\, dx.\end{aligned}$$

Then, using (10) and the orthogonality of P_n with respect to the weight function e^{-x^4}, we have:

$$\begin{aligned}\kappa_n(\lambda_n) = {} & \|P_n\|^2 + n^2\,\lambda_n\,\|P_{n-1}\|^2 - 8(n-2)\,\lambda_n\,\alpha_{n-2}(\lambda_n)\,\|P_n\|^2 \\ & + 16\,\lambda_n\,\frac{\|P_n\|^4}{\|P_{n-3}\|^2} + \alpha_{n-2}^2(\lambda_n)\,\kappa_{n-2}(\lambda_n)\,.\end{aligned}$$

Taking into account the value of $\alpha_{n-2}(\lambda_n)$ given by (11), we get

$$\kappa_n(\lambda_n) = \|P_n\|^2 \left(B_n(\lambda_n) - A_n(\lambda_n) \frac{\|P_{n-2}\|^2}{\kappa_{n-2}(\lambda_{n-2})} \right), \; n \ge 3, \tag{13}$$

where

$$A_n(\lambda_n) = 16{\lambda_n}^2\,(n-2)^2\,\frac{\kappa_{n-2}(\lambda_{n-2})}{\kappa_{n-2}(\lambda_n)}\frac{\|P_n\|^2}{\|P_{n-2}\|^2}$$

$$B_n(\lambda_n) = 1 + \lambda_n\,n^2\,\frac{\|P_{n-1}\|^2}{\|P_n\|^2} + 16\lambda_n\,\frac{\|P_n\|^2}{\|P_{n-3}\|^2}.$$

Next, we study $\lim_n B_n(\lambda_n)$ and $\lim_n A_n(\lambda_n)$. First, recall that the polynomials P_n satisfy (see [12])

$$\lim_{n\to\infty}\frac{\sqrt{n}\|P_{n-1}\|^2}{\|P_n\|^2} = 2\sqrt{3}. \tag{14}$$

On the other hand, $\lim_{n\to\infty}\frac{\kappa_{n-2}(\lambda_{n-2})}{\kappa_{n-2}(\lambda_n)} = 1$. Indeed, from the assumptions on λ_n and using the extremal property of the norms of monic orthogonal polynomials, we have

$$\begin{aligned}\kappa_{n-2}(\lambda_n) &\le \kappa_{n-2}(\lambda_{n-2}) \le \langle S_{n-2,\lambda_n}, S_{n-2,\lambda_n}\rangle_{\lambda_{n-2}}\\ &= \frac{\lambda_{n-2}}{\lambda_n}\left[\frac{\lambda_n}{\lambda_{n-2}}\|S_{n-2,\lambda_n}\|^2 + \lambda_n\|S'_{n-2,\lambda_n}\|^2\right] \le \frac{\lambda_{n-2}}{\lambda_n}\kappa_{n-2}(\lambda_n).\end{aligned}$$

Since $\frac{\lambda_{n-2}}{\lambda_n} \to 1$, it follows

$$\lim_{n\to\infty}\frac{\kappa_{n-2}(\lambda_{n-2})}{\kappa_{n-2}(\lambda_n)} = 1. \tag{15}$$

Firstly, let us suppose that $0 \le L < +\infty$. Then from (15) and (14) we deduce that

$$\lim_{n\to\infty} B_n(\lambda_n) = 1 + \frac{20}{9}\sqrt{3}L, \quad \text{and} \quad \lim_{n\to\infty} A_n(\lambda_n) = \frac{4}{3}L^2. \tag{16}$$

To obtain (9) observe that denoting $s_n = \kappa_n(\lambda_n)/\|P_n\|^2$, (13) becomes

$$s_n = B_n(\lambda_n) - A_n(\lambda_n)\frac{1}{s_{n-2}}. \tag{17}$$

Writing (17) for even indices and introducing a new sequence (q_n) by means of $q_{n+1} = s_{2n}q_n$, the above difference equation becomes

$$q_{n+1} - B_{2n}(\lambda_{2n})q_n + A_{2n}(\lambda_{2n})q_{n-1} = 0,$$

whose characteristic equation

$$q^2 - \left(1 + \frac{20}{9}\sqrt{3}L\right)q + \frac{4}{3}L^2 = 0 \tag{18}$$

has two simple and real roots with distinct moduli. Thus, Poincaré' s Theorem (see, e.g., [10]) assures that $\dfrac{q_{n+1}}{q_n} = s_{2n}$ converges to a root of (18). The extremal property of the norms yields

$$\kappa_n(\lambda_n) \geq \|P_n\|^2 + \lambda_n\, n^2 \|P_{n-1}\|^2,$$

and therefore, using (14)

$$l = \lim_{n\to\infty} s_{2n} \geq 1 + \lim_{n\to\infty} \lambda_{2n}(2n)^2 \frac{\|P_{2n-1}\|^2}{\|P_{2n}\|^2} = 1 + 2\sqrt{3}L.$$

So, it follows easily that $l = \frac{1}{18}\left[9 + 20\sqrt{3}\,L + \sqrt{768L^2 + 360\sqrt{3}L + 81}\right]$. Notice that, if $L \in (0, +\infty)$, then $l = \frac{2L}{\sqrt{3}}\,\varphi\!\left(\frac{20L+3\sqrt{3}}{12L}\right)$.

In a similar way, we also prove that s_{2n+1} converges to l. As a conclusion, there exists $\lim_n s_n = l = \kappa(L)$, and so for $L \in [0, +\infty)$ the Proposition follows.

To finish the proof, let us now assume that $L = +\infty$. From (15) and (14) we have

$$\lim_{n\to\infty} \frac{A_n(\lambda_n)}{\left(\lambda_n\, n^{3/2}\right)^2} = \frac{4}{3} \quad \text{and} \quad \lim_{n\to\infty} \frac{B_n(\lambda_n)}{\lambda_n\, n^{3/2}} = \frac{20}{9}\sqrt{3}.$$

Upon applying the same technique as in the case $L < +\infty$ and replacing s_n by $s_n/(\lambda_n\, n^{3/2})$, we obtain

$$\lim_{n\to\infty} \frac{s_n}{\lambda_n\, n^{3/2}} = \lim_{n\to\infty} \frac{\kappa_n(\lambda_n)}{\lambda_n\, n^{3/2}\|P_n\|^2} = 2\sqrt{3}. \tag{19}$$

Clearly, $\dfrac{\kappa_n(\lambda_n)}{\|P_n\|^2} \to +\infty$ when n tends to infinity and we conclude our statement. □

The main result of this section is the following:

Theorem 1 *Let (λ_n) be a decreasing sequence of positive real numbers such that* $\lim_n n^{7/4}(\lambda_{n-2} - \lambda_n) = 0 = \lim_n n^{1/4}\left(\frac{\lambda_{n-2}}{\lambda_n} - 1\right)$. *If*

$$\lim_{n\to\infty} n^{3/2}\lambda_n = L \in [0, +\infty],$$

then

$$\lim_{n\to\infty}\frac{S_{n,\lambda_n}(z)}{P_n(z)}=\begin{cases}1 & \text{if } L=0\\ \dfrac{1}{1-\left[\varphi\left(\frac{20\,L+3\sqrt{3}}{12\,L}\right)\right]^{-1}} & \text{if } 0<L<+\infty\\ 3/2 & \text{if } L=+\infty\,,\end{cases}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$.

Remarks. 1. The choice $\lambda_n\equiv$ constant, which corresponds to a non–balanced inner product, is a particular case of $L=+\infty$, and then Theorem 1 recovers the result already obtained in [3].

2. When $L\in(0,+\infty)$ the above result has also the following reading. Write

$$\langle P,Q\rangle_{\lambda_n}=\int_{\mathbb{R}}P(x)\,Q(x)\,W^2(x)\,dx+\lambda_n\int_{\mathbb{R}}P'(x)\,Q'(x)\,[L\,W^2(x)]\,dx.$$

If $\lambda_n=n^{-3/2}(1+o(1))$ then $\lim_{n\to\infty}\dfrac{S_{n,\lambda_n}}{P_n}$ depends on L, that is, on the ratio of the weights.

However, for any other choice of λ_n's the dependence on L disappears, in particular for $\lambda_n=n^{-2}$ (the right choice in the bounded case) and for $\lambda_n\equiv$ constant (the non–balanced case). This shows that our selection of λ_n is accurate since the asymptotic behavior of Sobolev orthogonal polynomials S_{n,λ_n} depends on both measures.

To prove Theorem 1 we will use the following result on the strong asymptotics of P_n which appears in [6, Section 3]:

$$\lim_{n\to\infty}\frac{P_n(z)}{\|P_n\|}\frac{D_n(z)}{\varphi^{n+1/2}(z/a_n)}=\frac{1}{\sqrt{2\pi}}\tag{20}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Here, a_n are the Mhaskar–Rakhmanov–Saff numbers associated with the weight function W, $\varphi(z)=z+\sqrt{z^2-1}$ is the conformal mapping from $\mathbb{C}\setminus[-1,1]$ onto the exterior of the unit circle, and

$$D_n(z)=\exp\left(\frac{\sqrt{z^2-a_n^2}}{2\pi}\int_{-a_n}^{a_n}\frac{-t^4}{(z-t)\sqrt{a_n^2-t^2}}dt\right),\qquad z\in\mathbb{C}\setminus[-a_n,a_n].\tag{21}$$

We would like to remark that, for $z \in \mathbb{C} \setminus [-a_n, a_n]$,

$$D_n(z) = D\left(\frac{1}{\varphi(z/a_n)}, W_n^2\right),$$

where W_n^2 is the weight function on the unit circle $\mathbb{T}$ defined by

$$W_n^2(e^{i\theta}) = W^2(a_n \cos\theta), \quad \theta \in [-\pi, \pi],$$

and

$$D(w, W_n^2) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+w}{e^{i\theta}-w} \log W_n(\theta) d\theta\right), \quad |w| < 1.$$

It is well known that $D(., W_n^2)$ is holomorphic in the open unit disk $\mathbb{D}$, belongs to the Hardy space $H^2(\mathbb{D})$, and satisfies:

(1) $D(w, W_n^2) \neq 0, \quad \text{for} \quad w \in \mathbb{D}$

(2) $D(0, W_n^2) > 0$

(3) for almost every ζ in the unit circle, $D(., W_n^2)$ has nontangential boundary values $D(\zeta, W_n^2)$ such that $|D(\zeta, W_n^2)|^2 = W_n^2(\zeta)$,

(see, for instance, [13]).

Next, we prove a technical result that will be also used in the proof of Theorem 1.

Lemma 1 *Assume that the sequence (λ_n) satisfies the same conditions as in Theorem 1, then*

$$\lim_{n\to\infty} \frac{S_{n,\lambda_{n-2}}(z) - S_{n,\lambda_n}(z)}{P_n(z)} = 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$.

Proof. On account of (20), it suffices to prove that

$$\lim_{n\to\infty} \frac{S_{n,\lambda_{n-2}}(z) - S_{n,\lambda_n}(z)}{\|P_n\| \, \varphi^{n+1/2}(z/a_n)} D_n(z) = 0$$

uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{R}$. To see this we will prove:

i) for every compact set K in $\mathbb{C} \setminus \mathbb{R}$, there exists a constant M_K, not depending on n, such that for n large enough

$$\sup_{z\in K}\left|\frac{S_{n,\lambda_{n-2}}(z)-S_{n,\lambda_n}(z)}{\|P_n\|\,\varphi^{n+1/2}(z/a_n)}D_n(z)\right|^2 \leq M_K\, a_n\int_{-a_n}^{a_n}\frac{|S_{n,\lambda_{n-2}}(x)-S_{n,\lambda_n}(x)|^2}{\|P_n\|^2}\,W^2(x)\,dx,$$

and

ii)

$$\lim_{n\to\infty} a_n\int_{-a_n}^{a_n}\frac{|S_{n,\lambda_{n-2}}(x)-S_{n,\lambda_n}(x)|^2}{\|P_n\|^2}\,W^2(x)\,dx=0.$$

The key idea to prove *i*) is to use the conformal mapping $\varphi(z/a_n)$ which applies $\mathbb{C}\setminus[-a_n,a_n]$ onto $\Omega=\{z\in\mathbb{C};|z|>1\}$, and the Cauchy integral representation for functions in $H^2(\Omega)$. Here, $H^2(\Omega)$ denotes the space of analytic functions f in Ω, with limit at ∞ and such that $f(\frac{1}{z})$ belongs to the Hardy space $H^2(\mathbb{D})$. From the Cauchy integral representation for functions in $H^2(\mathbb{D})$, see [13], we have that if $f\in H^2(\Omega)$ then

$$f(w)=-\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f^*(\zeta)}{\zeta-w}\,\frac{w}{\zeta}\,d\zeta,\quad w\in\Omega \tag{22}$$

where $f^*(\zeta)=\lim_{r\searrow 1}f(r\,\zeta)$ and the unit circle is positively oriented.

In order to prove *i*), given a compact set K in $\mathbb{C}\setminus\mathbb{R}$, there exists an absolute constant $C_K>0$ such that

$$|\sqrt{z^2-a_n^2}|\geq C_K,\quad \forall z\in K,\quad \forall n\geq 0.$$

Therefore, if $z\in K$,

$$\begin{aligned}&\left|\frac{S_{n,\lambda_{n-2}}(z)-S_{n,\lambda_n}(z)}{\|P_n\|\varphi^{n+1/2}(z/a_n)}D\left(\frac{1}{\varphi(z/a_n)},W_n^2\right)\right|^2\\ &\leq\frac{1}{C_K}\left|\frac{S_{n,\lambda_{n-2}}(z)-S_{n,\lambda_n}(z)}{\|P_n\|\varphi^{n+1/2}(z/a_n)}D\left(\frac{1}{\varphi(z/a_n)},W_n^2\right)\right|^2|\sqrt{z^2-a_n^2}|\\ &=\frac{1}{C_K}|F_n(w)|\end{aligned}$$

where

$$F_n(w)=\left[\frac{(S_{n,\lambda_{n-2}}-S_{n,\lambda_n})(a_n\,\varphi^{-1}(w))}{\|P_n\|w^{n+1/2}}D\left(\frac{1}{w},W_n^2\right)\right]^2 a_n\sqrt{(\varphi^{-1}(w))^2-1},$$

with $w = \varphi(z/a_n)$.

It is easy to check that $F_n \in H^2(\Omega)$ and its boundary values are

$$F_n^*(e^{i\theta}) = \frac{(S_{n,\lambda_{n-2}} - S_{n,\lambda_n})^2(a_n \cos\theta)}{\|P_n\|^2 e^{i(2n+1)\theta}} W^2(a_n\, \cos\theta) a_n \sqrt{\cos^2\theta - 1}.$$

Moreover, if we denote by $K_n = \{\varphi(z/a_n); z \in K\}$, straightforward computations yield that there exists an absolute constant $A_K > 0$ such that the distance between K_n and the unit circle satisfies $d(K_n, \mathbb{T}) \geq A_K/a_n$ for n large enough. Then, from the integral formula (22) applied to F_n we have for $w \in K_n$

$$\begin{aligned}
|F_n(w)| &\leq B_K\, a_n \int_{|\zeta|=1} |F_n^*(\zeta)|\, |d\,\zeta| \\
&= B_K\, a_n \int_{-\pi}^{\pi} \frac{(S_{n,\lambda_{n-2}} - S_{n,\lambda_n})^2(a_n\, \cos\theta)}{\|P_n\|^2} W^2(a_n\, \cos\theta)\, a_n |\operatorname{sen}\, \theta|\, d\theta \\
&= 2\, B_K\, a_n \int_{-a_n}^{a_n} \frac{(S_{n,\lambda_{n-2}} - S_{n,\lambda_n})^2(x)}{\|P_n\|^2} W^2(x)\, dx,
\end{aligned}$$

where B_K is an absolute positive constant depending only on K. So $i)$ is proved.

In order to deduce $ii)$, observe that

$$\begin{aligned}
&\int_{\mathbb{R}} |S_{n,\lambda_{n-2}}(x) - S_{n,\lambda_n}(x)|^2 W^2(x)\, dx \leq \langle S_{n,\lambda_{n-2}} - S_{n,\lambda_n}, S_{n,\lambda_{n-2}} - S_{n,\lambda_n}\rangle_{\lambda_n} \\
&= \langle S_{n,\lambda_{n-2}}, S_{n,\lambda_{n-2}}\rangle_{\lambda_n} - \langle S_{n,\lambda_n}, S_{n,\lambda_n}\rangle_{\lambda_n} \\
&= \kappa_n(\lambda_{n-2}) + (\lambda_n - \lambda_{n-2}) \int_{\mathbb{R}} |S'_{n,\lambda_{n-2}}(x)|^2\, W^2(x)\, dx - \kappa_n(\lambda_n) \\
&\leq \kappa_n(\lambda_{n-2}) - \kappa_n(\lambda_n).
\end{aligned}$$

Therefore, for every n we get

$$a_n \int_{-a_n}^{a_n} \frac{|S_{n,\lambda_{n-2}}(x) - S_{n,\lambda_n}(x)|^2}{\|P_n\|^2} W^2(x) dx \leq a_n\, \frac{\kappa_n(\lambda_{n-2}) - \kappa_n(\lambda_n)}{\|P_n\|^2}.$$

Finally, since $a_n \sim n^{1/4}$ it is enough to prove that $n^{1/4}\, \dfrac{\kappa_n(\lambda_{n-2}) - \kappa_n(\lambda_n)}{\|P_n\|^2}$ tends to 0 when n tends to infinity. Indeed, since

$$0 \leq \kappa_n(\lambda_{n-2}) - \kappa_n(\lambda_n) \leq \left(1 - \frac{\lambda_n}{\lambda_{n-2}}\right) \kappa_n(\lambda_{n-2}),$$

we have

$$n^{1/4}\,\frac{\kappa_n(\lambda_{n-2})-\kappa_n(\lambda_n)}{\|P_n\|^2}\le n^{1/4}\left(1-\frac{\lambda_n}{\lambda_{n-2}}\right)\frac{\kappa_n(\lambda_n)}{\|P_n\|^2}\frac{\kappa_n(\lambda_{n-2})}{\kappa_n(\lambda_n)}.$$

Now, taking into account that $\lim_n \frac{\kappa_n(\lambda_{n-2})}{\kappa_n(\lambda_n)}=1$, it suffices to keep in mind Proposition 1, (8), and (19) to conclude *ii*) and therefore the proof of the Lemma. □

Proof of Theorem 1. The algebraic relation between the polynomials P_n and the Sobolev polynomials given by (12) can be rewritten for λ_{n-2} as

$$\begin{aligned}P_n(z)&=S_{n,\lambda_{n-2}}(z)+\alpha_{n-2}(\lambda_{n-2})\,S_{n-2,\lambda_{n-2}}(z)\\&=S_{n,\lambda_n}(z)+S_{n,\lambda_{n-2}}(z)-S_{n,\lambda_n}(z)+\alpha_{n-2}(\lambda_{n-2})\,S_{n-2,\lambda_{n-2}}(z).\end{aligned}$$

Then, dividing both hand sides of the above expression by $P_n(z)$, we obtain

$$f_n(z)=b_n(z)f_{n-2}(z)+c_n(z),\quad z\in\mathbb{C}\setminus\mathbb{R} \tag{23}$$

where

$$\begin{aligned}f_n(z)&=\frac{S_{n,\lambda_n}(z)}{P_n(z)},\quad b_n(z)=-\alpha_{n-2}(\lambda_{n-2})\,\frac{P_{n-2}(z)}{P_n(z)},\\c_n(z)&=1-\frac{S_{n,\lambda_{n-2}}(z)-S_{n,\lambda_n}(z)}{P_n(z)}.\end{aligned}$$

Firstly, we study the limits of the sequences $(b_n(z))$ and $(c_n(z))$. As a consequence of Lemma 1 we know that

$$\lim_{n\to\infty}c_n(z)=1$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$.

With regard to $(b_n(z))$, if $L\in[0,+\infty)$ from Proposition 1 and (14)

$$\frac{\alpha_{n-2}(\lambda_{n-2})}{\sqrt{n-2}}=4\lambda_{n-2}(n-2)^{3/2}\frac{\|P_{n-2}\|^2}{\kappa_{n-2}(\lambda_{n-2})}\frac{\|P_n\|^2}{(n-2)\,\|P_{n-2}\|^2}\to\frac{L}{3\kappa(L)}.$$

Moreover, for the monic polynomials P_n it is known, (see [6]), that

$$\lim_{n\to\infty}\frac{\sqrt{n-2}\,P_{n-2}(z)}{P_n(z)}=-2\sqrt{3},$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Both results lead to

$$\lim_{n\to\infty}b_n(z)=\frac{2L}{\sqrt{3}\,\kappa(L)}=\begin{cases}0 & \text{if } L=0\\ \frac{1}{\varphi\left(\frac{20\,L+3\sqrt{3}}{12\,L}\right)} & \text{if } 0<L<+\infty\end{cases}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. In the case $L=+\infty$, using formula (19) we get $\lim_n b_n(z)=1/3$, uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$.

Finally, observe that the functions f_n, b_n and c_n are analytic in $\mathbb{C}\setminus\mathbb{R}$. Since for $L\in[0,+\infty]$ we have $\lim_n b_n(z)=b_L$, with $|b_L|<1$, and $\lim_n c_n(z)=1$ uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$, we can deduce that

$$\lim_{n\to\infty} f_n(z)=\frac{1}{1-b_L}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Indeed, for a fixed compact set $K\subset\mathbb{C}\setminus\mathbb{R}$ there exist constants $r\in(0,1)$, $R>1$ and a positive integer number n_0 such that

$$|b_n(z)|\le r,\quad |c_n(z)|\le R,\quad \text{for}\quad n\ge n_0,\quad z\in K.$$

Thus

$$|f_n(z)|\le r|f_{n-2}(z)|+R,\quad \text{for}\quad n\ge n_0,\quad z\in K,$$

and therefore we deduce that the sequence (f_n) is uniformly bounded on compact subsets of $\mathbb{C}\setminus\mathbb{R}$.

¿From (23), we can write

$$f_n(z)-\frac{1}{1-b_L}=b_L\left[f_{n-2}(z)-\frac{1}{1-b_L}\right]+\varepsilon_n(z),$$

with

$$\varepsilon_n(z)=(b_n(z)-b_L)f_{n-2}(z)+c_n(z)-1.$$

Notice that $\lim_n \varepsilon_n(z)=0$, uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. From the fact $|b_L|<1$, it is easy to deduce that

$$\lim_{n\to\infty} f_n(z)=\frac{1}{1-b_L},$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Taking into account the value of b_L with $L\in[0,+\infty]$, the Theorem is proved. $\square$

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