Tensor products of ideal codes over Hopf algebras

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Abstract

We study indecomposable codes over the well-known family of Radford Hopf algebras. We use properties of Hopf algebras to show that tensors of ideal codes are ideal codes, extending the corresponding result given in [4] and showing that in this case, semisimplicity is lost.

Keywords: Radford Hopf algebra, ideal code, tensor product

1   Introduction

The more mathematical structure one can add to a system the more information and description can be obtained. A clear example of this occurs with algebraic codes. Linearity on a code implies, between other things that one has not to compute all distances between every pair of codewords. If the code is also cyclic, then we get fast encoding/decoding, etc. As Berman in [1] observed, cyclic codes and Reed-Muller codes can be seen as ideals in the group ring $\mathbb{K}G$ (where $\mathbb{K}$ is a finite field and G is a finite cyclic group). This fact has led many authors to the study of codes from a point of view of Ring Theory (cf. [3] for instance). Recently Wood in [9] and [10] has stated the suitability of Frobenius rings when studing codes over a finite ring. More precisely he showed that a finite Frobenius ring is characterized by the fact of allowing the Extension Theorem for linear codes. One example of such rings are finite dimensional Hopf Algebras. In [4] the authors characterized all indecomposable codes over an important family of Hopf algebras. Codes in such a family of Hopf algebras are shown to be a concatenation of cyclic codes in $\mathbb{K}\mathbb{Z}_n$. As an application of identifying the concatenation of ideal / codes with an ideal / code in a bigger algebra is that we can describe easily duals and tensors of such codes. To do so, dual and tensor product of ideal codes are equipped with an additional module structure given by the antipode and the comultiplication of the underlying Hopf algebra respectively. In [5] the authors considered a larger family of Hopf algebras, known as Radford Hopf algebras, that contains the first one. They give a

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decomposition theorem relating these two families of Hopf algebras and using it, duals of indecomposable ideal codes are characterized. These new results extend those in [4], showing that these codes are also concatenation of cyclic codes in a bigger algebra. Our aim in this paper is to study the behavior of tensors of indecomposable ideal codes in the Radford Hopf algebras. The second section is devoted to recall the main definitions of the Taft and Radford Hopf algebras. We give a sketch of the proof of the Theorem appearing in [5] that relates both families in order to understand the structure of the indecomposables in the family of Radford Hopf algebras. Following we start the study of tensors of indecomposables. We recall that in [4] we get that the tensor product of two any indecomposables in the Taft Hopf algebras is always a semisimple ideal code. Now we detach the fact that semisimplicity is lost when considering the family of Radford Hopf algebras (3.3). For prerequisites on Hopf algebras we refer the reader to [2] or [6].

2   Two families of Hopf algebras

Let $n > 1$ be an integer and let $\mathbb{K}$ be a field containing a primitive n-th root of unity ω $(char(\mathbb{K}) \nmid n)$. The free algebra $\mathcal{H} = \mathbb{K}\langle g, x \rangle$ over the non-commutative indeterminates g and x is a bialgebra with comultiplication and counit given by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes g + 1 \otimes x$ and $\varepsilon(g) = 1$, $\varepsilon(x) = 0$ respectively. Let us consider the ideal Γ of $\mathcal{H}$ generated by $g^n - 1$, $xg - \omega gx$ and x^n. The quotient algebra, that we will denote by T_n, is a Hopf algebra (known as Taft Hopf algebra) with antipode given by $S(g) = g^{-1}$ and $S(x) = -xg^{-1}$. Observe that $\{g^i x^j : 0 \leq i, j, < n\}$ is a basis for T_n and so $dim(T_n) = n^2$. As noted before, indecomposable ideal codes over T_n where studied in [4].

Now let $p > 1$ be a prime number and assume $char(\mathbb{K}) \neq p$. Let Γ' be the ideal of $\mathcal{H}$ generated by $g^{pn} - 1$, $xg - \omega gx$, and $x^n - (g^n - 1)$ and let $A_p = \mathcal{H}/\Gamma'$. A_p is also a Hopf algebra with antipode given by $S(g) = g^{-1}$, $S(x) = -xg^{-1}$ (as above, now g and x denote the corresponding projections of the original elements g and x in $\mathcal{H}$). We easily see that the set $B = \{g^i x^j : 0 \leq i < pn, \ 0 \leq j < n\}$ is a basis for A_p and thus $dim(A_p) = pn^2$.

This family of Hopf algebras was introduced by Radford in [7] as an example of Hopf algebra H whose Jacobson radical $J(H)$ is not a Hopf ideal, i.e., the semisimple quotient algebra $H/J(H)$ does not admit a Hopf algebra structure making the canonical projection into a Hopf algebra map. We will refer to A_p as Radford Hopf algebra.

The following result appearing in [5, Theorem 2.2] relates these the two families of Hopf algebras defined above. We include a sketch of the proof here in order to understand better the incoming results.

Theorem 2.1. *Assume that* $\mathbb{K}$ *contains a primitive p-th root of unity θ and an n-th root of θ. Then*

$$A_p \cong T_n \oplus M_n(\mathbb{K}) \oplus \overset{(p-1)}{\cdots} \oplus M_n(\mathbb{K})$$

as algebras.

Proof. $e = \frac{1}{p} \sum_{k=0}^{p-1} g^{kn}$ is a central idempotent in A_p and it can be shown that $A_p e \cong T_n$ as algebras via the isomorphism $g \to ge$, $x \to xe$.

Now $A_p(1-e) \cong M_n(\mathbb{K}) \oplus \overset{(p-1)}{\cdots} \oplus M_n(\mathbb{K})$. To show this we can show that the set $\{f_i\}_{i=1}^{p-1}$ is a complete set of orthogonal central idempotents in $A_p(1-e)$, where

$$f_i = \frac{1}{p} \left(\sum_{j=0}^{p-1} \theta^{ij} g^{nj} \right) (1-e)$$

for $i = 1, \ldots, p-1$. Now considering the Pierce decomposition in $A_p e'$ corresponding to the set $\{f_i : i = 1, ..., p-1\}$, that is,

$$A_p(1-e) = \oplus_{i=1}^{p-1}(A_p(1-e))f_i = \oplus_{i=1}^{p-1} A_p f_i$$

we show that $A_p f_i \cong M_n(\mathbb{K})$ as algebras for every $i = 1, ..., p-1$ via the isomorphism

$$g f_i \longrightarrow M = \lambda^{p-i} diag(1, \omega^{-1}, \ldots, \omega^{-(n-1)})$$

$$x f_i \longrightarrow N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \theta^{p-i} - 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and where $\lambda \in \mathbb{K}$ is the n-th root of θ given by hypothesis.

3   Tensor Product of indecomposables

Our aim is to study the tensor product of indecomposables in A_p. Thus, by the precedent Theorem we have to consider, on one hand, those in T_n, and on the other, those corresponding to the matrix blocks. Indecomposable codes in T_n are discussed in [4] and we will refer to them as indecomposable Taft codes. Through the next subsection we recall their structure.

3.1   Indecomposables Taft codes

Recall that $T_n = \mathbb{K}\langle g, x : g^n = 1, \ x^n = 0, \ xg = \omega gx \rangle$, $dim(T_n) = n^2$ and a basis for T_n is $B = \{g^i x^j : \ 0 \le i, j, < n\}$. The Jacobson radical $J(T_n)$ of T_n coincides with the ideal generated by x. Then the semisimple quotient algebra $\overline{T}_n = T_n/J(T_n)$ is isomorphic to the group algebra $\mathbb{K}\mathbb{Z}_n$ via the map sending $g \mapsto g$ and $x \mapsto 0$. The primitive idempotents of $\mathbb{K}\mathbb{Z}_n$,

$$e_l = \frac{1}{n} \left(\sum_{i=0}^{n-1} \omega^{li} g^i \right), \qquad 0 \le l < n,$$

will give us the complete set $\{S_l : 0 \leq l < n\}$ of isomorphism classes of simple T_n-modules. Writing $u_l = e_l$ and $S_l = \overline{T}_n e_l = \mathbb{K} u_l$ we get that $g \cdot u_l = \omega^{-l} u_l$ and $x \cdot u_l = 0$. By lifting this set of orthogonal primitive idempotents to one of T_n we will obtain the set of isomorphism classes of projective indecomposable modules. Each of them is represented by the projective cover P_l of S_l. Observe that e_l, viewed as an element in T_n, provides a desired lifted idempotent of e_l. Therefore the above set is indeed a complete set of primitive orthogonal idempotents in T_n. Then $P_l = He_l$ is the projective cover of the simple $\overline{T}_n e_l$. Setting $v_i = x^i e_l$ for $i = 0, \ldots, n-1$, the set $\{v_0, \ldots, v_{n-1}\}$ is a basis of P_l. The action of T_n on these elements is given by:

$$
\begin{aligned}
g \cdot v_i &= \omega^{-(l+i)} v_i \\
x \cdot v_i &= v_{i+1}, \qquad i = 0, \cdots, n-2, \\
x \cdot v_{n-1} &= 0
\end{aligned}
\tag{1}
$$

As T_n is a serial algebra, the only submodules of P_l are $N_{l,j} = J(T_n)^j P_l$ for $j = 0, \ldots, n$. Taking into account (1), a basis for $N_{l,j}$ is $\{v_j, \ldots, v_{n-1}\}$ and so $dim(N_{l,j}) = n - j$. Then

$$
\{0\} = N_{l,n} \subset N_{l,n-1} \subset \cdots \subset N_{l,2} \subset N_{l,1} \subset N_{l,0} = P_l
$$

is a composition series of P_l and the composition factors are $N_{l,k}/N_{l,k+1} \cong S_{l+k}$ for $k = 0, \ldots, n-1$. We claim that the quotient module $P_l/N_{l,j}$ is isomorphic to $N_{l+j,n-j}$. For, denote by $[v]$ the class of $v \in P_l$. Then $\{[v_0], \ldots, [v_{j-1}]\}$ is a basis of $P_l/N_{l,j}$. The module structure is given by:

$$
\begin{aligned}
g \cdot [v_i] &= [g \cdot v_i] \\
&= [\omega^{-(i+l)} v_i] \\
&= \omega^{-(i+l)} [v_i], \qquad i = 0, \ldots, j-1, \\
x \cdot [v_i] &= [x \cdot v_i] \\
&= [v_{i+1}], \qquad i = 0, \ldots, j-2, \\
x \cdot [v_{j-1}] &= [x \cdot v_{j-1}] \\
&= [v_j] \\
&= [0]
\end{aligned}
$$

Hence the map $N_{l+j,n-j} \to P_l/N_{l,j}, v_{n-i} \mapsto [v_{j-i}]$ for $i = 1, \ldots, j$ is an isomorphism of modules.

For a characterization of the tensor product of two any indecomposables Taft codes we refer the reader to [4].

3.2 Product of two simples in the matrix part

We recall that any matrix block $M_n(\mathbb{K})$ is generated by a simple S_j, i.e., $M_n(\mathbb{K}) \cong S_j^{(n)}$ for $j = 1, \ldots, p-1$. By the proof of Theorem 2.1 it follows that the action of the generators of A_p over the basis of these simples is as follows: given $S_j = \mathbb{K}\{v_0, \ldots v_n\}$ $(j = 1, \ldots, p-1)$

$$
\begin{aligned}
g \cdot v_i &= \lambda^{(p-j)}\omega^{-i}v_i \\
x \cdot v_i &= v_{i+1}, \qquad i = 0, \ldots, n-2 \\
x \cdot v_{n-1} &= (\theta^{(p-j)} - 1)v_0
\end{aligned}
$$

Lemma 3.1. $S_j \otimes S_j \cong S_k^{(n)}$ for $j = 1, \ldots p-1$ and $k \equiv_p 2j$ as A_p-modules.

Proof. Let us write $S_j \otimes S_j = \mathbb{K}\{z_{r,s} : 0 \le r, s \le n-1\}$ for $z_{r,s} = v_r \otimes v_s$. Let us show the action of g on $z_{r,s}$.

$$
\begin{aligned}
gz_{r,s} &= gv_r \otimes gv_s = \omega^{-r}\lambda^{p-j}v_r \otimes \omega^{-s}\lambda^{p-j}v_s = \omega^{-(r+s)}\lambda^{2(p-j)}v_r \otimes v_s = \\
&= \omega^{-(r+s)}\lambda^{2(p-j)}z_{r,s}
\end{aligned}
$$

So let us consider $z_{k,l}$ such that $k+l \equiv_n 0$ and let $t_i = x^i z_{k,l}$ for $i = 0, \ldots, n-1$ and let us calculate the action of x and g on the elements of $B = \{t_0, \ldots, t_{n-1}\}$:

$$
\begin{aligned}
gt_i &= g(x^i z_{k,l}) = \omega^{-i}x^i(gz_{k,l}) = \omega^{-i}x^i(\omega^{-(k+l)}\lambda^{2(p-j)}z_{k,l}) = \omega^{-i}\lambda^{2(p-j)}x^i z_{k,l} \\
&= \omega^{-i}\lambda^{2(p-j)}t_i, \qquad\qquad \text{for } i = 0, \ldots, n-1
\end{aligned}
$$

$$
xt_i = x(x^i z_{k,l}) = x^{i+1}z_{k,l} = t_{i+1}, \qquad\qquad \text{for } i = 0, \ldots, n-2
$$
and
$$
xt_{n-1} = x(x^{n-1}z_{k,l}) = x^n z_{k,l} = (g^n - 1)z_{k,l} = (\theta^{2(p-j)} - 1)z_{k,l} = (\theta^{2(p-j)} - 1)t_0
$$

since $\lambda^n = \theta$. Thus if $z_{k,l}$ is such that $k + l \equiv_n 0$ we get, on one side, by the action of g on the elements of B, that B is a basis since every t_i $i = 0, \ldots, n-1$ is an eigenvector associated to a different eigenvalue and, on the other hand that $\mathbb{K}\{t_0, t_1, \ldots, t_{n-1}\} \cong S_{2j}$ as A_p-modules. By a reasoning on the dimension of the vector spaces we get that $S_j \otimes S_j \cong S_k^{(n)}$, $k = 2j(mod\ p)$ as A_p-modules.

Lemma 3.2. $S_j \otimes S_k \cong S_{j+k}^{(n)}$ for $j + k < p$ and $j \ne k$ as A_p-modules.

Proof. Analogous to Lemma 3.1.

Lemma 3.3. $S_j \otimes S_k \cong \oplus_{l=0}^{n-1} P_l$ for $j + k = p$ as A_p-modules, where P_l denotes the indecomposable projective ideal code corresponding to the simple ideal code S_l in T_n.

Proof. As in the previous results let us write $S_j \otimes S_k = \mathbb{K}\{z_{r,s} : 0 \le r, s \le n-1\}$ for $z_{r,s} = v_r \otimes v_s$ and let us show the action of g on $z_{r,s}$.

$$
gz_{r,s} = g\omega^{-(r+s)}\lambda^{(p-j+p-k)}z_{r,s} = \omega^{-(r+s)}\lambda^p z_{r,s} = \omega^{-(r+s+l)}z_{r,s}
$$

for some l since $1 = \theta^p = (\lambda^n)^p = (\lambda^p)^n$ and thus $\lambda^p = \omega^{-l}$ for some l.

Let us consider now $z_{r,0}$ for $r = 0, \ldots, n-1$. Then $gz_{r,0} = \omega^{-(r+l)}z_{r,0}$ for every $r = 0, \ldots, n-1$. Let us define $t_{i,r} = x^i z_{r,0}$ $i = 0, \ldots, n-1$. The action of g and x on these elements is

$$gt_{i,r} = g(x^i z_{r,0}) = \omega^{-i} x^i g z_{r,0} = \omega^{-i} x^i \omega^{-(r+l)} z_{r,0} = \omega^{-(i+r+l)} t_{i,r}$$

$$xt_{i,r} = x(x^i z_{r,0}) = x^{i+1} z_{r,0} t_{i+1,r} \qquad \text{for} \quad r = 0, \ldots, n-2$$

$$xt_{n-1,r} = x^n z_{r,0}(g^n - 1) z_{r,0} = (\omega^{-(i+r+l)n} - 1) z_{r,0} = 0$$

As in Lemma 3.1 we get that $\{t_{0,r}, t_{1,r}, \ldots, t_{n-1,r}\}$ is a linear independent set of vectors and from the definition of P_{r+l} we get that $\mathbb{K}\{t_{0,r}, t_{1,r}, \ldots, t_{n-1,r}\} \cong P_{r+l}$. Again, by an argument on the dimensions we get that

$$S_j \otimes S_k \cong \oplus_{l=0}^{n-1} P_{l+r} \cong \oplus_{l=0}^{n-1} P_l$$

Lemma 3.4. $S_j \otimes S_k \cong S_q^{(n)}$ for $j + k = p + q$ as A_p-modules.

Proof. Let q be such that $j + k = p + q$ and let $z_{r,s}$ the same elements considered in Lemma 3.3. Then

$$
\begin{aligned}
gz_{r,s} &= gv_r \otimes gw_s = \omega^{-r} \lambda^{p-j} v_r \otimes \omega^{-s} \lambda^{p-k} w_s = \omega^{-(r+s)} \lambda^{p-j+p-k} v_r \otimes v_s = \\
&= \omega^{-(r+s)} \lambda^{p-j+p-k} z_{r,s} = \omega^{-(r+s)} \lambda^{p-j-k} \lambda^p z_{r,s} \\
&= \omega^{-(r+s)} \lambda^{-q} \omega^{-l} z_{r,s}
\end{aligned}
$$

for some l, as in the previous result. Now let $t_{0,h} = x^h z_{r,s}$ such that $h + l \equiv_n 0$ and $r + s \equiv_n 0$. Thus we get that
$gt_{0,h} = gx^h z_{r,s} = \omega^{-h} x^h g z_{r,s} = \omega^{-h} x^h \omega^{-(r+s)} \lambda^{-q} \omega^{-l} z_{r,s} = \omega^{-(h+l)} \lambda^{-q} x^h z_{r,s} = \lambda^{-q} t_{0,h}$

Consider now the elements $t_{i,h} = x^i t_{0,h}$, $i = 0, \ldots, n-1$ and as before, let us show the actions of g and x on these.

$$gt_{i,h} = gx^i t_{0,h} = \omega^{-i} x^i gt_{0,h} = \omega^{-i} x^i \lambda^{-q} t_{0,h} = \omega^{-i} \lambda^{-q} x^i t_{0,h} = \omega^{-i} \lambda^{-q} t_{i,h}$$

$$xt_{i,h} = x(x^i t_{0,h}) = x^{i+1} t_{0,h} = t_{i+1,h}, \qquad i = 0, \ldots, n-2$$

$$xt_{n-1,h} = x(x^{n-1} t_{0,h}) = x^n t_{0,h} = (g^n - 1) t_{0,h} = (\lambda^{-qn} - 1) t_{0,h} = (\lambda^{p-q} - 1) t_{0,h}$$

Therefore we get that $\mathbb{K}\{t_{0,h}, xt_{0,h}, \ldots, x^{n-1} t_{0,h}\} \cong S_q$ and by a dimension argument $S_j \otimes S_k \cong S_q^{(n)}$.

By the precedent Lemmas we get the following:

Theorem 3.5. *The tensor product of two any simple ideal codes in the matrix part of the decomposition of A_p is either a (finite) direct sum of copies of a simple ideal code in the matrix part or a (finite) direct sum of projective ideal codes in the Taft algebra.*

3.3 Product of an indecomposable Taft code with a simple in the matrix part

Following the notation of Subsection 3.1 we will denote indecomposables Taft codes by $N_{r,s}$. The following Theorem characterizes the structure of a tensor product of such a indecomposable Taft code and a simple code in the matrix part of the decomposition of A_p, say S_j.

Theorem 3.6. $N_{r,s} \otimes S_j \cong S_j^{(n-s)} \cong S_j \otimes N_{r,s}$ for every $j = 1, \ldots, p-1$ and every $r, s = 0, \ldots, n-1$.

Proof. Let $z_{k,l} = u_k \otimes v_l$ $n - s \leq k \leq n - 1$ and $0 \leq l \leq n - 1$ an element of a basis of $N_{r,s} \otimes S_j$. Then the actions of g and x on these elements are

$$gz_{k,l} = gu_k \otimes gv_l = \omega^{-(r+k)}u_k \otimes \omega^{-l}\lambda^{p-j}v_l = \omega^{-(r+k+l)}\lambda^{p-j}u_k \otimes v_l$$

$$= \omega^{-(r+k+l)}\lambda^{p-j}z_{k,l}$$

for $n - s \leq k \leq n - 1$ and $0 \leq l \leq n - 1$.

$$xz_{k,l} = 1 \cdot u_k \otimes xv_l + xu_k \otimes gv_l = u_k \otimes v_{l+1} + u_{k+1} \otimes \omega^{-l}\lambda^{p-j}v_l = z_{k,l+1} + \omega^{-l}\lambda^{p-j}z_{k+1,l}$$

for $n - s \leq k \leq n - 2$ and $0 \leq l \leq n - 2$.

$$xz_{n-1,l} = 1 \cdot u_{n-1} \otimes xv_l + xu_{n-1} \otimes gv_l = u_{n-1} \otimes v_{l+1} + 0 \otimes \omega^{-l}\lambda^{p-j}v_l = z_{n-1,l+1}$$

$$l = 0, \ldots, n - 2$$

$$xz_{n-1,n-1} = 1 \cdot u_{n-1} \otimes xv_{n-1} + xu_{n-1} \otimes gv_{n-1} = u_{n-1} \otimes (\theta^{p-j} - 1)v_0 + 0 \otimes \omega^{-l}\lambda^{p-j}v_{n-1}$$

$$= z_{n-1,0}$$

Since ω is an n-th primitive root of unit we get that $\omega^{-(r+k+l)} = 1$ if and only if $r + k + l \equiv_n 0$. Now for fixed r, $r = 0, \ldots, n-1$, the equation $k + l \equiv_n n - r$ has a different solution for each k, $k = n - s, \ldots, n - 1$.

Let us consider now $z_{k,l}$ such that $r + k + l \equiv_n 0$ and define $t_i = x^i z_{k,l}$. Then we get that

$$gt_i = g(x^i z_{k,l}) = \omega^{-i}x^i gz_{k,l} = \omega^{-i}x^i \omega^{-(k+r+l)}\lambda^{p-j}z_{k,l} = \omega^{-i}x^i \lambda^{p-j}z_{k,l} =$$

$$= \omega^{-i}\lambda^{p-j}t_i$$

From this we get, as in previous reasonings, that $\{t_i : i = 0, \ldots, n-1\}$ is a basis.

The action of x on the elements of this basis is as follows:

$$xt_i = xx^i z_{k,l} = x^{i+1} z_{k,l} = t_{i+1} \quad \text{for } i = 0, \ldots, n-2$$

$$xt_{n-1} = x(x^{n-1} z_{k,l}) = x^n z_{k,l} = (g^n - 1)z_{k,l} = (g^n - 1)t_0 = \omega^{-(k+r+l)}(\lambda^{p-j})^n t_0 - t_0$$

$= (\theta^{p-j} - 1)t_0$ since $\lambda^n = \theta$ and $\omega^n = 1$.

In this way, for each of the s possible elections of $z_{k,l}$ we get that

$$\mathbb{K}\{z_{k,l}, xz_{k,l}, \ldots, x^{n-1} z_{k,l}\} \cong S_j$$

Thus $N_{r,s} \otimes S_j \cong S_j^{(n-s)}$.

An analogous reasoning gives that $S_j \otimes N_{r,s} \cong S_j^{(n-s)}$.

4 Some practical considerations

We recall from [4, Section 5] that indecomposable projective codes in the Taft Hopf algebra (those appearing in 3.3) are a concatenation of cyclic codes and a generator matrix of the indecomposable projective P_l $(l = 0, \cdots, n-1)$ is

$$\frac{1}{n} \begin{pmatrix} 1\,1\cdots 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0\,0\cdots 0 & 1 & \omega^{(l-1)} & \cdots & \omega^{(n-1)(l-1)} & \cdots & 0 & 0 & \cdots & 0 \\ & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0\,0\cdots 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \omega^{(l-n)} & \cdots & \omega^{(n-1)(l-n)} \end{pmatrix}$$

for ω being a n-th root of unity.

Simple codes in the matrix part may be also described easily and this is given in [5, Section IV]. A generator matrix for a simple code S_j in the matrix part is given, as above, by a block matrix,

$$N_j = \frac{1}{pn} \begin{pmatrix} B_0 & \overline{0} & \cdots & \overline{0} & \cdots & \overline{0} \\ \overline{0} & B_1 & \cdots & \overline{0} & \cdots & \overline{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{0} & \overline{0} & \cdots & \overline{0} & \cdots & B_{n-1} \end{pmatrix}$$

where $\overline{0}$ denotes a block of pn zeros and the block B_m, $m = 0, \ldots, n-1$ of length pn, is placed in position m. This block is given by $B_m = (B_{m\ 0}, \ldots, B_{m\ p-1})$ with each $B_{m\ l}$ being the block of length n given by

$$B_{m\ l} = \theta^{lj}(1, \lambda^{p-j}, \lambda^{2(p-j)}, \ldots, \lambda^{(n-1)(p-j)})$$

When considering linear codes, the tensor product of two linear codes can be viewed as a concatenation of these two linear codes (we are considering the usual linear structure) and, thus, the minimum distance of this new code is the minimum of the corresponding two minimum distances. What we have

considered through this paper is the tensor product of two ideal codes and got a structure as ideal code on this new code induced by the comultiplication of the considered Hopf algebra. Now we have got that this structure of ideal code can be characterized depending on the case and, from Lemmas 3.1, 3.2 and 3.4 we get codes that do not include the codes involved in the tensor product. Codes obtained in these Lemmas are concatenation of copies of one determined code in each case. The first difference with the linear case is that the dimension is not the sum of the codes. Secondly, by Lemmas 3.1, 3.2 and 3.4 and from the above matrix N_j, we get that the tensor codes obtained in these Lemmas have minimum distance pn, analogously to the case of linear codes. However, Lemma 3.3 offers interesting consequences from the Coding Theory and Ring Theory points of view respectively. From the Coding Theory point of view we have to remark that dimensions behaves as in Lemmas 3.1, 3.2 and 3.4, but the minimum distance decreases. From Lemma 3.3 and the matrix M_l we get that the minimum distance of the corresponding tensor product is n, being the minimum distance of the original codes appearing in the tensor product equal to pn. From the Ring Theory point of view we get that the tensor product of two ideal codes as in Lemma 3.3 is not semisimple.

Lastly, but not less important, we can observe that tensor products of indecomposables codes in the Radford algebra is always a concatenation of cyclic codes cf. [4] and [5]. Thus, although they do not have a great behavior concerning the minimum distance, they are of easy and fast implementation by blocks and suitable in situations where a limited number of errors can appear in every block, such as the so-called byte error correcting codes (cf. [8] for examples).

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